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Chlodowsky-type q-Bernstein-Stancu-Kantorovich operators

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Abstract
In this paper, we introduce Chlodowsky-type q-Bernstein-Stancu-Kantorovich operators on the unbounded domain. We should note that this generalization includes various kinds of operators which have not been introduced earlier. We calculate the error of approximation of these operators by using the modulus of continuity and Lipschitz-type functionals. Finally, we give generalization of the operators and investigate their approximations.
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1 Introduction

Generalizations of Bernstein polynomials and their q-analogues have been an intensive research area of approximation theory (see [1–19]). In this paper, we introduce the Chlodowsky-type q-Bernstein-Stancu-Kantorovich operators and investigate their approximation properties.

Firstly, let us recall the following notions of q-integers [20]. Let $q > 0$. For any integer $k \geq 0$, the q-integer $[k]_q = [k]$ is defined by

$$[k] = \begin{cases} (1-q^k)/(1-q), & q \neq 1, \\ k, & q = 1, \end{cases} \qquad [0] = 0,$$

the q-factorial $[k]_q! = [k]!$ is defined by

$$[k]! = \begin{cases} [k][k-1]\cdots[1], & k = 1,2,3,\ldots, \\ 1, & k = 0 \end{cases}$$

and for integers $n \geq k \geq 0$, q-binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[n-k]![k]!}.$$

In 2011, the q-based Bernstein-Schurer operators were defined by Muraru [21] as

$$B_n^p(f;q;x) = \sum_{k=0}^{n+p} f\left(\frac{[k]}{[n]}\right) \begin{bmatrix} n+p \\ k \end{bmatrix} x^k \prod_{s=0}^{n+p-k-1} \left(1-q^s x\right), \quad 0 \leq x \leq 1. \tag{1.1}$$

Then she obtained the Korovkin-type theorem and the order of convergence by using the modulus of continuity. She also mentioned that if $q \to 1^-$ in (1.1), the operators reduce to the Schurer operators considered by Schurer [22] and if $p = 0$ in (1.1), they contain the q-Bernstein operators [16]. After that, different approximation properties of the q-Bernstein-Schurer operators were studied in [23].

Recently, the q-Bernstein-Schurer-Kantorovich operators were defined [24] as

$$K_n^p(f;q;x) = \sum_{k=0}^{n+p} \begin{bmatrix} n+p \\ k \end{bmatrix} x^k \prod_{s=0}^{n+p-k-1} (1-q^s x) \times \int_0^1 f\left(\frac{[k]}{[n+1]} + \frac{1+(q-1)[k]}{[n+1]} t\right) d_q t. \tag{1.2}$$

Then, the approximation rates of the q-Bernstein-Schurer-Kantorovich operators were given by means of Lipschitz class functionals and the first and the second modulus of continuity.

Notice that if we choose $p = 0$ in (1.2), we get the q-Bernstein-Kantorovich operators which were defined by Mahmudov and Sabancıgil in [25]. We should also mention that in [26] the authors defined a different version of q-Bernstein-Kantorovich operators, where they used the usual integral instead of q-integral in the definition.

In 2013, the q-analogue of Bernstein-Schurer-Stancu operators $S_{n,p}^{\alpha,\beta} : C[0, 1 + p] \to C[0, 1]$ was introduced by Agrawal *et al.* in [2] by

$$S_{n,p}^{(\alpha,\beta)}(f;q;x) = \sum_{k=0}^{n+p} f\left(\frac{[k]+\alpha}{[n]+\beta}\right) \begin{bmatrix} n \\ k \end{bmatrix} x^k \prod_{s=0}^{n-k-1} (1-q^s x),$$

where α and β are real numbers which satisfy $0 \le \alpha \le \beta$ and also p is a non-negative integer.

Then, Ren and Zeng introduced the Kantorovich-type-q-Bernstein-Stancu operators [27]. They investigated the statistical approximation properties.

On the other hand, Karslı and Gupta [13] introduced q-Chlodowsky operators as follows:

$$C_{n,q}(f;x) = \sum_{k=0}^{n} f\left(\frac{[k]}{[n]} b_n\right) \begin{bmatrix} n \\ k \end{bmatrix} \left(\frac{x}{b_n}\right)^k \prod_{s=0}^{n-k-1} \left(1 - q^s \frac{x}{b_n}\right), \quad 0 \le x \le b_n,$$

where $n \in \mathbb{N}$ and (b_n) is a positive increasing sequence with $\lim_{n\to\infty} b_n = \infty$. Then, they investigated the approximation properties of $C_{n,q}(f;x)$.

