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Some identities of Barnes-type special polynomials

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Daegu, 702-701, South Korea**Abstract**

In this paper, we consider Barnes-type special polynomials and give some identities of their polynomials which are derived from the bosonic p -adic integral or the fermionic p -adic integral on \mathbb{Z}_p .

MSC: 11B68; 11S40**Keywords:** Barnes-type Bernoulli polynomial; higher-order Euler polynomial; multivariate p -adic fermionic integral; Barnes-type Euler polynomial

1 Introduction

As is known, the Bernoulli polynomials of order r are defined by the generating function to be

$$\left(\frac{t}{e^t - 1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x) \frac{t^n}{n!} \quad (\text{see [1-29]}). \quad (1)$$

When $x = 0$, $B_n^{(r)} = B_n^{(r)}(0)$ are called the Bernoulli numbers of order r .

For $a_1, a_2, \dots, a_r \neq 0 \in \mathbb{C}_p$, the Barnes-Bernoulli polynomials are defined by the generating function to be

$$\prod_{i=1}^r \left(\frac{t}{e^{a_i t} - 1}\right) e^{xt} = \sum_{n=0}^{\infty} B_n(x|a_1, a_2, \dots, a_r) \frac{t^n}{n!}.$$

When $x = 0$, $B_n(0|a_1, a_2, \dots, a_r) = B_n(a_1, a_2, \dots, a_r)$ are called Barnes Bernoulli numbers (see [14-33]).

Let p be a fixed odd prime number. Throughout this paper, \mathbb{Z}_p , \mathbb{Q}_p and \mathbb{C}_p will denote the ring of p -adic integers, the field of p -adic numbers and the completion of algebraic closure of \mathbb{Q}_p , respectively. The p -adic norm is defined as $|p|_p = 1/p$. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable function on \mathbb{Z}_p . For $f \in UD(\mathbb{Z}_p)$, the bosonic p -adic integral on \mathbb{Z}_p is defined by

$$\begin{aligned} I_0(f) &= \int_{\mathbb{Z}_p} f(x) d\mu_0(x) = \lim_{N \rightarrow \infty} \sum_{x=0}^{p^N-1} f(x) \mu_0(x + p^N \mathbb{Z}_p) \\ &= \lim_{N \rightarrow \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x) \quad (\text{see [16, 21]}). \end{aligned} \quad (2)$$

From (2), we have

$$I_0(f_1) = I_0(f) + f'(0), \tag{3}$$

where $f_1(x) = f(x + 1)$.

By using iterative method, we get

$$I_0(f_n) = I_0(f) + \sum_{i=0}^{n-1} f'(i), \tag{4}$$

where $f_n(x) = f(x + n)$ ($n \in \mathbb{N}$).

As is well known, the fermionic p -adic integral on \mathbb{Z}_p is defined by Kim to be

$$\begin{aligned} I_{-1}(f) &= \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N \rightarrow \infty} \sum_{x=0}^{p^N-1} f(x) \mu_{-1}(x + p^N \mathbb{Z}_p) \\ &= \lim_{N \rightarrow \infty} \sum_{x=0}^{p^N-1} f(x) (-1)^x \quad (\text{see [26, 27]}). \end{aligned} \tag{5}$$

From (5), we can derive

$$I_{-1}(f_n) + (-1)^{n-1} I_{-1}(f) = 2 \sum_{l=0}^{n-1} (-1)^{n-l-1} f(l). \tag{6}$$

In particular, $n = 1$, we have

$$I_{-1}(f_1) + I_{-1}(f) = 2f(0) \quad (\text{see [26, 27]}). \tag{7}$$

The purpose of this paper is to investigate several special polynomials related to Barnes-type polynomials and give some identities including Witt's formula of their polynomials.

Finally, we give some identities of mixed-type Bernoulli and Euler polynomials.

