View metadata, citation and similar papers at core.ac.uk

brought to you by CORE

provided by Crossref

Yang and Deng *Advances in Difference Equations* (2017) 2017:13
DOI 10.1186/s13662-017-1073-x

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Qualitative properties of a p-Laplacian population model with delay

Ying Yang[1] and Jian Deng[2*]

*Correspondence:
dengjian@scnu.edu.cn
[2]School of Mathematical Sciences, South China Normal University, Guangzhou, 510631, China
Full list of author information is available at the end of the article

Abstract

This paper is concerned with qualitative properties of the evolutionary p-Laplacian population model with delay. We first establish the existence of solutions of the model by using the method of parabolic regularization and energy estimate and give the uniqueness by a recursive process. Then, combining the upper and lower solution method and the oscillation theory of functional differential equations, we obtain the oscillation of all positive solutions about the positive equilibrium.

Keywords: p-Laplacian; delay; upper and lower solutions; oscillation

1 Introduction

This paper is concerned with the following evolutionary p-Laplacian with delay:

$$\frac{\partial u}{\partial t} - d(t)\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = u(x,t)\left(a + bu^m(x,t-\tau) - cu^n(x,t-\tau)\right), \quad (x,t) \in \Omega \times \mathbb{R}_+, \tag{1.1}$$

subject to the initial and boundary value conditions

$$|\nabla u|^{p-2}\nabla u \cdot \vec{n} = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}_+, \tag{1.2}$$

$$u(x,t) = \eta(x,t), \quad (x,t) \in \Omega \times [-\tau, 0], \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, $p \geq 2$, $a, b, c, \tau > 0$ are all constants, $m, n > 0$ are integers satisfying $m < n$, $0 < d(t) \in C([0, +\infty))$, and $\eta \in L^\infty(\Omega \times [-\tau, 0]) \cap L^p(-\tau, 0; W^{1,p}(\Omega))$ is a nonnegative function satisfying some suitable compatibility conditions.

The equations of the form (1.1) have been suggested as a mathematical model of the general Logistic model with delay in biology [1]. The functions $u(x,t)$ represent the spatial density of the population at space x and time t, the diffusion term $\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ represents the effect of dispersion in the habitat, which models a tendency to avoid crowding, and the speed of the diffusion is rather slow since $p > 2$, τ is the generation time, the boundary conditions (1.2) describe the living environment at the boundary and that there is no migration of individuals across the boundary $\partial\Omega$, a denotes the birth rate, b is the

© The Author(s) 2017. This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (http://creativecommons.org/licenses/by/4.0/), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

inter-specific competition, which represents an advantage to the species in grouping together, in that it adds to the growth rate in regions of high population density, whereas c is the intra-specific competition for resources that inhibits population growth.

In the last few decades, there are many works on the existence and uniqueness of solutions for parabolic equations with delay(s) (see [2–7] and the references therein). For example, Hino et al. [2] studied the existence of almost periodic solutions of parabolic equations with infinity delay in Banach spaces. In [3], the authors established the existence of positive travelling fronts for N-dimensional delayed reaction diffusion systems. Pao [4, 5] discussed the global existence and uniqueness of coupled system of nonlinear parabolic equations with both continuous and discrete delays. However, most of the results in the literature are concerned with linear and semilinear parabolic equations with delay(s). But for the quasilinear parabolic equation, especially for the degenerate or singular parabolic equations, with delay(s) in nonlinear source terms, as far as we know, there are very few results.

The oscillation of solutions for delayed evolutionary equations has received widespread attention; see, for example, [8–14] and the references therein. For the ODE model, Gopalsamy and Ladas [8] studied the oscillation and asymptotic behavior of

$$N'(t) = N(t)\big[a + bN(t-\tau) - cN^2(t-\tau)\big].$$

The oscillation of all positive solutions about the positive equilibrium N^* is established, and the equilibrium N^* attracts the solutions of the initial value problem globally. It is known that the density of the population is not only dependent on time but also on the position in space. So, taking the spatial structure into account, in [11], the author investigated the oscillation of the positive equilibrium for the general Logistic model with linear diffusion. Then, in 1997, the asymptotic stability of system (1.1)-(1.3) without delay was investigated [12]. It is worth mentioning the works by Wang and Wang et al. [15, 16], who studied the oscillation of the population model for the case $p = 2$ of (1.1). Using the upper and lower solution method and theory of functional differential equation, the authors showed that all positive solutions of the model oscillate about the positive equilibrium. However, comparing to the linear diffusion equation with the property of infinite speed of propagation of perturbations, it should be more reasonable to introduce the nonlinear diffusion version of equation (1.1), namely

$$\frac{\partial u}{\partial t} - d(t)\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) = u(x,t)\big(a + bu^m(x,t-\tau) - cu^n(x,t-\tau)\big).$$

The advantage of this modified version lies in that it involves non-Newtonian filtration diffusion, which is more suitable to the real-world applications.

Motivated by [15], in the present paper, we investigate the oscillation of (1.1)-(1.3). As far as we know, few works concerned with the oscillation property were obtained for the quasilinear parabolic equations such as a non-Newtonian filtration equation with delay. The biggest difficulty lies in that the oscillation property is studied on a point-to-point basis. In fact, for the linear partial differential systems with delay, we can discuss the oscillation theory of classical solutions (see, e.g., [11, 14–17], and the references therein). However, because of the degeneracy and the singularity, equation (1.1) might not have classical solutions in general. Therefore, before studying the oscillation property, we require that solutions of problem (1.1)-(1.3) are appropriately smooth, which makes us have to first discuss the existence and uniqueness of a Hölder continuous solution. Using the method of parabolic regularization method and energy estimate, we investigate the existence of solutions for (1.1)-(1.3). Especially, we show the Hölder continuity of solutions. Then, according to a recursive process, we also give the uniqueness of the solution. Based on these results, we find that, for the non-Newtonian filtration equation, the oscillation phenomenon may occur. By employing the upper and lower solution method and the oscillation theory of functional differential equation, we establish a sufficient condition for all positive solutions of the equation to oscillate about the positive equilibrium.

This paper is organized as follows. In Section 2, we introduce some basic assumptions and the definition of weak solutions. Section 3 is devoted to the study of the existence of solutions. In Section 4, we establish the uniqueness of the solution. In Section 5, we investigate the oscillation of the solution and present some examples that show the applicability of our results.

2 Preliminaries

As preliminaries, in this section, we present the basic assumptions and the definition of weak solutions. First, we introduce some notations. For any $T > 0$, let

$$Q_T = \Omega \times (0,T), \qquad Q_\tau = \Omega \times [-\tau,0], \qquad u(t-\tau) = u(x,t-\tau)$$

and denote

$$E = \left\{ w : w \in L^p\big(0,T; W^{1,p}(\Omega)\big) \cap L^\infty(Q_T) \cap C^{1,1/2}(\overline{Q_T}), \frac{\partial w}{\partial t} \in L^2(Q_T) \right\}.$$

Because of the degeneracy and singularity, equation (1.1) may not have classical solutions in general, and hence we consider nonnegative solutions of equation (1.1) in the following weak sense.

Definition 2.1 A function $u \in E$ is said to be a weak solution of problem (1.1)-(1.3) if for any $T > 0$, $\varphi \in \mathring{C}^\infty(\overline{Q_T})$, and $h(x) \in C_0^\infty(\Omega)$, we have the following integral equality:

$$\begin{aligned} &\iint_{Q_T} \left(\frac{\partial u}{\partial t}\varphi + d(t)|\nabla u|^{p-2}\nabla u \nabla\varphi \right) dx\,dt \\ &\quad = \iint_{Q_T} u\big(a + bu^m(t-\tau) - cu^n(t-\tau)\big)\varphi\,dx\,dt, \end{aligned} \tag{2.1}$$

and

$$\lim_{t\to 0^+} \int_\Omega u(x,t)h(x)\,dx = \int_\Omega \eta(x,0)h(x)\,dx, \tag{2.2}$$

$$u(x,\theta) = \eta(x,\theta), \quad (x,\theta) \in Q_\tau. \tag{2.3}$$

In what follows, we also give the definition of quasi-upper and quasi-lower solutions.

Definition 2.2 A pair of functions $\tilde{u}(x,t), \hat{u}(x,t) \in E$ is said to be coupled quasi-upper and quasi-lower solutions of equations (1.1)-(1.3) if $\tilde{u}(x,t) \geq \hat{u}(x,t)$ in $\Omega \times (-\tau, T)$ and they satisfy

$$\begin{aligned}
&\frac{\partial \tilde{u}}{\partial t} - d(t)\operatorname{div}\big(|\nabla \tilde{u}|^{p-2}\nabla \tilde{u}\big) \geq \tilde{u}\big(a + bw^m - cw^n\big) \quad \text{for all } w \in \langle \hat{u}, \tilde{u}\rangle, \\
&\frac{\partial \hat{u}}{\partial t} - d(t)\operatorname{div}\big(|\nabla \hat{u}|^{p-2}\nabla \hat{u}\big) \leq \hat{u}\big(a + bw^m - cw^n\big) \quad \text{for all } w \in \langle \hat{u}, \tilde{u}\rangle, \\
&\frac{\partial \tilde{u}}{\partial \vec{n}} \geq 0, \qquad \frac{\partial \hat{u}}{\partial \vec{n}} \leq 0, \quad (x,t) \in \partial\Omega \times (0, T], \\
&\hat{u}(x,\theta) \leq u(x,\theta) \leq \tilde{u}(x,\theta), \quad (x,\theta) \in \Omega \times [-\tau, 0),
\end{aligned}$$

in the weak sense, where $\langle \hat{u}, \tilde{u}\rangle \equiv \{u \in C(Q_T) : \hat{u} \leq u \leq \tilde{u}\}$.