Recently, the Chlodowsky variant of q-Bernstein-Schurer-Stancu operators was introduced by the authors in [28] as

$$C_{n,p}^{(\alpha,\beta)}(f;q;x) = \sum_{k=0}^{n+p} f\left(\frac{[k]+\alpha}{[n]+\beta} b_n\right) \begin{bmatrix} n+p \\ k \end{bmatrix} \left(\frac{x}{b_n}\right)^k \prod_{s=0}^{n+p-k-1} \left(1 - q^s \frac{x}{b_n}\right), \tag{1.3}$$

where $n \in \mathbb{N}$ and $p \in \mathbb{N}_0$ and $\alpha, \beta \in \mathbb{R}$ with $0 \le \alpha \le \beta$, $0 \le x \le b_n$, $0 < q < 1$, and Korovkin-type approximation theorems were proved in different function spaces. Moreover, the error of approximation was computed by using the modulus of continuity and Lipschitz-type functionals. Also, the generalization of the Chlodowsky variant of q-Bernstein-Schurer-Stancu operators was studied. Notice that $C_{n,p}^{(0,0)}(f;q;x)$ gives the q-Bernstein-Schurer-Chlodowsky operators which have not been defined yet, and additionally taking $p = 0$, we get the q-Bernstein-Chlodowsky operators [13]. On the other hand, from [28] the first three moments of $C_{n,p}^{(\alpha,\beta)}(f;q;x)$ are as follows:

(i) $C_{n,p}^{(\alpha,\beta)}(1;q;x) = 1$,

(ii) $C_{n,p}^{(\alpha,\beta)}(t;q;x) = \dfrac{[n+p]x + \alpha b_n}{[n] + \beta}$,

(iii) $C_{n,p}^{(\alpha,\beta)}(t^2;q;x) = \dfrac{1}{([n]+\beta)^2}\{[n+p-1][n+p]qx^2 + (2\alpha+1)[n+p]b_n x + \alpha^2 b_n^2\}$.

The organization of this paper is as follows. In Section 2, we introduce the Chlodowsky-type q-Bernstein-Stancu-Kantorovich operators and calculate the moments for them. In Section 3, Korovkin-type theorems are proved. In Section 4, we obtain the rate of convergence of the approximation process in terms of the first and the second modulus of continuity and also by means of Lipschitz class functions. In Section 5, we study the generalization of the Kantorovich-Stancu type generalization of q-Bernstein-Chlodowsky operators and study their approximation properties.

2 Construction of the operators

The Chlodowsky-type q-Bernstein-Stancu-Kantorovich operators are introduced as

$$K_{n,p}^{(\alpha,\beta)}(f;q;x) := \sum_{k=0}^{n+p} \begin{bmatrix} n+p \\ k \end{bmatrix} \left(\frac{x}{b_n}\right)^k \prod_{s=0}^{n+p-k-1}\left(1 - q^s \frac{x}{b_n}\right) \times \int_0^1 f\left(\frac{(1+(q-1)[k])t + [k] + \alpha}{[n+1]+\beta} b_n\right) dt, \tag{2.1}$$

where $n \in \mathbb{N}$ and $p \in \mathbb{N}_0$ and $\alpha, \beta \in \mathbb{R}$ with $0 \le \alpha \le \beta$, $0 \le x \le b_n$, $0 < q < 1$. Obviously, $K_{n,p}^{(\alpha,\beta)}$ is a linear and positive operator. We should notice that if we choose $p = \alpha = \beta = 0$ in (2.1) and taking into account that $(1 + (q-1)[k])t = q^k t$, the operator $K_{n,p}^{(\alpha,\beta)}(f;q;x)$ reduces to the Chlodowsky variant of the q-Bernstein Kantorovich operator [26].

First of all let us give the following lemma which will be used throughout the paper.

Lemma 2.1 *Let $K_{n,p}^{(\alpha,\beta)}(f;q;x)$ be given in* (2.1). *Then we have*

(i) $K_{n,p}^{(\alpha,\beta)}(1;q;x) = 1$,

(ii) $K_{n,p}^{(\alpha,\beta)}(t;q;x) = \dfrac{[n+p][2]x + (2\alpha+1)b_n}{2([n+1]+\beta)}$,

(iii) $$K_{n,p}^{(\alpha,\beta)}(t^2;q;x) = \frac{1}{([n+1]+\beta)^2}\left\{\frac{[3]}{3}[n+p-1][n+p]qx^2 + \left(\frac{q^2+3q+2}{3} + [2]\alpha\right)[n+p]b_n x + \left(\alpha^2 + \alpha + \frac{1}{3}\right)b_n^2\right\},$$

(iv) $K_{n,p}^{(\alpha,\beta)}((t-x);q;x) = \left(\dfrac{[n+p][2]}{2([n+1]+\beta)} - 1\right)x + \dfrac{(2\alpha+1)b_n}{2([n+1]+\beta)}$,

$$\text{(v)}\quad K_{n,p}^{(\alpha,\beta)}\big((t-x)^2;q;x\big) = \left(\frac{[3][n+p-1][n+p]q}{3([n+1]+\beta)^2} - \frac{[2][n+p]}{[n+1]+\beta} + 1\right)x^2 + \left(\frac{(q^2+3q+2+3[2]\alpha)[n+p]}{3([n+1]+\beta)^2} - \frac{(2\alpha+1)}{[n+1]+\beta}\right)b_n x + \frac{(3\alpha^2+3\alpha+1)b_n^2}{3([n+1]+\beta)^2}.$$