2 Barnes-type polynomials

Let $a_1, a_2, \dots, a_r \neq 0 \in \mathbb{C}_p$. Then, by (3), we get

$$\begin{aligned} &\int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p} e^{(a_1x_1+a_2x_2+\dots+a_r x_r+x)t} d\mu_0(x_1) \dots d\mu_0(x_r) \\ &= \left(\prod_{i=1}^r a_i \right) \left(\frac{t^r}{(e^{a_1t} - 1)(e^{a_2t} - 1) \dots (e^{a_rt} - 1)} \right) e^{xt} \\ &= \left(\prod_{i=1}^r a_i \right) \sum_{n=0}^{\infty} B_n(x|a_1, a_2, \dots, a_r) \frac{t^n}{n!}. \end{aligned} \tag{8}$$

From (8), we obtain the following Witt's formula for the Barnes-Bernoulli polynomials.

Theorem 1 For $a_1, a_2, \dots, a_r \neq 0 \in \mathbb{C}_p$, we have

$$B_n(x|a_1, \dots, a_r) = \left(\prod_{i=1}^r a_i \right)^{-1} \int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p} (a_1 x_1 + \dots + a_r x_r + x)^n d\mu_0(x_1) \dots d\mu_0(x_r).$$

Note that

$$\begin{aligned} (a_1 x_1 + \dots + a_r x_r)^n &= \sum_{l_1 + \dots + l_r = n} \binom{n}{l_1, \dots, l_r} a_1^{l_1} x_1^{l_1} \dots a_r^{l_r} x_r^{l_r} \\ &= \sum_{l_1 + \dots + l_r = n} \binom{n}{l_1, \dots, l_r} \left(\prod_{i=1}^r a_i^{l_i} \right) x_1^{l_1} \dots x_r^{l_r}. \end{aligned} \tag{9}$$

By (9) and Theorem 1, we obtain the following corollary.

Corollary 2 For $n \geq 2$, we have

$$B_n(a_1, \dots, a_r) = \sum_{l_1 + \dots + l_r = n} \binom{n}{l_1, \dots, l_r} \left(\prod_{i=1}^r a_i^{l_i - 1} \right) B_{l_1} \dots B_{l_r},$$

where $B_n = B_n(1)$ is the n th Bernoulli number.

From (2), we can easily derive the following integral equation:

$$\int_{\mathbb{Z}_p} f(x) d\mu_0(x) = \frac{1}{d} \sum_{a=0}^{d-1} \int_{\mathbb{Z}_p} f(a + dx) d\mu_0(x), \tag{10}$$

where $d \in \mathbb{N}$.

By (10), we get

$$\begin{aligned} &\int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p} e^{(a_1 x_1 + a_2 x_2 + \dots + a_r x_r + x)t} d\mu_0(x) \\ &= \frac{1}{d^r} \sum_{l_1=0}^{d-1} \dots \sum_{l_r=0}^{d-1} \int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p} e^{(l_1 a_1 + \dots + l_r a_r + a_1 dx_1 + \dots + a_r dx_r + x)t} d\mu_0(x_1) \dots d\mu_0(x_r) \\ &= \sum_{l_1=0}^{d-1} \dots \sum_{l_r=0}^{d-1} \sum_{n=0}^{\infty} \frac{d^n}{d^r} \int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p} \left(\frac{l_1 a_1 + \dots + l_r a_r}{d} \right. \\ &\quad \left. + a_1 x_1 + \dots + a_r x_r + \frac{x}{d} \right)^n d\mu_0(x_1) \dots d\mu_0(x_r) \frac{t^n}{n!}. \end{aligned} \tag{11}$$

By Theorem 1 and (11), we get

$$B_n(x|a_1, \dots, a_r) = d^{n-r} \sum_{l_1=0}^{d-1} \dots \sum_{l_r=0}^{d-1} B_n \left(\frac{l_1 a_1 + \dots + l_r a_r + x}{d} \mid a_1, \dots, a_r \right). \tag{12}$$

Therefore, by (12), we obtain the following distribution relation for a Barnes-type Bernoulli polynomial.