3 The existence of solutions

In this section, we study the existence of solutions for problem (1.1)-(1.3).

To study the existence of solutions, let us first consider the regularized problem

$$\begin{aligned}
&\frac{\partial u_\varepsilon}{\partial t} = d(t)\operatorname{div}\Big(\big(|\nabla u_\varepsilon|^2 + \varepsilon\big)^{(p-2)/2}\nabla u_\varepsilon\Big) + u_\varepsilon\big(a + bu_\varepsilon^m(t-\tau) - cu_\varepsilon^n(t-\tau)\big), \\
&\quad (x,t) \in \Omega \times \mathbb{R}_+, \qquad (3.1)\\
&\frac{\partial u_\varepsilon}{\partial \vec{n}} = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}_+, \qquad (3.2)\\
&u_\varepsilon(x,\theta) = \eta_\varepsilon(x,\theta), \quad (x,t) \in \Omega \times [-\tau, 0], \qquad (3.3)
\end{aligned}$$

where $\eta_\varepsilon(x,\theta)$ is a positive bounded function in $C^\infty(Q_\tau)$ satisfying the condition

$$0 < \eta_\varepsilon \leq \|\eta\|_{L^\infty(Q_\tau)}.$$

The desired solution of problem (1.1)-(1.3) will be obtained by the limit of some subsequence of solutions u_ε of the regularized problem (3.1)-(3.3). We first need to establish the existence of solutions u_ε, which can be done by using the method of upper and lower solutions and associated monotone iterations.

Theorem 3.1 *If* $\eta_\varepsilon(x,\theta) > 0$ *for* $(x,\theta) \in \Omega \times [-\tau, 0)$, *then* (3.1)-(3.3) *have a unique positive global solution in* $\Omega \times (-\tau, +\infty)$.

Proof Since $a > 0, c > 0$, and $m < n$, without loss of generality, we may suppose that the maximum of the function $f(s) = a + bs^m - cs^n$ is equal to M ($M \geq a > 0$) when $s > 0$. Let $\tilde{u} = M^* e^{M(t+\tau)}$, $\hat{u} = 0$, where $M^* \geq \max_{(x,t)\in\Omega\times[-\tau,0)} \eta(x,t)$. It is easy to verify that, for any $T > 0$, two functions $\tilde{u}, \hat{u}$ in $\Omega \times (-\tau, T)$ are a couple of upper and lower solutions of (3.1)-(3.3). By Theorem 2.2 of [10], (3.1)-(3.3) have a unique solution $u_\varepsilon(x,t)$ in $\Omega \times (-\tau, T)$ and $u_\varepsilon(x,t) \in \langle 0, \tilde{u}\rangle$. Since T is arbitrary and $\eta_\varepsilon(x,\theta)$ is positive, (3.1)-(3.3) have a unique continuous positive global solution $u_\varepsilon(x,t)$ in $\Omega \times (-\tau, +\infty)$. □

We need the following lemma for the a priori estimates on solutions u_ε.

Lemma 3.1 *There exists a positive constant C_{GN} such that, for all $u \in W^{1,p}(\Omega)$,*

$$\|u\|_s \le C_{GN}\big(\|\nabla u\|_p^a \|u\|_\theta^{1-a} + \|u\|_\theta\big)$$

with $0 < \theta \le s < p^*$ and $a = (\frac{N}{\theta} - \frac{N}{s})(1 - \frac{N}{p} + \frac{N}{\theta})^{-1}$, where $p^* = \frac{Np}{N-p}$ when $N > p$ and $p^* = \infty$ when $N \le p$.

Lemma 3.2 *Assume that u_ε is a solution of* (3.1)-(3.3). *Then there exists a positive constant C, independent of ε, such that*

$$\|u_\varepsilon\|_{L^\infty(Q_T)} \le C.$$

Proof Multiplying equation (3.1) by u_ε^s ($s \ge 0$) and integrating over Ω, we have

$$\begin{aligned}&\frac{1}{s+1}\frac{\partial}{\partial t}\int_\Omega u_\varepsilon^{s+1}\,dx + sd(t)\int_\Omega \big(|\nabla u_\varepsilon|^2 + \varepsilon\big)^{(p-2)/2} u_\varepsilon^{s-1}|\nabla u_\varepsilon|^2\,dx \\ &\quad= \int_\Omega u_\varepsilon^{s+1}\big(a + bu_\varepsilon^m(t-\tau) - cu_\varepsilon^n(t-\tau)\big)\,dx.\end{aligned}$$

Here

$$\frac{1}{s+1}\frac{\partial}{\partial t}\int_\Omega u_\varepsilon^{s+1}\,dx + sd(t)\int_\Omega |\nabla u_\varepsilon|^p u_\varepsilon^{s-1}\,dx \le M\int_\Omega u_\varepsilon^{s+1}\,dx.$$

Therefore, we have

$$\frac{\partial}{\partial t}\int_\Omega u_\varepsilon^{s+1}\,dx + \frac{s(s+1)d(t)}{(\frac{s-1}{p}+1)^p}\int_\Omega \big|\nabla u_\varepsilon^{\frac{s-1}{p}+1}\big|^p\,dx \le M(s+1)\int_\Omega u_\varepsilon^{s+1}\,dx.$$

Then,

$$\frac{\partial}{\partial t}\int_\Omega u_\varepsilon^{s+1}\,dx + \frac{s(s+1)d_m}{(s+p-1)^p}\int_\Omega \big|\nabla u_\varepsilon^{\frac{p+s-1}{p}}\big|^p\,dx \le M(s+1)\int_\Omega u_\varepsilon^{s+1}\,dx, \tag{3.4}$$

where $d_m = \min_{[0,T]} d(t)$. Let

$$s_k = p^k - \frac{1}{p-1}, \qquad \alpha_k = \frac{p(s_k+1)}{s_k+p-1}, \qquad u_k(t) = u_\varepsilon^{(s_k+p-1)/p}(t) \quad (k = 0,1,\ldots).$$

Then inequality (3.4) with $s = s_k$ becomes

$$\frac{d}{dt}\big\|u_k(t)\big\|_{\alpha_k}^{\alpha_k} + \frac{s_k(s_k+1)d_m}{(s_k+p-1)^p}\|\nabla u_k\|_p^p \le M(s_k+1)\big\|u_k(t)\big\|_{\alpha_k}^{\alpha_k}. \tag{3.5}$$

To estimate the terms on the right-hand side of inequality (3.5), we apply Lemma 3.1 with $\theta = 1, s = \alpha_k$, and $a = \frac{1-\frac{1}{\alpha_k}}{1-\frac{1}{p}+\frac{1}{N}}$ to get

$$\big\|u_k(t)\big\|_{\alpha_k}^{\alpha_k} \le (2C_{GN})^{\alpha_k}\big(\big\|\nabla u_k(t)\big\|_p^{a\alpha_k}\big\|u_k(t)\big\|_1^{(1-a)\alpha_k} + \big\|u_k(t)\big\|_1^{\alpha_k}\big),$$

and thus

$$\begin{aligned}
&\frac{d}{dt}\|u_k(t)\|_{\alpha_k}^{\alpha_k} + \frac{s_k(s_k+1)d_m}{(s_k+p-1)^p}\|\nabla u_k(t)\|_p^p \\
&\quad\le C_1(s_k+1)\big(\|\nabla u_k(t)\|_p^{a\alpha_k}\|u_k(t)\|_1^{(1-a)\alpha_k} + \|u_k(t)\|_1^{\alpha_k} \\
&\qquad + \|\nabla u_k(t)\|_p^{\frac{a\alpha_k s_k}{s_k+1}}\|u_k(t)\|_1^{\frac{(1-a)\alpha_k s_k}{s_k+1}} + \|u_k(t)\|_1^{\frac{\alpha_k s_k}{s_k+1}}\big),
\end{aligned} \tag{3.6}$$

where C_1 is a constant independent of k.

Since $a\alpha_k \in (0, \frac{Np(p-1)}{N(p-1)+p})$, we can apply Young's inequality to estimate

$$\begin{aligned}
&C_1(s_k+1)\big(\|\nabla u_k(t)\|_p^{a\alpha_k}\|u_k(t)\|_1^{(1-a)\alpha_k} + \|\nabla u_k(t)\|_p^{\frac{a\alpha_k s_k}{s_k+1}}\|u_k\|_1^{\frac{(1-a)\alpha_k s_k}{s_k+1}}\big) \\
&\quad\le \frac{s_k(s_k+1)d_m}{2(s_k+p-1)^p}\|\nabla u_k(t)\|_p^p + \left(\frac{4(s_k+p-1)^p}{s_k(s_k+1)d_m}\right)^{\frac{a\alpha_k}{p-a\alpha_k}}\big\{C_1(s_k+1)\|u_k\|_1^{(1-a)\alpha_k}\big\}^{\frac{p}{p-a\alpha_k\frac{s_k}{s_k+1}}} \\
&\qquad + \left(\frac{4(s_k+p-1)^p}{s_k(s_k+1)d_m}\right)^{\frac{a\alpha_k}{p\frac{s_k+1}{s_k}-a\alpha_k}}\big\{C_1(s_k+1)\|u_k(t)\|_1^{(1-a)\alpha_k\frac{s_k}{s_k+1}}\big\}^{\frac{p}{p-a\alpha_k\frac{s_k}{s_k+1}}}
\end{aligned}$$

and get

$$\begin{aligned}
&\frac{d}{dt}\|u_k(t)\|_{\alpha_k}^{\alpha_k} + \frac{s_k(s_k+1)d_m}{2(s_k+p-1)^p}\|\nabla u_k(t)\|_p^p \\
&\quad\le C_1(s_k+1)\big(\|u_k\|_1^{\alpha_k} + \|u_k(t)\|_1^{\frac{\alpha_k s_k}{s_k+1}}\big) + \left(\frac{4(s_k+p-1)^p}{s_k(s_k+1)d_m}\right)^{\frac{a\alpha_k}{p-a\alpha_k}} \\
&\qquad\times\big\{\big(C_1(s_k+1)\|u_k(t)\|_1^{(1-a)\alpha_k}\big)^{\frac{p}{p-a\alpha_k}} + \big(C_1(s_k+1)\|u_k(t)\|_1^{(1-a)\alpha_k\frac{s_k}{s_k+1}}\big)^{\frac{p}{p-a\alpha_k\frac{s_k}{s_k+1}}}\big\}.
\end{aligned}$$