Proof (i) Using (2.1) and $C_{n,q}(1;x) = 1$, we get

$$K_{n,p}^{(\alpha,\beta)}(1;q;x) = \sum_{k=0}^{n+p} \begin{bmatrix} n+p \\ k \end{bmatrix} \left(\frac{x}{b_n}\right)^k \prod_{s=0}^{n+p-k-1} \left(1 - q^s \frac{x}{b_n}\right) = C_{n,p}^{(\alpha,\beta)}(1;q;x) = 1. \tag{2.2}$$

(ii) After some calculations, we obtain

$$\begin{aligned} &K_{n,p}^{(\alpha,\beta)}(t;q;x) \\ &= \sum_{k=0}^{n+p} \begin{bmatrix} n+p \\ k \end{bmatrix} \left(\frac{x}{b_n}\right)^k \prod_{s=0}^{n+p-k-1} \left(1 - q^s \frac{x}{b_n}\right) \int_0^1 \left(\frac{(1+(q-1)[k])t+[k]+\alpha}{[n+1]+\beta} b_n\right) dt \\ &= \sum_{k=0}^{n+p} \begin{bmatrix} n+p \\ k \end{bmatrix} \left(\frac{x}{b_n}\right)^k \prod_{s=0}^{n+p-k-1} \left(1 - q^s \frac{x}{b_n}\right) \left(\frac{[k]+\alpha}{[n+1]+\beta} b_n + \frac{1+(q-1)[k]}{2([n+1]+\beta)} b_n\right) \\ &= \frac{[n+p][2]x + (2\alpha+1)b_n}{2([n+1]+\beta)}. \end{aligned}$$

Whence the result.

(iii) By (2.1) we can write

$$\begin{aligned} &K_{n,p}^{(\alpha,\beta)}\big(t^2;q;x\big) \\ &= \sum_{k=0}^{n+p} \begin{bmatrix} n+p \\ k \end{bmatrix} \left(\frac{x}{b_n}\right)^k \prod_{s=0}^{n+p-k-1} \left(1 - q^s \frac{x}{b_n}\right) \int_0^1 \left(\frac{(1+(q-1)[k])t+[k]+\alpha}{[n+1]+\beta} b_n\right)^2 dt \\ &= \sum_{k=0}^{n+p} \begin{bmatrix} n+p \\ k \end{bmatrix} \left(\frac{x}{b_n}\right)^k \prod_{s=0}^{n+p-k-1} \left(1 - q^s \frac{x}{b_n}\right) \frac{b_n^2}{([n+1]+\beta)^2} \left\{\left(\frac{2(q-1)^2}{3} + q\right)[k]^2 \right. \\ &\quad \left. + \big(1+[2]\alpha\big)[k] + \alpha^2 + \alpha + \frac{1}{3}\right\}. \end{aligned}$$

After some calculations as in (i) and (ii), we get the desired result.

(iv) Using (i) and (ii), we get

$$\begin{aligned} K_{n,p}^{(\alpha,\beta)}\big((t-x);q;x\big) &= K_{n,p}^{(\alpha,\beta)}(t;q;x) - xK_{n,p}^{(\alpha,\beta)}(1;q;x) \\ &= \left(\frac{[n+p][2]}{2([n+1]+\beta)} - 1\right)x + \frac{(2\alpha+1)b_n}{2([n+1]+\beta)}. \end{aligned}$$

(v) It is known that

$$K_{n,p}^{(\alpha,\beta)}\big((t-x)^2;q;x\big) = K_{n,p}^{(\alpha,\beta)}\big(t^2;q;x\big) - 2xK_{n,p}^{(\alpha,\beta)}(t;q;x) + x^2 K_{n,p}^{(\alpha,\beta)}(1;q;x).$$

Then we obtain the result directly. □

Lemma 2.2 *For the second central moment, if we take supremum on* $[0, b_n]$, *we get the following estimate*:

$$K_{n,p}^{(\alpha,\beta)}\big((t-x)^2;q;x\big) \leq b_n^2\left\{\left|\frac{[3][n+p-1][n+p]q}{3([n+1]+\beta)^2} - \frac{[2][n+p]}{[n+1]+\beta} + 1\right| + \left|\frac{(q^2+3q+2+3[2]\alpha)[n+p]}{3([n+1]+\beta)^2} - \frac{(2\alpha+1)}{[n+1]+\beta}\right| + \frac{(3\alpha^2+3\alpha+1)}{3([n+1]+\beta)^2}\right\}.$$

3 Korovkin-type approximation theorem

In this section, we study Korovkin-type approximation theorems of Chlodowsky-type q-Bernstein-Stancu-Kantorovich operators. Let C_ρ denote the space of all continuous functions f such that the following condition

$$\big|f(x)\big| \leq M_f\rho(x), \quad -\infty < x < \infty$$

is satisfied.

It is clear that C_ρ is a linear normed space with the norm

$$\|f\|_\rho = \sup_{-\infty<x<\infty} \frac{|f(x)|}{\rho(x)}.$$

The following theorems play an important role in our investigations.