Theorem 3 For $n \geq 0$, we have

$$B_n(x|a_1, \dots, a_r) = d^{n-r} \sum_{l_1=0}^{d-1} \cdots \sum_{l_r=0}^{d-1} B_n\left(\frac{l_1 a_1 + \cdots + l_r a_r + x}{d} \mid a_1, \dots, a_r\right).$$

From (4), we note that

$$\int_{\mathbb{Z}_p} e^{a_1(x_1+n)t} d\mu_0(x_1) - \int_{\mathbb{Z}_p} e^{a_1 x_1 t} d\mu_0(x_1) = a_1 t \sum_{l=0}^{n-1} e^{a_1 l t}. \tag{13}$$

By (13), we get

$$\int_{\mathbb{Z}_p} e^{a_1 x_1 t} d\mu_0(x_1) = \frac{a_1 t}{e^{a_1 n t} - 1} \sum_{l=0}^{n-1} e^{a_1 l t}. \tag{14}$$

From (14), we can derive

$$\begin{aligned} & \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{(a_1 x_1 + \cdots + a_r x_r)t} d\mu_0(x_1) \cdots d\mu_0(x_r) \\ &= \left(\frac{a_1 t}{e^{na_1 t} - 1}\right) \cdots \left(\frac{a_r t}{e^{na_r t} - 1}\right) \sum_{l_1, \dots, l_r=0}^{n-1} e^{(a_1 l_1 + \cdots + a_r l_r)t} \\ &= \left(\prod_{i=1}^r a_i\right) \sum_{l_1, \dots, l_r=0}^{n-1} \left(\sum_{k=0}^{\infty} B_k(na_1, \dots, na_r) \frac{t^k}{k!}\right) \sum_{j=0}^{\infty} (a_1 l_1 + \cdots + a_r l_r)^j \frac{t^j}{j!} \\ &= \left(\prod_{i=1}^r a_i\right) \sum_{m=0}^{\infty} \left\{ \sum_{l_1, \dots, l_r=0}^{n-1} \sum_{j=0}^m (a_1 l_1 + \cdots + a_r l_r)^j B_{m-j}(na_1, \dots, na_r) \binom{m}{j} \right\} \frac{t^m}{m!}. \end{aligned} \tag{15}$$

By (15), we get

$$\begin{aligned} & \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (a_1 x_1 + \cdots + a_r x_r)^m d\mu_0(x_1) \cdots d\mu_0(x_r) \\ &= \left(\prod_{i=1}^r a_i\right) \sum_{l_1, \dots, l_r=0}^{n-1} \sum_{j=0}^m (a_1 l_1 + \cdots + a_r l_r)^j B_{m-j}(na_1, \dots, na_r) \binom{m}{j}, \end{aligned} \tag{16}$$

where $n \in \mathbb{N}$ and $m \in \mathbb{Z} \geq 0$.

Therefore, by Theorem 1 and (16), we obtain the following theorem.

Theorem 4 For $n \in \mathbb{N}$ and $m \in \mathbb{Z}$ with $m \geq 0$, we have

$$B_m(a_1, \dots, a_r) = \sum_{l_1, \dots, l_r=0}^{n-1} \sum_{j=0}^m (a_1 l_1 + \cdots + a_r l_r)^j B_{m-j}(na_1, \dots, na_r) \binom{m}{j}.$$

Moreover,

$$B_m(x|a_1, \dots, a_r) = \sum_{l_1, \dots, l_r=0}^{n-1} \sum_{j=0}^m (a_1 l_1 + \cdots + a_r l_r + x)^j B_{m-j}(na_1, \dots, na_r) \binom{m}{j}.$$