Applying the Poincaré inequality

$$C_2\|u_k(t)\|_{\alpha_k}^p \le \|\nabla u_k(t)\|_p^p + \|u_k(t)\|_1^p,$$

we have

$$\begin{aligned}
&\frac{d}{dt}\|u_k(t)\|_{\alpha_k}^{\alpha_k} + \frac{C_2 s_k(s_k+1)d_m}{2(s_k+p-1)^p}\|\nabla u_k(t)\|_p^p \\
&\quad\le C_1(s_k+1)\big(\|u_k\|_1^{\alpha_k} + \|u_k(t)\|_1^{\frac{\alpha_k s_k}{s_k+1}}\big) + \frac{s_k(s_k+1)d_m}{2(s_k+p-1)^p}\|u_k(t)\|_1^p \\
&\qquad + \left(\frac{4(s_k+p-1)^p}{s_k(s_k+1)d_m}\right)^{\frac{a\alpha_k}{p-a\alpha_k}}\big\{\big(C_1(s_k+1)\|u_k(t)\|_1^{(1-a)\alpha_k}\big)^{\frac{p}{p-a\alpha_k}} \\
&\qquad + \big(C_1(s_k+1)\|u_k(t)\|_1^{(1-a)\alpha_k\frac{s_k}{s_k+1}}\big)^{\frac{p}{p-a\alpha_k\frac{s_k}{s_k+1}}}\big\}.
\end{aligned}$$

Since $\|u_k(t)\|_1 = \|u_{k-1}(t)\|_{\alpha_{k-1}}^{\alpha_{k-1}}$, we have

$$\frac{d}{dt}\|u_k(t)\|_{\alpha_k}^{\alpha_k} + \frac{C_2 s_k(s_k+1)d_m}{2(s_k+p-1)^p}\|u_k(t)\|_{\alpha_k}^p \le C_3 s_k^{\frac{p+(p-2)a\alpha_k}{p-a\alpha_k}}\chi_{k-1}^{p\alpha_{k-1}},$$

where C_3 is a constant independent of k, and $\chi_{k-1} = \max\{1, \sup_{t\in[0,T]} \|u_{k-1}(t)\|_{\alpha_{k-1}}\}$. Taking the continuity of $\|u_k(t)\|_{\alpha_k}$ into account, we get that there exists t_0 at which $\|u_k(t)\|_{\alpha_k}$ reaches its maximum value, and then we have

$$\sup_{t\in[0,T]} \|u_k(t)\|_{\alpha_k} \le C_4 \lambda^k \chi_{k-1}^p,$$

where C_4 is a constant independent of k, and $\lambda = p^{\frac{(p-1)^2 N}{p}} > 1$. Therefore,

$$\begin{aligned}
\ln \chi_k &\le \ln C_4 + k \ln \lambda + p \ln \chi_{k-1} \\
&\le \ln C_4 \sum_{i=0}^{k-1} p^i + p^k \ln \chi_0 + \ln \lambda \sum_{j=0}^{k-1} (k-j) p^j \\
&\le (\ln C_4 + \ln \chi_0) p^k + f(k) \ln \lambda,
\end{aligned}$$

namely,

$$\sup_{t\in[0,T]} \|u_\varepsilon(t)\|_{s_k+1} \le \{C_4^{p^k} \chi_0^{p^k} \lambda^{f(k)}\}^{\frac{p}{s_k+p-1}},$$

where $f(k) = p^{k+1} - p^{k-1} - k - 2$. Letting $k \to \infty$, we get

$$\sup_{t\in[0,T]} \|u_\varepsilon(t)\|_\infty \le C_5 \chi_0^p \le C_5 \max\Big\{1, \sup_{t\in[0,T]} \|u_\varepsilon(t)\|_{2-\frac{1}{p-1}}^{p-\frac{1}{p-1}}\Big\}, \tag{3.7}$$

where C_5 is a constant independent of k and ε.

In what follows, we estimate $\sup_{t\in[0,T]} \|u_\varepsilon(t)\|_2$.

Multiplying equation (3.1) by u_ε and integrating the resulting relation over Ω, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega u_\varepsilon^2\,dx + \int_\Omega d(t)\big(|\nabla u_\varepsilon|^2 + \varepsilon\big)^{(p-2)/2} |\nabla u_\varepsilon|^2\,dx \\
&\quad \le \int_\Omega u_\varepsilon^2\big(a + b u_\varepsilon^m(t-\tau) - c u_\varepsilon^n(t-\tau)\big)\,dx,
\end{aligned} \tag{3.8}$$

which, together with $\int_\Omega d(t)(|\nabla u_\varepsilon|^2 + \varepsilon)^{(p-2)/2} |\nabla u_\varepsilon|^2\,dx \ge 0$, implies that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u_\varepsilon^2\,dx \le M \int_\Omega u_\varepsilon^2\,dx.$$

By Gronwall's inequality we have

$$\int_\Omega u_\varepsilon^2(x,t)\,dx \le e^{2MT} \int_\Omega u_\varepsilon^2(x,0)\,dx = e^{2MT} \int_\Omega \eta_\varepsilon^2(x,0)\,dx.$$

Therefore,

$$\sup_{t\in[0,T]} \int_\Omega u_\varepsilon^2(x,t)\,dx \le e^{2MT} \int_\Omega \eta_\varepsilon^2(x,0)\,dx \le e^{2MT} \|\eta\|_{L^\infty(Q_\tau)}^2 |\Omega|,$$

which, together with (3.7), gives $\|u_\varepsilon\|_{L^\infty(Q_T)} \le C$ for some C independent of ε. Thus, the proof of this lemma is completed. □

Lemma 3.3 *Assume that u_ε is a solution of problem* (3.1)-(3.3). *Then there exists a positive constant C, independent of ε, such that*

$$\iint_{Q_T} |\nabla u_\varepsilon|^p \,dx\,dt \le C.$$

Proof Integrating (3.8) over Ω, we have

$$\frac{1}{2}\iint_{Q_T} \frac{\partial u_\varepsilon^2}{\partial t}\,dx\,dt + \iint_{Q_T} d(t)\bigl(|\nabla u_\varepsilon|^2+\varepsilon\bigr)^{(p-2)/2}|\nabla u_\varepsilon|^2\,dx\,dt \le M\iint_{Q_T} u_\varepsilon^2\,dx\,dt.$$

Then,

$$\frac{1}{2}\iint_{Q_T} \frac{\partial u_\varepsilon^2}{\partial t}\,dx\,dt + \iint_{Q_T} d(t)|\nabla u_\varepsilon|^p\,dx\,dt \le M\iint_{Q_T} u_\varepsilon^2\,dx\,dt,$$

that is,

$$\frac{1}{2}\int_\Omega u_\varepsilon^2(x,T)\,dx - \frac{1}{2}\int_\Omega u_\varepsilon^2(x,0)\,dx + d_m\iint_{Q_T} |\nabla u_\varepsilon|^p\,dx\,dt \le M\iint_{Q_T} u_\varepsilon^2\,dx\,dt,$$

which implies

$$d_m\iint_{Q_T} |\nabla u_\varepsilon|^p\,dx\,dt \le M\iint_{Q_T} u_\varepsilon^2\,dx\,dt + \frac{1}{2}\int_\Omega \eta_\varepsilon^2(x,0)\,dx \le C.$$

Therefore, we get

$$\iint_{Q_T} |\nabla u_\varepsilon|^p\,dx\,dt \le C.$$

Thus, the proof of this lemma is completed. □

Lemma 3.4 *Assume that u_ε is a solution of problem* (3.1)-(3.3). *Then there exists a positive constant C, independent of ε, such that*

$$\iint_{Q_T} \left|\frac{\partial u_\varepsilon}{\partial t}\right|^2 dx\,dt \le C.$$

Proof Multiplying equation (3.1) by $\frac{\partial u_\varepsilon}{\partial t}/d(t)$ and integrating over Q_T, we have

$$\begin{aligned}
&\iint_{Q_T} \frac{1}{d(t)}\left|\frac{\partial u_\varepsilon}{\partial t}\right|^2 dx\,dt \\
&\quad= \iint_{Q_T} \operatorname{div}\bigl(\bigl(|\nabla u_\varepsilon|^2+\varepsilon\bigr)^{(p-2)/2}\nabla u_\varepsilon\bigr)\frac{\partial u_\varepsilon}{\partial t}\,dx\,dt \\
&\qquad+ \iint_{Q_T} \frac{u_\varepsilon}{d(t)}\frac{\partial u_\varepsilon}{\partial t}\bigl(a+bu_\varepsilon^m(t-\tau)-cu_\varepsilon^n(t-\tau)\bigr)\,dx\,dt.
\end{aligned}$$