Theorem 3.1 (See [9]) *There exists a sequence of positive linear operators* U_n, *acting from* C_ρ *to* C_ρ, *satisfying the conditions*

$$\lim_{n\to\infty}\big\|U_n(1;x)-1\big\|_\rho = 0, \tag{3.1}$$

$$\lim_{n\to\infty}\big\|U_n(\phi;x)-\phi\big\|_\rho = 0, \tag{3.2}$$

$$\lim_{n\to\infty}\big\|U_n\big(\phi^2;x\big)-\phi^2\big\|_\rho = 0, \tag{3.3}$$

where $\phi(x)$ *is a continuous and increasing function on* $(-\infty,\infty)$ *such that* $\lim_{x\to\pm\infty}\phi(x) = \pm\infty$ *and* $\rho(x) = 1+\phi^2$, *and there exists a function* $f^* \in C_\rho$ *for which* $\overline{\lim}_{n\to\infty}\|U_nf^* - f^*\|_\rho > 0$.

Theorem 3.2 (See [9]) *Conditions* (3.1), (3.2), (3.3) *imply*

$$\lim_{n\to\infty}\|U_nf - f\|_\rho = 0$$

for any function f *belonging to the subset* $C_\rho^0 := \{f \in C_\rho : \lim_{|x|\to\infty}\frac{f(x)}{\rho(x)}$ *is finite*$\}$.

Let us choose $\rho(x) = 1+x^2$ and consider the operators:

$$U_n^{(\alpha,\beta)}(f;q;x) = \begin{cases} K_{n,p}^{(\alpha,\beta)}(f;q;x) & \text{if } x \in [0,b_n], \\ f(x) & \text{if } x \in [0,\infty)/[0,b_n]. \end{cases}$$

It should be mentioned that the operators $U_n^{(\alpha,\beta)}$ act from C_{1+x^2} to C_{1+x^2}. For all $f \in C_{1+x^2}$, we have

$$\begin{aligned} \left\| U_n^{(\alpha,\beta)}(f;q;\cdot)\right\|_{1+x^2} &\le \sup_{x\in[0,b_n]} \frac{|K_{n,p}^{(\alpha,\beta)}(f;q;x)|}{1+x^2} + \sup_{b_n<x<\infty} \frac{|f(x)|}{1+x^2} \\ &\le \|f\|_{1+x^2}\left[\sup_{x\in[0,\infty)} \frac{|K_{n,p}^{(\alpha,\beta)}(1+t^2;q;x)|}{1+x^2} + 1\right]. \end{aligned}$$

Therefore, it is clear from Lemma 2.1 that

$$\left\| U_n^{(\alpha,\beta)}(f;q;\cdot)\right\|_{1+x^2} \le M\|f\|_{1+x^2}$$

provided that $q := (q_n)$ with $0 < q_n < 1$, $\lim_{n\to\infty} q_n = 1$, $\lim_{n\to\infty} q_n^n = N < \infty$ and $\lim_{n\to\infty} \frac{b_n}{[n]} = 0$. We have the following approximation theorem.

Theorem 3.3 *For all $f \in C_{1+x^2}^0$, we have*

$$\lim_{n\to\infty} \left\| U_n^{(\alpha,\beta)}(f;q_n;\cdot) - f(\cdot)\right\|_{1+x^2} = 0$$

provided that $q := (q_n)$ with $0 < q_n < 1$, $\lim_{n\to\infty} q_n = 1$, $\lim_{n\to\infty} q_n^n = N < \infty$ and $\lim_{n\to\infty} \frac{b_n}{[n]} = 0$.

Proof With help of Theorem 3.2 and Lemma 2.1(i), (ii) and (iii), we have the following estimates, respectively:

$$\sup_{x\in[0,\infty)} \frac{|U_n^{(\alpha,\beta)}(1;q_n;x) - 1|}{1+x^2} = \sup_{0\le x\le b_n} \frac{|K_n^{(\alpha,\beta)}(1;q_n;x) - 1|}{1+x^2} = 0,$$

$$\begin{aligned} &\sup_{x\in[0,\infty)} \frac{|U_n^{(\alpha,\beta)}(t;q_n;x) - t|}{1+x^2} \\ &\quad = \sup_{0\le x\le b_n} \frac{|K_n^{(\alpha,\beta)}(t;q_n;x) - x|}{1+x^2} \le \sup_{0\le x\le b_n} \frac{|\frac{[n+p][2]}{2([n+1]+\beta)} - 1|x + \frac{(2\alpha+1)b_n}{2([n+1]+\beta)}}{(1+x^2)} \\ &\quad \le \left|\frac{[n+p][2]}{2([n+1]+\beta)} - 1\right| + \frac{(2\alpha+1)b_n}{2([n+1]+\beta)} \to 0 \end{aligned}$$

and

$$\begin{aligned} &\sup_{x\in[0,\infty)} \frac{|U_n^{(\alpha,\beta)}(t^2;q_n;x) - t^2|}{1+x^2} \\ &\quad = \sup_{0\le x\le b_n} \frac{|K_n^{(\alpha,\beta)}(t^2;q_n;x) - x^2|}{1+x^2} \\ &\quad \le \sup_{0\le x\le b_n} \frac{1}{1+x^2}\left\{\left|\frac{[3]}{3}\frac{[n+p-1][n+p]q}{([n+1]+\beta)^2} - 1\right|x^2 \right. \\ &\qquad \left. + \left(\frac{q^2+3q+2}{3} + [2]\alpha\right)\frac{[n+p]b_n}{([n+1]+\beta)^2}x + \left(\alpha^2+\alpha+\frac{1}{3}\right)\frac{b_n^2}{([n+1]+\beta)^2}\right\} \end{aligned}$$