From (15), we observe that

$$\begin{aligned}
 & \int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p} e^{(a_1x_1+\dots+a_rx_r)t} d\mu_0(x_1) \dots d\mu_0(x_r) \\
 &= \left(\frac{a_1t}{e^{na_1t}-1}\right) \dots \left(\frac{a_rt}{e^{na_rt}-1}\right) \sum_{l_1, \dots, l_r=0}^{n-1} e^{(a_1l_1+\dots+a_rl_r)t} \\
 &= \sum_{l_1, \dots, l_r=0}^{n-1} \frac{a_1t \dots a_rt}{(e^{na_1t}-1) \dots (e^{na_rt}-1)} e^{(\frac{a_1l_1+\dots+a_rl_r}{n})nt} \\
 &= \left(\prod_{i=1}^r a_i\right) \sum_{l_1, \dots, l_r=0}^{n-1} \sum_{m=0}^{\infty} B_m\left(\frac{a_1l_1+\dots+a_rl_r}{n} \mid a_1, \dots, a_r\right) n^m \frac{t^m}{m!} \\
 &= \sum_{m=0}^{\infty} \left(\prod_{i=1}^r a_i\right) n^m \sum_{l_1, \dots, l_r=0}^{n-1} B_m\left(\frac{a_1l_1+\dots+a_rl_r}{n} \mid a_1, \dots, a_r\right) \frac{t^m}{m!}.
 \end{aligned} \tag{17}$$

Thus, by (17), we get

$$\begin{aligned}
 & \int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p} (a_1x_1+\dots+a_rx_r)^m d\mu_0(x_1) \dots d\mu_0(x_r) \\
 &= \left(\prod_{i=1}^r a_i\right) n^m \sum_{l_1, \dots, l_r=0}^{n-1} B_m\left(\frac{a_1l_1+\dots+a_rl_r}{n} \mid a_1, \dots, a_r\right),
 \end{aligned} \tag{18}$$

where $n \in \mathbb{N}$ and $m \in \mathbb{Z} \geq 0$.

Therefore, by Theorem 1 and (17), we obtain the following theorem.

Theorem 5 For $n \in \mathbb{N}$ and $m \geq 0$, we have

$$B_m(a_1, \dots, a_r) = n^m \sum_{l_1, \dots, l_r=0}^{n-1} B_m\left(\frac{a_1l_1+\dots+a_rl_r}{n} \mid a_1, \dots, a_r\right).$$

Moreover,

$$B_m(x \mid a_1, \dots, a_r) = n^m \sum_{l_1, \dots, l_r=0}^{n-1} B_m\left(\frac{a_1l_1+\dots+a_rl_r+x}{n} \mid a_1, \dots, a_r\right).$$

Remark Let $a_1 = 1$ and $r = 1$. Then we have

$$\int_{\mathbb{Z}_p} e^{(x+n)t} d\mu_0(x) - \int_{\mathbb{Z}_p} e^{xt} d\mu_0(x) = t \sum_{l=0}^{n-1} e^{lt}.$$

Thus, we have

$$\sum_{m=0}^{\infty} \left\{ \int_{\mathbb{Z}_p} (x+n)^m d\mu_0(x) - \int_{\mathbb{Z}_p} x^m d\mu_0(x) \right\} \frac{t^m}{m!} (x) = t \sum_{l=0}^{n-1} \sum_{m=0}^{\infty} l^m \frac{t^m}{m!}. \tag{19}$$

By (19), we get

$$\frac{1}{m+1} \left\{ \int_{\mathbb{Z}_p} (x+n)^{m+1} d\mu_0(x) - \int_{\mathbb{Z}_p} x^{m+1} d\mu_0(x) \right\} = \sum_{l=0}^{n-1} l^m, \tag{20}$$

where $n \in \mathbb{N}$ and $m \in \mathbb{Z} \geq 0$.

It is easy to show that

$$\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_0(y) = \frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \tag{21}$$

where $B_n(x)$ is the n th Bernoulli polynomial.

Thus, by (21), we get

$$\int_{\mathbb{Z}_p} (x+y)^n d\mu_0(y) = B_n(x) \quad (n \geq 0). \tag{22}$$

From (20) and (21), we note that

$$\frac{1}{m+1} \{B_{m+1}(n) - B_{m+1}\} = \sum_{l=0}^{n-1} l^m,$$

where $m \in \mathbb{Z} \geq 0$ and $n \in \mathbb{N}$.

From (5) and (6), we can derive the following equation:

$$\int_{\mathbb{Z}_p} e^{(x+1)t} d\mu_{-1}(x) + \int_{\mathbb{Z}_p} e^{xt} d\mu_{-1}(x) = 2.$$

Thus, we have

$$\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_{-1}(y) = \frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!},$$

where $E_n(x)$ is the n th Euler polynomial.