Then,

$$\iint_{Q_T} \frac{1}{d(t)} \left| \frac{\partial u_\varepsilon}{\partial t} \right|^2 dx\,dt + \iint_{Q_T} \left(|\nabla u_\varepsilon|^2 + \varepsilon \right)^{(p-2)/2} \nabla u_\varepsilon \frac{\partial \nabla u_\varepsilon}{\partial t}\,dx\,dt$$
$$= \iint_{Q_T} \frac{u_\varepsilon}{d(t)} \frac{\partial u_\varepsilon}{\partial t} \left(a + b u_\varepsilon^m (t-\tau) - c u_\varepsilon^n (t-\tau) \right) dx\,dt,$$

that is,

$$\iint_{Q_T} \frac{1}{d(t)} \left| \frac{\partial u_\varepsilon}{\partial t} \right|^2 dx\,dt + \frac{1}{p} \iint_{Q_T} \frac{\partial}{\partial t} \left(|\nabla u_\varepsilon|^2 + \varepsilon \right)^{p/2} dx\,dt \le C \iint_{Q_T} \frac{1}{d(t)} \left| \frac{\partial u_\varepsilon}{\partial t} \right| dx\,dt,$$

which implies

$$\iint_{Q_T} \frac{1}{d(t)} \left| \frac{\partial u_\varepsilon}{\partial t} \right|^2 dx\,dt \le C \iint_{Q_T} \frac{1}{d(t)} \left| \frac{\partial u_\varepsilon}{\partial t} \right| dx\,dt + \frac{1}{p} \int_\Omega \left(\left| \nabla u_\varepsilon(x,0) \right|^2 + \varepsilon \right)^{p/2} dx.$$

Then, we have

$$\iint_{Q_T} \frac{1}{2d(t)} \left| \frac{\partial u_\varepsilon}{\partial t} \right|^2 dx\,dt \le \frac{C}{2} \iint_{Q_T} \frac{1}{d(t)}\,dx\,dt + \frac{C}{p} \int_\Omega \left| \nabla \eta_\varepsilon(x,0) \right|^p dx + C.$$

So, we have

$$\frac{1}{2d_M} \iint_{Q_T} \left| \frac{\partial u_\varepsilon}{\partial t} \right|^2 dx\,dt \le \frac{C}{2d_m} |Q_T| + \frac{C}{p} \int_\Omega \left| \nabla \eta_\varepsilon(x,0) \right|^p dx + C.$$

Therefore, we get

$$\iint_{Q_T} \left| \frac{\partial u_\varepsilon}{\partial t} \right|^2 dx\,dt \le C.$$

Thus, the proof of this lemma is completed. □

We are now in a position to present the proof of the existence of generalized solutions for problem (1.1)-(1.3).

Theorem 3.2 *The initial and boundary value problem* (1.1)-(1.3) *admits at least one solution.*

Proof Let $\varepsilon = 1/h$ ($h = 1, 2, \ldots$), and let u_h be a solution of problem (3.1)-(3.3). According to Lemmas 3.2, 3.2, and 3.4, we see that

$$\|u_h\|_{L^\infty(Q_T)} \le C,$$
$$\|\nabla u_h\|_{L^p(Q_T)}^p \le C,$$
$$\left\| \frac{\partial u_h}{\partial t} \right\|_{L^2(Q_T)}^2 \le C.$$

Furthermore, we can obtain the Hölder norm estimate of solutions

$$\left|u_h(x_2,t_2)-u_h(x_1,t_1)\right| \le C\left(|x_2-x_1|+|t_2-t_1|^{1/2}\right). \tag{3.9}$$

It suffices to prove that, for any $t_0 \in (0,T)$, u satisfies (3.9) on $\Omega \times (t_0,T)$.

Consider the mollifier

$$u_{h\epsilon} = J_\epsilon u_h(x,t) = \int_0^T \int_{\mathbb{R}^N} j_\epsilon(x-y,t-\gamma)u_h(y,\gamma)\,dy\,d\gamma, \quad 0<\epsilon<t_0<t<T-\epsilon.$$

For any $x_1, x_2 \in \Omega$,

$$\begin{aligned}
&u_{h\epsilon}(x_1,t)-u_{h\epsilon}(x_2,t)\\
&\quad= \int_0^T \int_{\mathbb{R}^N} \int_0^1 \frac{d}{ds} j_\epsilon\left(sx_1+(1-s)x_2-y,t-\gamma\right)u_h(y,\gamma)\,ds\,dy\,d\gamma\\
&\quad= \int_0^T \int_{\mathbb{R}^N} \int_0^1 \nabla_x j_\epsilon\left(sx_1+(1-s)x_2-y,t-\gamma\right)u_h(y,\gamma)\,ds\,dy\,d\gamma \cdot (x_1-x_2)\\
&\quad= -\int_0^1 \int_{\mathbb{R}^N} \int_0^T \nabla_y j_\epsilon\left(sx_1+(1-s)x_2-y,t-\gamma\right)u_h(y,\gamma)\,dy\,d\gamma\,ds \cdot (x_1-x_2)\\
&\quad= \int_0^1 \int_{\mathbb{R}^N} \int_0^T j_\epsilon\left(sx_1+(1-s)x_2-y,t-\gamma\right)\cdot \nabla_y u_h(y,\gamma)\,dy\,d\gamma\,ds \cdot (x_1-x_2).
\end{aligned}$$

Hence, by Lemma 3.3,

$$\begin{aligned}
&\left|u_{h\epsilon}(x_1,t)-u_{h\epsilon}(x_2,t)\right|\\
&\quad\le \int_0^1 \int_{\Omega} \int_0^T \left|j_\epsilon\left(sx_1+(1-s)x_2-y,t-\gamma\right)\right| \cdot \left|\nabla_y u_h(y,\gamma)\right|\,dy\,d\gamma\,ds \cdot |x_1-x_2|\\
&\quad\le \int_0^1 \left(\iint_{Q_T} \left|j_\epsilon\left(sx_1+(1-s)x_2-y,t-\gamma\right)\right|^q\,dy\,d\gamma\right)^{1/q}\\
&\qquad \cdot \left(\iint_{Q_T} \left|\nabla u_h(y,\gamma)\right|^p\,dy\,d\gamma\right)^{1/p} ds|x_1-x_2|\\
&\quad\le C|x_1-x_2|.
\end{aligned} \tag{3.10}$$

Here and below, C denotes a constant independent of ϵ.

Let $0<\epsilon<t_0<t_1<t_2<T$, $B(\Delta t) = B_{(\Delta t)^{1/2}}(x_0)$, $\zeta \in C_0^1(B(\Delta t))$, $x_0 \in \Omega$, $\Delta t = t_2 - t_1$. Then

$$\begin{aligned}
&\int_{B(\Delta t)} \zeta(x)\left(u_{h\epsilon}(x,t_2)-u_{h\epsilon}(x,t_1)\right)dx\\
&\quad= \int_{B(\Delta t)} \zeta(x) \int_0^1 \frac{d}{ds} u_{h\epsilon}\left(x,st_2+(1-s)t_1\right)ds\,dx\\
&\quad= \Delta t \int_{B(\Delta t)} \zeta(x) \int_0^1 \int_0^T \int_{\mathbb{R}^N} j_{\epsilon t}\left(x-y,st_2+(1-s)t_1-\gamma\right)\cdot u_h(y,\gamma)\,dy\,d\gamma\,ds\,dx\\
&\quad= -\Delta t \int_{B(\Delta t)} \zeta(x) \int_0^1 \int_0^T \int_{\mathbb{R}^N} j_{\epsilon\gamma}\left(x-y,st_2+(1-s)t_1-\gamma\right)\\
&\qquad \cdot u_h(y,\gamma)\,dy\,d\gamma\,ds\,dx.
\end{aligned} \tag{3.11}$$

Noting that, for any fixed $(x,t) \in Q_T$ with $0 < \epsilon < t_0 < t < T - \epsilon$, $J_\epsilon(x-y,t-\gamma) \in C_0^1(Q_T)$, from the regularized problem (3.1)-(3.3) we obtain

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^N} j_{\epsilon\gamma}\big(x-y, st_2 + (1-s)t_1 - \gamma\big) u_h(y,\gamma)\,dy\,d\gamma \\
&= -\int_0^T \int_{\mathbb{R}^N} j_\epsilon\big(x-y, st_2 + (1-s)t_1 - \gamma\big) \frac{\partial}{\partial\gamma} u_h(y,\gamma)\,dy\,d\gamma \\
&= -\int_0^T \int_{\mathbb{R}^N} j_\epsilon\big(x-y, st_2 + (1-s)t_1 - \gamma\big) \\
&\quad \cdot \left[\operatorname{div}\left(\left(|\nabla_y u_h|^2 + \frac{1}{h}\right)^{(p-2)/2} \nabla_y u_h\right) + u_h\big(a + bu_h^m(\gamma-\tau) - cu_h^n(\gamma-\tau)\big)\right] dy\,d\gamma \\
&= \int_0^T \int_{\mathbb{R}^N} \left(|\nabla_y u_h|^2 + \frac{1}{h}\right)^{(p-2)/2} \nabla_y u_h \nabla_y j_\epsilon\big(x-y, st_2 + (1-s)t_1 - \gamma\big)\,dy\,d\gamma \\
&\quad - \int_0^T \int_{\mathbb{R}^N} j_\epsilon\big(x-y, st_2 + (1+s)t_1 - \gamma\big) u_h\big(a + bu_h^m(\gamma-\tau) - cu_h^n(\gamma-\tau)\big)\,dy\,d\gamma.
\end{aligned}$$