$$\leq \left\{ \left| \frac{[3]}{3} \frac{[n+p-1][n+p]q}{([n+1]+\beta)^2} - 1 \right| + \left(\frac{q^2+3q+2}{3} + [2]\alpha \right) \frac{[n+p]b_n}{([n+1]+\beta)^2} + \left(\alpha^2 + \alpha + \frac{1}{3} \right) \frac{b_n^2}{([n+1]+\beta)^2} \right\} \to 0$$

whenever $n \to \infty$, since $\lim_{n\to\infty} q_n = 1$ and $\frac{b_n}{[n]} = 0$ as $n \to \infty$. □

Lemma 3.4 *Let A be a positive real number independent of n and f be a continuous function which vanishes on $[A, \infty]$. Assume that $q := (q_n)$ with $0 < q_n < 1$, $\lim_{n\to\infty} q_n = 1$, $\lim_{n\to\infty} q_n^n = N < \infty$ and $\lim_{n\to\infty} \frac{b_n^2}{[n]} = 0$. Then we have*

$$\lim_{n\to\infty} \sup_{0\leq x\leq b_n} \left| K_{n,p}^{(\alpha,\beta)}(f; q_n; x) - f(x) \right| = 0.$$

Proof From the hypothesis on f, one can write $|f(x)| \leq M$ $(M > 0)$. For arbitrary small $\varepsilon > 0$, we have

$$\left| f\left(\frac{(1+(q-1)[k])t + [k] + \alpha}{[n+1]+\beta} b_n \right) - f(x) \right| < \varepsilon + \frac{2M}{\delta^2} \left(\frac{(1+(q-1)[k])t + [k] + \alpha}{[n+1]+\beta} b_n - x \right)^2$$

for $x \in [0, b_n]$ and $\delta = \delta(\varepsilon)$. Using the following equality

$$\sum_{k=0}^{n+p} \begin{bmatrix} n+p \\ k \end{bmatrix} \left(\frac{x}{b_n} \right)^k \prod_{s=0}^{n+p-k-1} \left(1 - q^s \frac{x}{b_n} \right) \int_0^1 \left(\frac{(1+(q-1)[k])t + [k] + \alpha}{[n+1]+\beta} b_n - x \right)^2 dt = K_{n,p}^{(\alpha,\beta)}\left((t-x)^2; q_n; x \right),$$

we get by Lemma 2.2 that

$$\sup_{0\leq x\leq b_n} \left| K_{n,p}^{(\alpha,\beta)}(f; q_n; x) - f(x) \right| \leq \varepsilon + \frac{2M}{\delta^2} \left[\left| \frac{[3][n+p-1][n+p]q_n}{3([n+1]+\beta)^2} - \frac{[2][n+p]}{[n+1]+\beta} + 1 \right| b_n^2 + \left| \frac{(q_n^2 + 3q_n + 2 + 3[2]\alpha)[n+p]}{([n+1]+\beta)^2} - \frac{(2\alpha+1)}{[n+1]+\beta} \right| b_n^2 + \frac{(3\alpha^2 + 3\alpha + 1)b_n^2}{3([n+1]+\beta)^2} \right].$$

Since $0 < q_n < 1$, $\lim_{n\to\infty} q_n = 1$, $\lim_{n\to\infty} q_n^n = N < \infty$ and $\lim_{n\to\infty} \frac{b_n^2}{[n]} = 0$, we have the desired result. □

Theorem 3.5 *Let f be a continuous function on the semi-axis $[0, \infty)$ and*

$$\lim_{x\to\infty} f(x) = k_f < \infty.$$

Assume that $q := (q_n)$ with $0 < q_n < 1$, $\lim_{n\to\infty} q_n = 1$, $\lim_{n\to\infty} q_n^n = K < \infty$ and $\lim_{n\to\infty} \frac{b_n^2}{[n]} = 0$. Then

$$\lim_{x\to\infty} \sup_{0\leq x\leq b_n} \left| K_{n,p}^{(\alpha,\beta)}(f; q_n; x) - f(x) \right| = 0.$$

Proof If we apply the same techniques as in the proof of Theorem 3.5 in [28] and use Lemma 3.4, we obtain the desired result. □

4 Order of convergence

In this section, we study the rate of convergence of the operators in terms of the elements of Lipschitz classes and the first and the second modulus of continuity of the function.

Firstly, we give the rate of convergence of the operators $K_{n,p}^{(\alpha,\beta)}$ in terms of the Lipschitz class $\mathrm{Lip}_M(\gamma)$. Let $C_B[0,\infty)$ denote the space of bounded continuous functions on $[0,\infty)$ endowed with the usual supremum norm. A function $f \in C_B[0,\infty)$ belongs to $\mathrm{Lip}_M(\gamma)$ $(0 < \gamma \leq 1)$ if the condition

$$\bigl|f(t) - f(x)\bigr| \leq M|t-x|^{\gamma} \quad \bigl(t,x \in [0,\infty)\bigr)$$

is satisfied.