Witt's formula for the Euler polynomials is given by

$$\int_{\mathbb{Z}_p} (x+y)^n d\mu_{-1}(y) = E_n(x) \quad (n \geq 0) \text{ (see [26, 27])}. \tag{23}$$

When $x = 0$, $E_n = E_n(0)$ are called the Euler numbers.

For $r \in \mathbb{N}$, the generating function of higher-order Euler polynomials can be derived from the multivariate p -adic fermionic integral on \mathbb{Z}_p as follows:

$$\begin{aligned} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{(x_1 + \cdots + x_r + x)t} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) &= \left(\frac{2}{e^t + 1} \right)^r e^{xt} \\ &= \sum_{n=0}^{\infty} E_n^{(r)}(x) \frac{t^n}{n!}. \end{aligned}$$

Thus we get

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_r + x)^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) = E_n^{(r)}(x) \quad (n \in \mathbb{Z} \geq 0). \tag{24}$$

It is easy to show that

$$\begin{aligned} & \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_r + x)^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \\ &= \sum_{l_1 + \cdots + l_r = n} \binom{n}{l_1, \dots, l_r} \int_{\mathbb{Z}_p} x_1^{l_1} d\mu_{-1}(x_1) \cdots \int_{\mathbb{Z}_p} x_{r-1}^{l_{r-1}} d\mu_{-1}(x_{r-1}) \\ & \quad \times \int_{\mathbb{Z}_p} (x_r + x)^{l_r} d\mu_{-1}(x_r) \\ &= \sum_{l_1 + \cdots + l_r = n} \binom{n}{l_1, \dots, l_r} E_{l_1} E_{l_2} \cdots E_{l_{r-1}} E_{l_r}(x). \end{aligned} \tag{25}$$

From (24) and (25), we have

$$E_n^{(r)}(x) = \sum_{l_1 + \cdots + l_r = n} \binom{n}{l_1, \dots, l_r} E_{l_1} \cdots E_{l_{r-1}} E_{l_r}(x). \tag{26}$$

When $x = 0$, $E_n^{(r)} = E_n^{(r)}(0)$ are called the higher-order Euler numbers.

From (6), we note that

$$\int_{\mathbb{Z}_p} e^{(x+n)t} d\mu_{-1}(x) + (-1)^{n-1} \int_{\mathbb{Z}_p} e^{xt} d\mu_{-1}(x) = 2 \sum_{l=0}^{n-1} (-1)^{n-1-l} e^{lt} \quad (n \in \mathbb{N}). \tag{27}$$

Thus, by (27), we get

$$\int_{\mathbb{Z}_p} e^{xt} d\mu_{-1}(x) = \frac{2}{e^{nt} + (-1)^{n-1}} \sum_{l=0}^{n-1} (-1)^{n-1-l} e^{lt}, \tag{28}$$

and

$$\begin{aligned} & \sum_{m=0}^{\infty} \left\{ \int_{\mathbb{Z}_p} (x+n)^m d\mu_{-1}(x) + (-1)^{n-1} \int_{\mathbb{Z}_p} x^m d\mu_{-1}(x) \right\} \frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty} \left\{ 2 \sum_{l=0}^{n-1} (-1)^{n-1-l} l^m \right\} \frac{t^m}{m!} \quad (n \in \mathbb{N}). \end{aligned} \tag{29}$$

By comparing the coefficients on the both sides of (29), we get

$$\int_{\mathbb{Z}_p} (x+n)^m d\mu_{-1}(x) + (-1)^{n-1} \int_{\mathbb{Z}_p} x^m d\mu_{-1}(x) = 2 \sum_{l=0}^{n-1} (-1)^{n-1-l} l^m, \tag{30}$$

where $n \in \mathbb{N}$ and $m \in \mathbb{Z} \geq 0$.

Therefore, by (23) and (30), we obtain the following lemma.