Substituting this into (3.11) gives

$$\begin{aligned}
&\int_{B(\Delta t)} \zeta(x)\big(u_{h\epsilon}(x,t_2) - u_{h\epsilon}(x,t_1)\big)\,dx \\
&= -\Delta t \int_{B(\Delta t)} \zeta(x) \int_0^1 \int_0^T \int_{\mathbb{R}^N} \left[\left(|\nabla_y u_h|^2 + \frac{1}{h}\right)^{p-2/2}\right. \\
&\quad \times \nabla_y u_h \nabla_y j_\epsilon\big(x-y, st_2 + (1-s)t_1 - \gamma\big) - j_\epsilon\big(x-y, st_2 + (1+s)t_1 - \gamma\big) \\
&\quad \left.\times\, u_h\big(a + bu_h^m(\gamma-\tau) - cu_h^n(\gamma-\tau)\big)\right] dy\,d\gamma\,ds\,dx \\
&= -\Delta t \int_{B(\Delta t)} \zeta(x) \int_0^1 \int_0^T \int_{\mathbb{R}^N} \left(|\nabla_y u_h|^2 + \frac{1}{h}\right)^{(p-2)/2} \\
&\quad \times \nabla_y u_h \nabla_y j_\epsilon\big(x-y, st_2 + (1-s)t_1 - \gamma\big)\,dy\,d\gamma\,ds\,dx \\
&\quad + \Delta t \int_{B(\Delta t)} \zeta(x) \int_0^1 \int_0^T \int_{\mathbb{R}^N} j_\epsilon\big(x-y, st_2 + (1+s)t_1 - \gamma\big) \\
&\quad \times u_h\big(a + bu_h^m(\gamma-\tau) - cu_h^n(\gamma-\tau)\big)\,dy\,d\gamma\,ds\,dx \\
&= -\Delta t \int_0^1 \int_0^T \int_{\mathbb{R}^N} \left(|\nabla_y u_h|^2 + \frac{1}{h}\right)^{(p-2)/2} \\
&\quad \times \nabla_y u_h \left(\int_{B(\Delta t)} \nabla_x \zeta(x) j_\epsilon\big(x-y, st_2 + (1-s)t_1 - \gamma\big)\,dx\right) dy\,d\gamma\,ds \\
&\quad + \Delta t \int_{B(\Delta t)} \zeta(x) \int_0^1 \int_0^T \int_{\mathbb{R}^N} j_\epsilon\big(x-y, st_2 + (1+s)t_1 - \gamma\big) \\
&\quad \times u_h\big(a + bu_h^m(\gamma-\tau) - cu_h^n(\gamma-\tau)\big)\,dy\,d\gamma\,ds\,dx \\
&= -\Delta t \int_0^1 \int_{B(\Delta t)} \nabla_x \zeta \cdot J_\epsilon\left(\left(|\nabla u_h|^2 + \frac{1}{h}\right)^{(p-2)/2} \nabla u_h\right)\big(x, st_2 + (1-s)t_1\big)\,dx\,ds
\end{aligned}$$

$$+ \Delta t \int_{B(\Delta t)} \zeta(x) \int_0^1 \int_0^T \int_{\mathbb{R}^N} j_\epsilon\bigl(x-y, st_2 + (1+s)t_1 - \gamma\bigr) \times u_h\bigl(a + bu_h^m(\gamma-\tau) - cu_h^n(\gamma-\tau)\bigr)\,dy\,d\gamma\,ds\,dx. \tag{3.12}$$

Now choose $\delta(s) \in C_0^1(\mathbb{R})$ such that $\delta(s) \geq 0$, $\delta(s) = 0$ for $|s| \geq 1$, and $\int_{\mathbb{R}} \delta(s)\,ds = 1$. For $l > 0$, define $\delta_l(s) = \frac{1}{l}\delta(\frac{s}{l})$. By approximation we see that (3.12) holds for $\zeta \in W_0^{1,1}(B(\Delta t))$. Thus, if we choose

$$\zeta = \zeta_l(x) = \int_{-l}^{(\Delta t)^{1/2} - |x-x_0| - 2l} \delta_l(s)\,ds$$

in (3.12), then we have

$$\begin{aligned}
&\int_{B(\Delta t)} \zeta_l(x)\bigl(u_{h\epsilon}(x,t_2) - u_{h\epsilon}(x,t_1)\bigr)\,dx \\
&\quad = -\Delta t \int_0^1 \int_{B(\Delta t)} \delta_l\bigl((\Delta t)^{1/2} - |x-x_0| - 2l\bigr) \cdot \frac{x_{0i} - x_i}{|x-x_0|} \\
&\quad\quad \cdot J_\epsilon\left(\left(|\nabla u_h|^2 + \frac{1}{h}\right)^{(p-2)/2} u_{hx_i}\right)\bigl(x, st_2 + (1-s)t_1\bigr)\,dx\,ds \\
&\quad\quad + \Delta t \int_{B(\Delta t)} \zeta_l(x) \int_0^1 \int_0^T \int_{\mathbb{R}^N} j_\epsilon\bigl(x-y, st_2 + (1+s)t_1 - \gamma\bigr) \\
&\quad\quad \times u_h\bigl(a + bu_h^m(\gamma-\tau) - cu_h^n(\gamma-\tau)\bigr)\,dy\,d\gamma\,ds\,dx.
\end{aligned} \tag{3.13}$$

Note that, for $x \in B(\Delta t)$, $\lim_{l\to 0} \zeta_l(x) = 1$, and if $|x-x_0| < (\Delta t)^{1/2} - h$, then $\delta_l((\Delta t)^{1/2} - |x - x_0| - 2l) = 0$, $\delta_l \leq \frac{C}{l}$, and

$$\operatorname{mes}\bigl(B(\Delta t)\backslash B_{(\Delta t)^{1/2}-l}(x_0)\bigr) \leq Ch(\Delta t)^{(N-1)/2}.$$

From (3.13) by Lemma 3.3 we obtain

$$\begin{aligned}
&\left|\int_{B(\Delta t)} \zeta_l(x)\bigl(u_{h\epsilon}(x,t_2) - u_{h\epsilon}(x,t_1)\bigr)\,dx\right| \\
&\quad = \left| -\Delta t \int_0^1 \int_{B(\Delta t)} \delta_l\bigl((\Delta t)^{1/2} - |x-x_0| - 2l\bigr) \cdot \frac{x_{0i} - x_i}{|x-x_0|}\right. \\
&\quad\quad \cdot \int_0^T \int_{\mathbb{R}^n} \left(|\nabla_y u_h|^2 + \frac{1}{h}\right)^{(p-2)/2} \nabla_{y_i} u_h(y)) j_\epsilon\bigl(x-y, st_2 + (1-s)t_1 - \gamma\bigr)\,dy\,d\gamma\,dx\,ds \\
&\quad\quad + \Delta t \int_{B(\Delta t)} \zeta_l(x) \int_0^1 \int_0^T \int_{\mathbb{R}^N} j_\epsilon\bigl(x-y, st_2 + (1+s)t_1 - \gamma\bigr) \\
&\quad\quad \left. \times u_h\bigl(a + bu_h^m(\gamma-\tau) - cu_h^n(\gamma-\tau)\bigr)\,dy\,d\gamma\,ds\,dx \right| \\
&\quad \leq |\Delta t| \frac{C}{l} \int_0^1 \int_{B(\Delta t)\backslash B_{(\Delta t)^{1/2}-l}(x_0)} \int_0^T \int_\Omega \left(|\nabla u_h|^2 + \frac{1}{h}\right)^{(p-2)/2} |\nabla u_h| \\
&\quad\quad \times \bigl|j_\epsilon\bigl(x-y, st_2 + (1-s)t_1 - \gamma\bigr)\bigr|\,dy\,d\gamma\,dx\,ds
\end{aligned}$$

$$
\begin{aligned}
&+ |\Delta t| \int_{B(\Delta t)} \big|\zeta_l(x)\big| \int_0^1 \int_0^T \int_\Omega \big|j_\epsilon\big(x-y, st_2 + (1+s)t_1 - \gamma\big)\big| \\
&\times \big|u_h\big(a + bu_h^m(\gamma-\tau) - cu_h^n(\gamma-\tau)\big)\big|\,dy\,d\gamma\,ds\,dx \\
\le{}& |\Delta t| \frac{C}{h} Dh(\Delta t)^{(N-1)/2} \iint_{Q_T} \left(|\nabla u_h|^2 + \frac{1}{h}\right)^{(p-2)/2} |\nabla u_h|\,dy\,d\gamma \\
&+ |\Delta t| C \int_{B(\Delta t)} \big|\zeta_l(x)\big|\,dx \\
\le{}& C(\Delta t)^{(N+1)/2} + |\Delta t| C \int_{B(\Delta t)} \big|\zeta_l(x)\big|\,dx.
\end{aligned}
$$

Letting $l \to 0$ yields

$$
\left|\int_{B(\Delta t)} \big(u_h(x,t_2) - u_h(x,t_1)\big)\right| \le C(\Delta t)^{(N+1)/2} + C(\Delta t)(\Delta t)^{N/2} \le C(\Delta t)^{(N+1)/2},
$$

from which by the mean value theorem it follows that there exists $x^* \in B(\Delta t)$ such that

$$
\big|u_h\big(x^*, t_2\big) - u_h\big(x^*, t_1\big)\big| \le C(\Delta t)^{1/2}.
$$

Using this inequality and (3.10), we derive

$$
\begin{aligned}
&\big|u_h(x_0,t_2) - u_h(x_0,t_1)\big| \\
&\quad\le \big|u_h(x_0,t_2) - u_h\big(x^*,t_2\big)\big| + \big|u_h\big(x^*,t_2\big) - u_h\big(x^*,t_1\big)\big| + \big|u_h\big(x^*,t_1\big) - u_h(x_0,t_1)\big| \\
&\quad\le C\big|x_0 - x^*\big| + C(\Delta t)^{1/2} + C\big|x^* - x_0\big| \\
&\quad\le C(\Delta t)^{1/2}.
\end{aligned}
\tag{3.14}
$$

Combining (3.10) with (3.14), we have

$$
\begin{aligned}
&\big|u_h(x_1,t_1) - u_h(x_2,t_2)\big| \\
&\quad\le \big|u_h(x_1,t_1) - u_h(x_2,t_1)\big| + \big|u_h(x_2,t_1) - u_h(x_2,t_2)\big| \le C|x_1 - x_2| + C|t_2 - t_1|^{1/2} \\
&\quad\le C\big(|x_1 - x_2| + C|t_2 - t_1|^{1/2}\big),
\end{aligned}
$$

where C is independent of h. The Hölder norm estimate of solutions is completed.