Theorem 4.1 *Let $f \in \mathrm{Lip}_M(\gamma)$. Then we have*

$$\bigl|K_{n,p}^{(\alpha,\beta)}(f;q;x) - f(x)\bigr| \leq M\bigl(\delta_{n,q}(x)\bigr)^{\gamma/2},$$

where

$$\delta_{n,q}(x) = \biggl(\frac{[3][n+p-1][n+p]q}{3([n+1]+\beta)^2} - \frac{[2][n+p]}{[n+1]+\beta} + 1\biggr)x^2 + \biggl(\frac{(q^2+3q+2+3[2]\alpha)[n+p]}{([n+1]+\beta)^2} - \frac{(2\alpha+1)}{[n+1]+\beta}\biggr)b_n x + \frac{(3\alpha^2+3\alpha+1)b_n^2}{3([n+1]+\beta)^2}.$$

Proof Using the monotonicity and the linearity of the operators and taking into account that $f \in \mathrm{Lip}_M(\gamma)$, we get

$$\begin{aligned}
&\bigl|K_{n,p}^{(\alpha,\beta)}(f;q;x) - f(x)\bigr| \\
&\quad= \Biggl|\sum_{k=0}^{n+p} \begin{bmatrix} n+p \\ k \end{bmatrix} \biggl(\frac{x}{b_n}\biggr)^k \prod_{s=0}^{n+p-k-1}\biggl(1 - q^s\frac{x}{b_n}\biggr) \\
&\qquad\times \int_0^1 \biggl(f\biggl(\frac{(1+(q-1)[k])t+[k]+\alpha}{[n+1]+\beta}b_n\biggr) - f(x)\biggr)\,dt\Biggr| \\
&\quad\leq \sum_{k=0}^{n+p} \begin{bmatrix} n+p \\ k \end{bmatrix} \biggl(\frac{x}{b_n}\biggr)^k \prod_{s=0}^{n+p-k-1}\biggl(1 - q^s\frac{x}{b_n}\biggr) \\
&\qquad\times \int_0^1 \biggl|f\biggl(\frac{(1+(q-1)[k])t+[k]+\alpha}{[n+1]+\beta}b_n\biggr) - f(x)\biggr|\,dt \\
&\quad\leq M\sum_{k=0}^{n+p} \begin{bmatrix} n+p \\ k \end{bmatrix} \biggl(\frac{x}{b_n}\biggr)^k \prod_{s=0}^{n+p-k-1}\biggl(1 - q^s\frac{x}{b_n}\biggr) \\
&\qquad\times \int_0^1 \biggl|\frac{(1+(q-1)[k])t+[k]+\alpha}{[n+1]+\beta}b_n - x\biggr|^{\gamma}\,dt.
\end{aligned}$$

Applying Hölder's inequality with $p = \frac{2}{\gamma}$ and $q = \frac{2}{2-\gamma}$, we have the following inequalities by (2.2):

$$\begin{aligned}\int_0^1 \left| \frac{(1+(q-1)[k])t+[k]+\alpha}{[n+1]+\beta} b_n - x \right|^\gamma dt &\le \left\{ \int_0^1 \left(\frac{(1+(q-1)[k])t+[k]+\alpha}{[n+1]+\beta} b_n - x \right)^2 dt \right\}^{\frac{\gamma}{2}} \left\{ \int_0^1 dt \right\}^{\frac{2-\gamma}{2}} \\ &= \left\{ \int_0^1 \left(\frac{(1+(q-1)[k])t+[k]+\alpha}{[n+1]+\beta} b_n - x \right)^2 dt \right\}^{\frac{\gamma}{2}}.\end{aligned}$$

Then we get

$$\left| K_{n,p}^{(\alpha,\beta)}(f;q;x) - f(x) \right| \le M \sum_{k=0}^{n+p} \left\{ \int_0^1 \left(\frac{(1+(q-1)[k])t+[k]+\alpha}{[n+1]+\beta} b_n - x \right)^2 dt \right\}^{\frac{\gamma}{2}} p_{n,k}(q;x),$$

where $p_{n,k}(q;x) = \sum_{k=0}^{n+p} \left[{n+p \atop k} \right] (\frac{x}{b_n})^k \prod_{s=0}^{n+p-k-1} (1 - q^s \frac{x}{b_n})$. Again using Hölder's inequality with $p = \frac{2}{\gamma}$ and $q = \frac{2}{2-\gamma}$, we have

$$\begin{aligned}&\left| K_{n,p}^{(\alpha,\beta)}(f;q;x) - f(x) \right| \\ &\quad \le M \left\{ \sum_{k=0}^{n+p} \int_0^1 \left(\frac{(1+(q-1)[k])t+[k]+\alpha}{[n+1]+\beta} b_n - x \right)^2 dt\, p_{n,k}(q;x) \right\}^{\frac{\gamma}{2}} \left\{ \sum_{k=0}^{n+p} p_{n,k}(q;x) \right\}^{\frac{2-\gamma}{2}} \\ &\quad = M \left\{ \sum_{k=0}^{n+p} p_{n,k}(q;x) \int_0^1 \left(\frac{(1+(q-1)[k])t+[k]+\alpha}{[n+1]+\beta} b_n - x \right)^2 dt \right\}^{\frac{\gamma}{2}} \\ &\quad = M \bigl(\delta_{n,q}(x)\bigr)^{\gamma/2},\end{aligned}$$