Lemma 6 For $m \geq 0, n \in \mathbb{N}$, we have

$$E_m(n) + (-1)^{n-1}E_m = 2 \sum_{l=0}^{n-1} (-1)^{n-1-l} l^m.$$

Let us assume that $n \in \mathbb{N}$ with $n \equiv 1 \pmod{2}$.

Then, by (28), we get

$$\int_{\mathbb{Z}_p} e^{xt} d\mu_{-1}(x) = \frac{2}{e^{nt} + 1} \sum_{l=0}^{n-1} (-1)^l e^{lt}. \tag{31}$$

Now, we consider the multivariate p -adic fermionic integral on \mathbb{Z}_p related to the higher-order Euler numbers as follows:

$$\begin{aligned} & \int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p} e^{(x_1+x_2+\dots+x_r)t} d\mu_{-1}(x_1) \dots d\mu_{-1}(x_r) \\ &= \left(\frac{2}{e^{nt} + 1} \right)^r \sum_{l_1=0}^{n-1} \dots \sum_{l_r=0}^{n-1} (-1)^{l_1+\dots+l_r} e^{(l_1+\dots+l_r)t} \\ &= \left(\sum_{l=0}^{\infty} E_l^{(r)} n^l \frac{t^l}{l!} \right) \sum_{l_1=0}^{n-1} \dots \sum_{l_r=0}^{n-1} (-1)^{l_1+\dots+l_r} \sum_{k=0}^{\infty} (l_1 + \dots + l_r)^k \frac{t^k}{k!} \\ &= \sum_{l_1=0}^{n-1} \dots \sum_{l_r=0}^{n-1} (-1)^{l_1+\dots+l_r} \sum_{m=0}^{\infty} \left\{ \sum_{k=0}^m \binom{m}{k} (l_1 + \dots + l_r)^k n^{m-k} E_{m-k}^{(r)} \right\} \frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty} \left\{ \sum_{l_1=0}^{n-1} \dots \sum_{l_r=0}^{n-1} (-1)^{l_1+\dots+l_r} \sum_{k=0}^m \binom{m}{k} n^{m-k} E_{m-k}^{(r)} (l_1 + \dots + l_r)^k \right\} \frac{t^m}{m!}. \end{aligned} \tag{32}$$

Thus, from (32), we can derive the following equation:

$$\begin{aligned} & \int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p} (x_1 \dots + x_r)^m d\mu_{-1}(x_1) \dots d\mu_{-1}(x_r) \\ &= \sum_{l_1=0}^{n-1} \dots \sum_{l_r=0}^{n-1} \sum_{k=0}^m (-1)^{l_1+\dots+l_r} \binom{m}{k} E_{m-k}^{(r)} n^{m-k} (l_1 + \dots + l_r)^k, \end{aligned} \tag{33}$$

where $m \geq 0$ and $n \in \mathbb{N}$ with $n \equiv 1 \pmod{2}$.

Therefore, by (24) and (33), we obtain the following theorem.

Theorem 7 For $m \geq 0$ and $n \in \mathbb{N}$ with $n \equiv 1 \pmod{2}$, we have

$$E_m^{(r)} = \sum_{l_1=0}^{n-1} \dots \sum_{l_r=0}^{n-1} \sum_{k=0}^m (-1)^{l_1+\dots+l_r} \binom{m}{k} E_{m-k}^{(r)} n^{m-k} (l_1 + \dots + l_r)^k,$$

Moreover,

$$E_m^{(r)}(x) = \sum_{l_1=0}^{n-1} \dots \sum_{l_r=0}^{n-1} \sum_{k=0}^m (-1)^{l_1+\dots+l_r} \binom{m}{k} E_{m-k}^{(r)} n^{m-k} (l_1 + \dots + l_r + x)^k.$$