So by the Arzelà-Ascoli theorem there exist a subsequence of $\{u_h\}_{h=1}^\infty$, supposed to be $\{u_h\}_{h=1}^\infty$ itself, and a function

$$
u \in \left\{u : u \in L\big(0,T;W^{1,p}(\Omega)\big) \cap L^\infty(Q_T) \cap C^{1,1/2}(\overline{Q_T}), \frac{\partial u}{\partial t} \in L^2(Q_T)\right\}
$$

such that

$$
\begin{aligned}
&u_h(x,t) \longrightarrow u(x,t) \quad \text{uniformly in } Q_T, \\
&\nabla u_h \rightharpoonup \nabla u \quad \text{in } L^p(Q_T), \\
&\frac{\partial u_h}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} \quad \text{in } L^2(Q_T).
\end{aligned}
$$

Furthermore, for any $\varphi \in C^1(\overline{Q}_T)$, we have

$$\iint_{Q_T} \left(|\nabla u_h|^2 + \frac{1}{h}\right)^{(p-2)/2} \nabla u_h \nabla \varphi \,dx\,dt \longrightarrow \iint_{Q_T} |\nabla u|^{p-2} \nabla u \nabla \varphi \,dx\,dt$$

(see Chapter 2 in [18]). Letting $h \to \infty$ in

$$\begin{aligned}\iint_{Q_T} \biggl(& \frac{\partial u_h}{\partial t}\varphi - d(t)\left(|\nabla u_h|^2 + \frac{1}{h}\right)^{(p-2)/2} \nabla u_h \nabla \varphi \\ & + u_h\bigl(a + bu_h^m(t-\tau) - cu_h^n(t-\tau)\bigr)\varphi\biggr)\,dx\,dt = 0,\end{aligned}$$

we see that u satisfies the integral identity in the definition of generalized solutions. So, problem (1.1)-(1.3) admits a solution u that satisfies

$$\begin{aligned}&\|u\|_{L^\infty(Q_T)} \le C_0, \\ &\|\nabla u\|^p_{L^p(Q_T)} \le C_1, \\ &\left\|\frac{\partial u}{\partial t}\right\|^2_{L^2(Q_T)} \le C.\end{aligned}$$

Moreover, from the arbitrariness of $T > 0$ we easily to see that the solution exists globally. □

4 The uniqueness of a solution

In this section, we study the uniqueness of a solution for problem (1.1)-(1.3). Our main result is the following.

Theorem 4.1 *The solution of the initial and boundary value problem* (1.1)-(1.3) *is unique.*

Proof Assume that there exist nonnegative bounded functions $u_1(x,t)$ and $u_2(x,t)$ satisfying (1.1)-(1.3). We can see that

$$\begin{cases} \frac{\partial u_1}{\partial t} - d(t)\operatorname{div}(|\nabla u_1|^{p-2}\nabla u_1) \\ \quad = u_1(x,t)(a + bu_1^m(x,t-\tau) - cu_1^n(x,t-\tau)), & (x,t) \in \Omega \times \mathbb{R}_+, \\ \frac{\partial u_1}{\partial \vec{n}} = 0, & (x,t) \in \partial\Omega \times \mathbb{R}_+, \\ u_1(x,t) = \eta(x,t), & (x,t) \in \Omega \times [-\tau, 0], \end{cases}$$

and

$$\begin{cases} \frac{\partial u_2}{\partial t} - d(t)\operatorname{div}\bigl(|\nabla u_2|^{p-2}\nabla u_2\bigr) \\ \quad = u_2(x,t)\bigl(a + bu_2^m(x,t-\tau) - cu_2^n(x,t-\tau)\bigr), & (x,t) \in \Omega \times \mathbb{R}_+, \\ \frac{\partial u_2}{\partial \vec{n}} = 0, & (x,t) \in \partial\Omega \times \mathbb{R}_+, \\ u_2(x,t) = \eta(x,t), & (x,t) \in \Omega \times [-\tau, 0]. \end{cases}$$

In fact, when $t \in [0,\tau]$, that is, $t-\tau \in [-\tau,0]$, we have $u(x,t-\tau) = \eta(x,t-\tau)$. Then

$$\begin{cases} \frac{\partial(u_1-u_2)}{\partial t} = d(t)\bigl[\operatorname{div}\bigl(|\nabla u_1|^{p-2}\nabla u_1\bigr) - \operatorname{div}\bigl(|\nabla u_2|^{p-2}\nabla u_2\bigr)\bigr] & \\ \qquad + (u_1-u_2)\bigl(a + b\eta^m(x,t-\tau) - c\eta^n(x,t-\tau)\bigr), & (x,t) \in \Omega \times [0,\tau], \\ \frac{\partial(u_1-u_2)}{\partial \vec{n}} = 0, & (x,t) \in \partial\Omega \times [0,\tau], \\ (u_1-u_2)(x,t) = \eta(x,t) - \eta(x,t) = 0, & (x,t) \in \Omega \times [-\tau,0]. \end{cases} \tag{4.1}$$

Let $H_\varepsilon(s) = \int_0^s h_\varepsilon \, d\rho$, $h_\varepsilon(s) = \frac{2}{\varepsilon}(1 - \frac{|s|}{\varepsilon})_+$. Clearly, we can see that $h_\varepsilon \in C(\mathbb{R})$ and, for all $s \in \mathbb{R}$,

$$\begin{aligned} &h_\varepsilon(s) \geq 0, \qquad \bigl|sh_\varepsilon(s)\bigr| \leq 1, \qquad \bigl|H_\varepsilon(s)\bigr| \leq 1, \\ &\lim_{\varepsilon\to 0} H_\varepsilon(s) = \operatorname{sgn}(s), \qquad \lim_{\varepsilon\to 0} sh_\varepsilon(s) = 0. \end{aligned}$$

Multiplying the equation in (4.1) by $H_\varepsilon(u_1-u_2)$ and integrating by parts over $\Omega \times [0,t]$, we have

$$\begin{aligned} &\int_\Omega \int_0^t \frac{\partial(u_1-u_2)}{\partial s} \dot{H}_\varepsilon(u_1-u_2)\,dx\,ds \\ &\qquad + \int_\Omega \int_0^t d(s)\bigl(|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2\bigr)\nabla H_\varepsilon(u_1-u_2)\,dx\,ds \\ &\quad = \int_\Omega \int_0^t (u_1-u_2)H_\varepsilon(u_1-u_2)\bigl(a + b\eta^m(x,s-\tau) - c\eta^n(x,s-\tau)\bigr)\,dx\,ds. \end{aligned} \tag{4.2}$$

Letting $\varepsilon \to 0$ in (4.2) (see Chapter 3 in [18]), we obtain

$$\int_\Omega \int_0^t \frac{\partial}{\partial s}(u_1-u_2)_+ \,dx\,ds \leq \int_\Omega \int_0^t (u_1-u_2)_+\bigl(a + b\eta^m(x,s-\tau) - c\eta^n(x,s-\tau)\bigr)\,dx\,ds.$$

Since the maximum of the function $f(s) = a + bs^m - cs^n$ is equal to M ($M \geq a > 0$) when $s > 0$, we have

$$\int_\Omega \int_0^t \frac{\partial}{\partial s}(u_1-u_2)_+ \,dx\,ds \leq M \int_\Omega \int_0^t (u_1-u_2)_+ \,dx\,ds.$$

Therefore,

$$\int_\Omega (u_1-u_2)_+ \,dx \leq M \int_\Omega \int_0^t (u_1-u_2)_+ \,dx\,ds.$$

It follows from Gronwall's inequality that

$$\int_\Omega \int_0^t (u_1-u_2)_+ \,dx\,ds \leq 0,$$

which implies that

$$\bigl(u_1(x,t) - u_2(x,t)\bigr)_+ = 0, \quad (x,t) \in \Omega \times [0,\tau].$$

So we have

$$u_1(x,t) \leq u_2(x,t), \quad (x,t) \in \Omega \times [0,\tau].$$

As in the previous proof, we arrive at

$$u_1(x,t) \geq u_2(x,t), \quad (x,t) \in \Omega \times [0,\tau].$$

Thus,

$$u_1(x,t) = u_2(x,t), \quad (x,t) \in \Omega \times [0,\tau],$$

that is, the solution of problem (1.1)-(1.3) is unique in $\Omega \times [0,\tau]$.

When $t \in [\tau, 2\tau]$, that is, $t-\tau \in [0,\tau]$, we have $u_1(x,t-\tau) = u_2(x,t-\tau)$, $(x,t) \in \Omega \times [0,\tau]$. Similarly to the previous proof, we easily obtain

$$u_1(x,t) = u_2(x,t), \quad (x,t) \in \Omega \times [\tau,2\tau].$$

By a recursive process we obtain

$$u_1(x,t) = u_2(x,t), \quad (x,t) \in \Omega \times [-\tau,+\infty).$$

Thus, the uniqueness of the solution for (1.1)-(1.3) is obtained. The proof of Theorem 4.1 is completed. □

5 The oscillation of solutions

In this section, we show the oscillation of all positive solutions for problem (1.1)-(1.3) about the positive equilibrium. Due to the degeneracy of the equation, the oscillation is required to be analyzed in the frame of weak solutions rather than classical solutions. However, for simplicity of our arguments, we may assume that the solutions are appropriately smooth since we may consider approximate solutions of the approximate problem (3.1)-(3.3) and finally get the desired oscillation properties of problem (1.1)-(1.3) after a limit process.