where $\delta_{n,q}(x) := K_{n,p}^{(\alpha,\beta)}((t-x)^2;q;x)$. □

Now, we give the rate of convergence of the operators by means of the modulus of continuity $\omega(f;\delta)$. Let $f \in C_B[0,\infty)$ such that f is uniformly continuous and $x \ge 0$. The modulus of continuity of f is given as

$$\omega(f;\delta) = \max_{\substack{|t-x|\le\delta \\ t,x\in[0,\infty)}} \left| f(t) - f(x) \right|. \tag{4.1}$$

It is known that for any $\delta > 0$ the following inequality

$$\left| f(x) - f(y) \right| \le \omega(f;\delta) \left(\frac{|x-y|}{\delta} + 1 \right) \tag{4.2}$$

is satisfied [8].

Theorem 4.2 *If $f \in C_B[0,\infty)$, we have*

$$\left| K_{n,p}^{(\alpha,\beta)}(f;q;x) - f(x) \right| \le 2\omega\bigl(f; \sqrt{\delta_{n,q}(x)}\bigr),$$

where $\delta_{n,q}(x)$ is the same as in Theorem 4.1.

Proof From monotonicity, we have

$$\left|K_{n,p}^{(\alpha,\beta)}(f;q;x)-f(x)\right| \leq \sum_{k=0}^{n+p}\begin{bmatrix} n+p \\ k \end{bmatrix}\left(\frac{x}{b_n}\right)^k \prod_{s=0}^{n+p-k-1}\left(1-q^s\frac{x}{b_n}\right) \times \int_0^1 \left|f\left(\frac{(1+(q-1)[k])t+[k]+\alpha}{[n+1]+\beta}b_n\right)-f(x)\right|dt.$$

Now by (4.2) we get

$$\begin{aligned}
&\left|K_{n,p}^{(\alpha,\beta)}(f;q;x)-f(x)\right| \\
&\quad\leq \sum_{k=0}^{n+p}\int_0^1\left(\frac{|\frac{(1+(q-1)[k])t+[k]+\alpha}{[n+1]+\beta}b_n-x|}{\delta}+1\right) \\
&\qquad\times \omega(f;\delta)\begin{bmatrix} n+p \\ k \end{bmatrix}\left(\frac{x}{b_n}\right)^k \prod_{s=0}^{n+p-k-1}\left(1-q^s\frac{x}{b_n}\right)dt \\
&\quad= \omega(f;\delta)\sum_{k=0}^{n+p}\begin{bmatrix} n+p \\ k \end{bmatrix}\left(\frac{x}{b_n}\right)^k \prod_{s=0}^{n+p-k-1}\left(1-q^s\frac{x}{b_n}\right) \\
&\qquad+\frac{\omega(f;\delta)}{\delta}\sum_{k=0}^{n+p}\int_0^1\left|\frac{(1+(q-1)[k])t+[k]+\alpha}{[n+1]+\beta}b_n-x\right| \\
&\qquad\times\begin{bmatrix} n+p \\ k \end{bmatrix}\left(\frac{x}{b_n}\right)^k \prod_{s=0}^{n+p-k-1}\left(1-q^s\frac{x}{b_n}\right)dt.
\end{aligned}$$

Then, using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\left|K_{n,p}^{(\alpha,\beta)}(f;q;x)-f(x)\right| \\
&\quad\leq \omega(f;\delta)+\frac{\omega(f;\delta)}{\delta}\left\{\sum_{k=0}^{n+p}p_{n,k}(q;x)\int_0^1\left(\frac{(1+(q-1)[k])t+[k]+\alpha}{[n+1]+\beta}b_n-x\right)^2dt\right\}^{\frac{1}{2}} \\
&\qquad\times\left\{\sum_{k=0}^{n+p}p_{n,k}(q;x)\right\}^{\frac{1}{2}} \\
&\quad= \omega(f;\delta)+\frac{\omega(f;\delta)}{\delta}\left\{K_{n,p}^{(\alpha,\beta)}\left((t-x)^2\right);q;x\right\}^{1/2}.
\end{aligned}$$

Finally, let us choose $\delta_{n,q}(x)$ the same as in Theorem 4.1. Then we get

$$\left|K_{n,p}^{(\alpha,\beta)}(f;q;x)-f(x)\right| \leq 2\omega\left(f;\sqrt{\delta_{n,q}(x)}\right). \qquad \square$$