From (32), we note that

$$\begin{aligned}
 & \int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p} e^{(x_1+\dots+x_r)t} d\mu_{-1}(x_1) \dots d\mu_{-1}(x_r) \\
 &= \left(\frac{2}{e^{nt} + 1}\right)^r \sum_{l_1=0}^{n-1} \dots \sum_{l_r=0}^{n-1} (-1)^{l_1+\dots+l_r} e^{(l_1+\dots+l_r)t} \\
 &= \sum_{l_1=0}^{n-1} \dots \sum_{l_r=0}^{n-1} (-1)^{l_1+\dots+l_r} \left(\frac{2}{e^{nt} + 1}\right)^r e^{\left(\frac{l_1+\dots+l_r}{n}\right)nt} \\
 &= \sum_{m=0}^{\infty} \sum_{l_1=0}^{n-1} \dots \sum_{l_r=0}^{n-1} (-1)^{l_1+\dots+l_r} E_m^{(r)}\left(\frac{l_1 + \dots + l_r}{n}\right) n^m \frac{t^m}{m!},
 \end{aligned} \tag{34}$$

where $n \in \mathbb{N}$ with $n \equiv 1 \pmod{2}$.

Thus, by (34), we get

$$\begin{aligned}
 & \int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p} (x_1 + \dots + x_r)^m d\mu_{-1}(x_1) \dots d\mu_{-1}(x_r) \\
 &= \sum_{l_1=0}^{n-1} \dots \sum_{l_r=0}^{n-1} (-1)^{l_1+\dots+l_r} E_m^{(r)}\left(\frac{l_1 + \dots + l_r}{n}\right) n^m,
 \end{aligned} \tag{35}$$

where $m \in \mathbb{Z} \geq 0$, $n \in \mathbb{N}$ with $n \equiv 1 \pmod{2}$.

Therefore by (24) and (35), we obtain the following theorem.

Theorem 8 For $m \in \mathbb{Z} \geq 0$, $n \in \mathbb{N}$ with $n \equiv 1 \pmod{2}$, we have

$$E_m^{(r)} = n^m \sum_{l_1=0}^{n-1} \dots \sum_{l_r=0}^{n-1} (-1)^{l_1+\dots+l_r} E_m^{(r)}\left(\frac{l_1 + \dots + l_r}{n}\right),$$

Moreover,

$$E_m^{(r)}(x) = n^m \sum_{l_1=0}^{n-1} \dots \sum_{l_r=0}^{n-1} (-1)^{l_1+\dots+l_r} E_m^{(r)}\left(\frac{l_1 + \dots + l_r + x}{n}\right).$$

For $a_1, a_2, \dots, a_r \in \mathbb{C}_p \setminus \{0\}$, let us consider the Barnes-type multiple Euler polynomials as follows:

$$\begin{aligned}
 & \int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p} e^{(a_1x_1+\dots+a_r x_r+x)t} d\mu_{-1}(x_1) \dots d\mu_{-1}(x_r) \\
 &= \left(\frac{2}{e^{a_1t} + 1}\right) \times \dots \times \left(\frac{2}{e^{a_r t} + 1}\right) e^{xt} \\
 &= \sum_{n=0}^{\infty} E_n(x|a_1, \dots, a_r) \frac{t^n}{n!}.
 \end{aligned} \tag{36}$$

When $x = 0$, $E_n(a_1, \dots, a_r) = E_n(0|a_1, \dots, a_r)$ is called the n th Barnes-type Euler number.

For $d \in \mathbb{N}$ with $d \equiv 1 \pmod{2}$, we observe that

$$\int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \sum_{a=0}^{d-1} (-1)^a \int_{\mathbb{Z}_p} f(a + dx) d\mu_{-1}(x). \tag{37}$$

From (37), we can derive the following equation:

$$\begin{aligned} & \int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p} e^{(a_1x_1 + \dots + a_rx_r)t} d\mu_{-1}(x_1) \dots d\mu_{-1}(x_r) \\ &= \sum_{l_1=0}^{d-1} \dots \sum_{l_r=0}^{d-1} (-1)^{l_1 + \dots + l_r} \int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p} e^{\{a_1l_1 + \dots + a_rl_r + (a_1x_1 + \dots + a_rx_r)d\}t} d\mu_{-1}(x_1) \dots d\mu_{-1}(x_r) \\ &= \sum_{n=0}^{\infty} d^n \sum_{l_1=0}^{d-1} \dots \sum_{l_r=0}^{d-1} (-1)^{l_1 + \dots + l_r} \int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p} \left(\frac{a_1l_1 + \dots + a_rl_r}{d} \right. \\ & \quad \left. + a_1x_1 + \dots + a_rx_r \right)^n d\mu_{-1}(x_1) \dots d\mu_{-1}(x_r) \frac{t^n}{n!}. \end{aligned} \tag{38}$$