Now, we give the definition of oscillation properties of solutions, as did in [15].

Definition 5.1 We say that the solution $u(x,t)$ in $\Omega \times \mathbb{R}_+$ of (1.1)-(1.3) oscillates about the positive equilibrium u^* if for any $T > 0$, there exists $(x_0,t_0) \in \Omega \times [T,+\infty)$ such that $u(x_0,t_0) = u^*$; otherwise, we say that $u(x,t)$ does not oscillate about u^*.

In order to obtain the oscillation of solutions, we first consider the existence and uniqueness conditions of the positive equilibrium of equation (1.1) and give conditions for nonexistence of ultimately positive solution or negative solution of the evolutionary p-Laplacian with delay.

Lemma 5.1 *Equation* (1.1) *has a unique positive equilibrium* u^* *that satisfies*

$$a + bu^{*m} - cu^{*n} = 0. \tag{5.1}$$

Moreover, $a + bu^m - cu^n < 0$ *if* $u > u^*$ *and* $a + bu^m - cu^n > 0$ *if* $0 < u < u^*$.

Proof By the zero point theorem, since $a > 0$ and $c > 0$, it is clear that equation (1.1) has a unique positive equilibrium. Then, we also have

$$a + bu^m - cu^n < 0, \quad u \in \left(u^*, +\infty\right),$$

and

$$a + bu^m - cu^n > 0, \quad u \in \left(0, u^*\right).$$

The proof of Lemma 5.1 is completed. □

The following lemma is from p.84 of [17].

Lemma 5.2 *Assume that*

(1) $f \in C[-R, R], \quad yf(y) > 0 \quad (y \neq 0)$;

(2) $\lim\limits_{t\to+\infty} \int_{t-\tau}^{t} p(s)\,ds > \frac{M}{e}, \quad \textit{where } M = \lim\limits_{y\to 0} \frac{y}{f(y)}.$

Then

(1) *the nonlinear differential inequality with delay* $y'(t) + p(t)f(y(t-\tau)) \leq 0$ *has no ultimately positive solution*;

(2) *the nonlinear differential inequality with delay* $y'(t) + p(t)f(y(t-\tau)) \geq 0$ *has no ultimately negative solution*.

In the following, we investigate a sufficient condition for all positive solutions of (1.1)-(1.3) to oscillate about the positive equilibrium.

Theorem 5.1 *Suppose that*

$$\left(2cu^{*(n-m)} - bm\right)u^{*m}\tau > \frac{1}{e}. \tag{5.2}$$

Then all positive solutions of problem (1.1)-(1.3) *oscillate about the positive equilibrium*.

Proof Let $u(x,t)$ be a nonoscillatory solution of system (1.1)-(1.3). By Definition 5.1 there exists $T > 0$ such that $u(x,t) > u^*$ or $u(x,t) < u^*$ when $(x,t) \in \Omega \times [T, +\infty)$. Therefore, we need to consider these two cases.

Case 1: $u(x,t) > u^*$ for $(x,t) \in \Omega \times [T, +\infty)$. In general, we may suppose that $u(x, t-\tau) > u^*$.

Let $M(x,t) = u(x,t) - u^* > 0$, $(x,t) \in \Omega \times [T, +\infty)$. We can see that

$$M(x, t-\tau) > 0, \quad (x,t) \in \Omega \times [T, +\infty),$$

and

$$\frac{\partial M(x,t)}{\partial t} = \frac{\partial u(x,t)}{\partial t}, \qquad \operatorname{div}\left(|\nabla M|^{p-2}\nabla M\right) = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right).$$

Substituting $M(x,t) = u(x,t) - u^*$ into (1.1), since $a + bu^m - cu^n < 0$ for $u > u^*$ and $cu^{*(n-m)} - b > 0$, by (5.1) we get

$$\begin{aligned}\frac{\partial M}{\partial t} &= d(t)\operatorname{div}\bigl(|\nabla M|^{p-2}\nabla M\bigr) + M(x,t)\bigl\{a + b\bigl[M(x,t-\tau)+u^*\bigr]^m \\ &\quad - c\bigl[M(x,t-\tau)+u^*\bigr]^n\bigr\} + u^*\bigl\{a + b\bigl[M(x,t-\tau)+u^*\bigr]^m - c\bigl[M(x,t-\tau)+u^*\bigr]^n\bigr\} \\ &< d(t)\operatorname{div}\bigl(|\nabla M|^{p-2}\nabla M\bigr) + u^*\bigl[bM^m(x,t-\tau) + bC_m^1 M^{m-1}(x,t-\tau)u^* + \cdots \\ &\quad + bC_m^{m-2}M^2(x,t-\tau)u^{*(m-2)} + bC_m^{m-1}M(x,t-\tau)u^{*(m-1)} + bu^{*m} \\ &\quad - cM^n(x,t-\tau) - cC_n^1 M^{n-1}(x,t-\tau)u^* - \cdots - cC_n^{n-m}M^m(x,t-\tau)u^{*(n-m)} - \cdots \\ &\quad - cC_n^{n-2}M^2(x,t-\tau)u^{*(n-2)} - cC_n^{n-1}M(x,t-\tau)u^{*(n-1)} - cu^{*n}\bigr].\end{aligned}$$

Since

$$\begin{aligned}& bM^m(x,t-\tau) + bC_m^1 M^{m-1}(x,t-\tau)u^* + \cdots + bC_m^{m-2}M^2(x,t-\tau)u^{*(m-2)} \\ &\quad - cC_n^{n-m}M^m(x,t-\tau)u^{*(n-m)} - cC_n^{n-m+1}M^{m-1}(x,t-\tau)u^{*(n-m+1)} - \cdots \\ &\quad - cC_n^{n-2}M^2(x,t-\tau)u^{*(n-2)} \\ &= \sum_{k=0}^{m-2}\bigl[bC_m^k M^{m-k}(x,t-\tau)u^{*k} - cC_n^{n-m+k}M^{m-k}(x,t-\tau)u^{*(n-m+k)}\bigr] \\ &= \sum_{k=0}^{m-2}M^{m-k}(x,t-\tau)u^{*k}\bigl[bC_m^k - cC_n^{n-m+k}u^{*(n-m)}\bigr] \\ &< \sum_{k=0}^{m-2}M^{m-k}(x,t-\tau)u^{*k}C_n^{n-m+k}\bigl[b - cu^{*(n-m)}\bigr] < 0,\end{aligned}$$

it is clear that $C_m^k < C_n^{n-m+k}$. At the same time, we observe from condition (5.2) that $b - cu^{*(n-m)} < 0$, which implies that

$$\begin{aligned}\frac{\partial M}{\partial t} &< d(t)\operatorname{div}\bigl(|\nabla M|^{p-2}\nabla M\bigr) \\ &\quad + u^*\bigl[bC_m^{m-1}M(x,t-\tau)u^{*(m-1)} + bu^{*m} - cC_n^{n-1}M(x,t-\tau)u^{*(n-1)} - cu^{*n}\bigr] \\ &= d(t)\operatorname{div}\bigl(|\nabla M|^{p-2}\nabla M\bigr) \\ &\quad + u^*\bigl[bmM(x,t-\tau)u^{*(m-1)} - cnM(x,t-\tau)u^{*(n-1)} + bu^{*m} - cu^{*n}\bigr] \\ &= d(t)\operatorname{div}\bigl(|\nabla M|^{p-2}\nabla M\bigr) \\ &\quad + u^*\bigl(bmu^{*(m-1)} - cnu^{*(n-1)}\bigr)M(x,t-\tau) + u^*u^{*m}\bigl(b - cu^{*(n-m)}\bigr),\end{aligned}$$

which gives

$$\frac{\partial M}{\partial t} < d(t)\operatorname{div}\bigl(|\nabla M|^{p-2}\nabla M\bigr) + u^*\bigl(bmu^{*(m-1)} - cnu^{*(n-1)}\bigr)M(x,t-\tau). \tag{5.3}$$

Integrating this inequality over Ω, we have

$$\begin{aligned}&\frac{\partial}{\partial t}\int_\Omega M(x,t)\,dx\\ &\quad< d(t)\int_\Omega \operatorname{div}\bigl(|\nabla M|^{p-2}\nabla M\bigr)\,dx + u^*\bigl[bmu^{*(m-1)} - cnu^{*(n-1)}\bigr]\int_\Omega M(x,t-\tau)\,dx. \end{aligned}\tag{5.4}$$

By the "Green formula" and boundary condition (1.2) we get

$$\int_\Omega \operatorname{div}\bigl(|\nabla M|^{p-2}\nabla M\bigr)\,dx = \int_\Omega |\nabla M|^{p-2}\frac{\partial M}{\partial \vec{n}}\,ds = 0. \tag{5.5}$$

Therefore, if $v(t) = \frac{1}{|\Omega|}\int_\Omega M(x,t)\,dx$ $(t \ge T)$, then $v(t) > 0$. It follows from (5.4) and (5.5) that

$$v'(t) + u^*\bigl(cnu^{*(n-1)} - bmu^{*(m-1)}\bigr)v(t-\tau) < 0, \tag{5.6}$$

which implies that $v(t)$ is an ultimately positive solution of inequality (5.6).

On the other hand, condition (5.2) gives $(cnu^{*(n-1)} - mb)u^{*m}\tau > \frac{1}{e}$. This, combined with Lemma 5.2, yields that the differential inequality with delay (5.6) has no ultimately positive solution. This is a contradiction. Therefore, case 1 does not hold.