By (38), we get

$$\begin{aligned} & \int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p} (a_1x_1 + \dots + a_rx_r)^n d\mu_{-1}(x_1) \dots d\mu_{-1}(x_r) \\ &= d^n \sum_{l_1=0}^{d-1} \dots \sum_{l_r=0}^{d-1} (-1)^{l_1 + \dots + l_r} \int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p} \left(\frac{a_1l_1 + \dots + a_rl_r}{d} \right. \\ & \quad \left. + a_1x_1 + \dots + a_rx_r \right)^n d\mu_{-1}(x_1) \dots d\mu_{-1}(x_r). \end{aligned} \tag{39}$$

Therefore, by (36) and (39), we obtain the following theorem.

Theorem 9 For $d \in \mathbb{N}$ with $d \equiv 1 \pmod{2}$, $n \geq 0$, we have

$$E_n(a_1, \dots, a_r) = d^n \sum_{l_1=0}^{d-1} \dots \sum_{l_r=0}^{d-1} (-1)^{l_1 + \dots + l_r} E_n \left(\frac{a_1l_1 + \dots + a_rl_r}{d} \mid a_1, \dots, a_r \right).$$

Moreover,

$$E_n(x \mid a_1, \dots, a_r) = d^n \sum_{l_1=0}^{d-1} \dots \sum_{l_r=0}^{d-1} (-1)^{l_1 + \dots + l_r} E_n \left(\frac{x + a_1l_1 + \dots + a_rl_r}{d} \mid a_1, \dots, a_r \right).$$

Remark Note that

$$\begin{aligned} E_n(x \mid a_1, \dots, a_r) &= \sum_{l=0}^n \binom{n}{l} x^l E_{n-l}(a_1, \dots, a_r) \\ &= \sum_{l=0}^n \binom{n}{l} x^{n-l} E_l(a_1, \dots, a_r). \end{aligned}$$

Thus, we have

$$\begin{aligned}
 & \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (a_1x_1 + \cdots + a_r x_r \\
 & \quad + b_1y_1 + \cdots + b_s y_s)^n d\mu_{-1}(y_1) \cdots d\mu_{-1}(y_s) d\mu_0(x_1) \cdots d\mu_0(x_r) \\
 &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} E_n(a_1x_1 + \cdots + a_r x_r | b_1, \dots, b_s) d\mu_0(x_1) \cdots d\mu_0(x_r) \\
 &= \sum_{l=0}^n \binom{n}{l} E_{n-l}(b_1, \dots, b_s) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (a_1x_1 + \cdots + a_r x_r)^l d\mu_0(x_1) \cdots d\mu_0(x_r) \\
 &= \sum_{l=0}^n \binom{n}{l} E_{n-l}(b_1, \dots, b_s) B_l(a_1, \dots, a_r). \tag{40}
 \end{aligned}$$

Now, we define mixed-type Barnes-type Euler and Bernoulli numbers as follows:

$$\begin{aligned}
 & EB_n(b_1, \dots, b_s; a_1, \dots, a_r) \\
 &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (a_1x_1 + \cdots + a_r x_r \\
 & \quad + b_1y_1 + \cdots + b_s y_s)^n d\mu_{-1}(y_1) \cdots d\mu_{-1}(y_s) d\mu_0(x_1) \cdots d\mu_0(x_r), \tag{41}
 \end{aligned}$$

where $a_1, \dots, a_r, b_1, \dots, b_s \neq 0$.

By (40) and (41), we get

$$EB_n(b_1, \dots, b_s; a_1, \dots, a_r) = \sum_{l=0}^n \binom{n}{l} E_{n-l}(b_1, \dots, b_s) B_l(a_1, \dots, a_r).$$

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to this work. All authors read and approved the final manuscript.

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