Case 2: $0 < u(x,t) < u^*$ for $(x,t) \in \Omega \times [T,+\infty)$. Without loss of generality, we may also suppose that $0 < u(x,t-\tau) < u^*$.

Let $u(x,t) = u^*e^{w(x,t)}$. Then we have $w(x,t) < 0$ and $w(x,t-\tau) < 0$. Taking $u(x,t) = u^*e^{w(x,t)}$ in (1.1), with the help of (5.1) and the condition $cu^{*(n-m)} - b > 0$ $(m \ge 2)$, we easily see that

$$\begin{aligned}\frac{\partial w}{\partial t} &= \frac{d(t)}{u}\operatorname{div}\bigl(|\nabla u|^{p-2}\nabla u\bigr) + \bigl(a + bu^{*m}e^{mw(x,t-\tau)} - cu^{*n}e^{nw(x,t-\tau)}\bigr)\\ &= \frac{d(t)}{u}\operatorname{div}\bigl(|\nabla u|^{p-2}\nabla u\bigr)\\ &\quad + \bigl[\bigl(a + bu^{*m} - cu^{*n}\bigr) + bu^{*m}\bigl(e^{mw(x,t-\tau)} - 1\bigr) - cu^{*n}\bigl(e^{nw(x,t-\tau)} - 1\bigr)\bigr]\\ &= \frac{d(t)}{u}\operatorname{div}\bigl(|\nabla u|^{p-2}\nabla u\bigr)\\ &\quad + \bigl(e^{w(x,t-\tau)} - 1\bigr)\bigl\{\bigl(bu^{*m} - cu^{*n}\bigr)\bigl[e^{(m-1)w(x,t-\tau)} + e^{(m-2)w(x,t-\tau)} + \cdots + e^{w(x,t-\tau)}\bigr]\\ &\quad + bu^{*m} - cu^{*n}\bigl[e^{(n-1)w(x,t-\tau)} + e^{(n-2)w(x,t-\tau)} + \cdots + e^{mw(x,t-\tau)} + 1\bigr]\bigr\}.\end{aligned}$$

Since $bu^{*m} - cu^{*n} < 0$ (due to $cu^{*(n-m)} - b > 0$) and $w(x,t-\tau) < 0$, we obtain

$$\frac{\partial w}{\partial t} > \frac{d(t)}{u}\operatorname{div}\bigl(|\nabla u|^{p-2}\nabla u\bigr) + \bigl(e^{w(x,t-\tau)} - 1\bigr)\bigl[bu^{*m} - cu^{*n}\bigl(e^{(n-1)w(x,t-\tau)} + 1\bigr)\bigr].$$

Notice that $a + bu^{*m} - cu^{*n} = 0$. Then

$$\frac{\partial w}{\partial t} > \frac{d(t)}{u}\operatorname{div}\bigl(|\nabla u|^{p-2}\nabla u\bigr) - \bigl(a + cu^{*n}e^{(n-1)w(x,t-\tau)}\bigr)\bigl(e^{w(x,t-\tau)} - 1\bigr). \tag{5.7}$$

Integrating (5.7) over Ω, we have

$$\begin{aligned}&\frac{d}{dt}\int_\Omega w(x,t)\,dx+\int_\Omega\big(a+cu^{*n}e^{(n-1)w(x,t-\tau)}\big)\big(e^{w(x,t-\tau)}-1\big)\,dx\\&\quad>\int_\Omega\frac{d(t)}{u}\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big)\,dx=d(t)\int_\Omega\frac{1}{u^2}|\nabla u|^{p-2}\,dx>0.\end{aligned}$$

Denote $v(t)=\frac{1}{|\Omega|}\int_\Omega w(x,t)\,dx<0$ ($t\ge T$). Then $v(t)$ is the ultimately negative solution of the following differential inequality with delay:

$$v'(t)+f\big(v(t-\tau)\big)>0. \tag{5.8}$$

Here $f(v(t))=\frac{1}{|\Omega|}\int_\Omega(a+cu^{*n}e^{(n-1)w(x,t)})(e^{w(x,t)}-1)\,dx$.

On the other hand,

$$\lim_{y\to 0}\frac{y}{f(y)}=\frac{1}{a+cu^{*n}}$$

and

$$a+cu^{*n}=\big(2cu^{*(n-m)}-b\big)u^{*m}.$$

According to condition (5.2) and Lemma 5.2, we see that the differential inequality with delay (5.8) has no ultimately negative solution. This is a contradiction. Therefore, case 2 does not hold, too. That is to say, all positive solutions of problem (1.1)-(1.3) oscillate about the positive equilibrium. The proof of Theorem 5.1 is completed. □

Next, we give an example to show this phenomenon.

Example Consider the following p-Laplacian population model with delay:

$$\begin{aligned}&\frac{\partial u}{\partial t}=\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big)+u(x,t)\left[2+u^3\left(x,t-\frac{1}{e}\right)-3u^4\left(x,t-\frac{1}{e}\right)\right],\\&\quad(x,t)\in\Omega\times\mathbb{R}_+,\end{aligned}\tag{5.9}$$

subject to the initial and boundary value conditions

$$|\nabla u|^{p-2}\nabla u\cdot\vec{n}=0,\quad (x,t)\in\partial\Omega\times\mathbb{R}_+, \tag{5.10}$$

$$u(x,\theta)=\eta(x,\theta),\qquad \eta(x,\theta)\ge 0\text{ and }\eta(x,\theta)\not\equiv 0,\quad (x,\theta)\in\Omega\times[-\tau,0). \tag{5.11}$$

It is obvious that $u^*=1$ is the unique positive equilibrium of equation (5.9). For any initial $\eta(x,\theta)\ge 0$ with $\eta(x,\theta)\not\equiv 0$, we deduce from Theorem 3.2 that problem (5.9)-(5.11) has a unique continuous positive global solution $u(x,t)$. It is not hard to check that problem (5.9)-(5.11) satisfies the conditions of Theorem 5.1.

In fact, we can take $a=1$, $b=1$, $c=3$, $d(t)=1$, $\tau=\frac{1}{e}$, $m=3$, $n=4$. Then,

$$\big(2cu^{*(n-m)}-bm\big)u^{*m}\tau=3\times\frac{1}{e}>\frac{1}{e}.$$

It follows from Theorem 5.1 that all positive solutions of this equation oscillate about the positive equilibrium $u^* = 1$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally to the writing of this paper. Both authors read and approved the manuscript.

Author details
[1]School of Mathematics and Statistics, Shenzhen University, Shenzhen, 518060, China. [2]School of Mathematical Sciences, South China Normal University, Guangzhou, 510631, China.

Acknowledgements
This work is partially supported by the National Science Foundation of China (11301345, 11671155, 11471127), the Project of Young Creative Talents in Higher Education of Guangdong (No. 2015KQNCX019) and Natural Science Foundation of SZU (201545).

Received: 7 June 2016 Accepted: 26 December 2016 Published online: 13 January 2017

References
1. Watt, KDF: Ecology and Resource Management. McGraw Hill, New York (1968)
2. Hino, Y, Naito, T, Minh, NV, Shin, JS: Almost Periodic Solutions of Differential Equations in Banach Spaces. Taylor & Francis, London (2002)
3. Faria, T, Trofimchuk, S: Positive travelling fronts for reaction-diffusion systems with distributed delay. Nonlinearity **23**, 2457-2481 (2010)
4. Pao, CV: Coupled nonlinear parabolic systems with time delays. J. Math. Anal. Appl. **196**, 237-265 (1995)
5. Pao, CV: Systems of parabolic equations with continuous and discrete delays. J. Math. Anal. Appl. **205**, 157-185 (1997)
6. Wu, J: Theory and Application of Partial Functional Differential Equations. Springer, New York (1996)
7. Hale, JK, Verduyn Lunel, SM: Introduction to Function Differential Equations. Springer, Berlin (1993)
8. Gopalsamy, K, Ladas, G: On the oscillation and asymptotic behavior of $N'(t) = N(t)[a + bN(t-\tau) - cN^2(t-\tau)]$. Q. Appl. Math. **48**(3), 433-440 (1990)
9. Rodrigues, IW: Oscillation and attraction in a differential equation with piecewise constant argument. Rocky Mt. J. Math. **24**(1), 261-271 (1994)
10. Yan, JR, Feng, QX: Global attraction and oscillation in a nonlinear delay equation. Nonlinear Anal. TMA **43**(1), 101-108 (2001)
11. Luo, Y: Oscillation on a partial functional population model. Acta Math. Sci. Ser. A Chin. Ed. **13**(3), 345-352 (1993) (in Chinese)
12. Liao, XX, Li, J: Stability in Gilpin-Ayala competition models with diffusion. Nonlinear Anal. TMA **28**(10), 1751-1758 (1997)
13. Gourley, SA, Brition, NF: On a modified Volterra population equation with diffusion. Nonlinear Anal. TMA **21**(2), 389-395 (1993)
14. He, MX, Liu, AP: Oscillation of hyperbolic partial functional differential equations. Appl. Math. Comput. **142**, 205-224 (2003)
15. Wang, CY, Wang, S: Oscillation of partial population model with diffusion and delay. Appl. Math. Lett. **22**, 1793-1797 (2009)
16. Wang, CY, Wang, S, Yan, XP, Li, LR: Oscillation of a class of partial functional population model. J. Math. Anal. Appl. **368**, 32-42 (2010)
17. Ladde, GS, Lakshmikantham, V, Zhang, BG: Oscillation Theory of Differential Equations with Deviating Arguments. Dekker, New York (1987)
18. Wu, ZQ, Zhao, JN, Yin, JX, Li, HL: Nonlinear Diffusion Equations. World Scientific, Singapore (2001)