Now let us denote by $C_B^2[0,\infty)$ the space of all functions $f \in C_B[0,\infty)$ such that $f', f'' \in C_B[0,\infty)$. Let $\|f\|$ denote the usual supremum norm of f. The classical Peetre's K-functional and the second modulus of smoothness of the function $f \in C_B[0,\infty)$ are defined respectively as

$$K(f,\delta) := \inf_{g\in C_B^2[0,\infty)}\left[\|f-g\|+\delta\left\|g''\right\|\right]$$

and

$$\omega_2(f,\delta) := \sup_{\substack{0<h<\delta,\\ x,x+h\in I}} \left|f(x+2h) - 2f(x+h) + f(x)\right|,$$

where $\delta > 0$. It is known that (see [8], p.177) there exists a constant $A > 0$ such that

$$K(f,\delta) \le A\omega_2(f,\sqrt{\delta}). \tag{4.3}$$

Theorem 4.3 *Let $q \in (0,1)$, $x \in [0,b_n]$ and $f \in C_B[0,\infty)$. Then, for fixed $p \in \mathbb{N}_0$, we have*

$$\left|K_{n,p}^{(\alpha,\beta)}(f;q;x) - f(x)\right| \le C\omega_2\bigl(f,\sqrt{\alpha_{n,q}(x)}\bigr) + \omega\bigl(f,\beta_{n,q}(x)\bigr)$$

for some positive constant C, where

$$\begin{aligned}\alpha_{n,q}(x) := \Bigg[&\left(\left(\frac{[3]}{3} + \frac{[2]^2}{4}\right)\frac{[n+p]^2}{([n+1]+\beta)^2} - 2\frac{[2][n+p]}{[n+1]+\beta} + 2\right)x^2 \\ &+ \left(\frac{q^2+3q+2+3[2]\alpha}{3([n+1]+\beta)^2} + \frac{(2\alpha+1)[2][n+p]}{2([n+1]+\beta)} - \frac{2(2\alpha+1)}{[n+1]+\beta}\right)b_n x \\ &+ \left(\frac{24\alpha^2+24\alpha+7}{12}\right)\frac{b_n^2}{([n+1]+\beta)^2}\Bigg]\end{aligned} \tag{4.4}$$

and

$$\beta_{n,q}(x) := \left|\frac{[2][n+p]}{2([n+1]+\beta)} - 1\right|x + \frac{(2\alpha+1)b_n}{2([n+1]+\beta)}. \tag{4.5}$$

Proof Define an auxiliary operator $K_{n,p}^*(f;q;x) : C_B[0,\infty) \to C_B[0,\infty)$ by

$$K_{n,p}^*(f;q;x) := K_{n,p}^{(\alpha,\beta)}(f;q;x) - f\left(\frac{[2][n+p]x + (2\alpha+1)b_n}{2([n+1]+\beta)}\right) + f(x). \tag{4.6}$$

Then by Lemma 2.1 we get

$$\begin{aligned} &K_{n,p}^*(1;q;x) = 1, \\ &K_{n,p}^*\bigl((t-x);q;x\bigr) = 0. \end{aligned} \tag{4.7}$$

For given $g \in C_B^2[0,\infty)$, it follows by the Taylor formula that

$$g(y) - g(x) = (y-x)g'(x) + \int_x^y (y-u)g''(u)\,du.$$

Taking into account (4.5) and using (4.7), we get

$$\begin{aligned}\left|K_{n,p}^*(g;q;x) - g(x)\right| &= \left|K_{n,p}^*\bigl(g(y)-g(x);q;x\bigr)\right| \\ &= \left|g'(x)K_{n,p}^*\bigl((u-x);q;x\bigr) + K_{n,p}^*\left(\int_x^y (y-u)g''(u)\,du;q;x\right)\right| \\ &= \left|K_{n,p}^*\left(\int_x^y (y-u)g''(u)\,du;q;x\right)\right|.\end{aligned}$$

Then by (4.6)

$$\begin{aligned}
&\left|K_{n,p}^{*}(g;q;x)-g(x)\right| \\
&\quad =\left|K_{n,p}^{(\alpha,\beta)}\left(\int_x^y (y-u)g''(u)\,du;q;x\right)\right. \\
&\qquad \left. -\int_x^{\frac{[2][n+p]x+(2\alpha+1)b_n}{2([n+1]+\beta)}}\left(\frac{[2][n+p]x+(2\alpha+1)b_n}{2([n+1]+\beta)}-u\right)g''(u)\,du\right| \\
&\quad \le \left|K_{n,p}^{(\alpha,\beta)}\left(\int_x^y (y-u)g''(u)\,du;q;x\right)\right| \\
&\qquad +\left|\int_x^{\frac{[2][n+p]x+(2\alpha+1)b_n}{2([n+1]+\beta)}}\left(\frac{[2][n+p]x+(2\alpha+1)b_n}{2([n+1]+\beta)}-u\right)g''(u)\,du\right|.
\end{aligned}$$

Since

$$\left|K_{n,p}^{(\alpha,\beta)}\left(\int_x^y (y-u)g''(u)\,du;q;x\right)\right| \le \|g''\|K_{n,p}^{(\alpha,\beta)}\left((y-x)^2;q;x\right)$$

and

$$\begin{aligned}
&\left|\int_x^{\frac{[2][n+p]x+(2\alpha+1)b_n}{2([n+1]+\beta)}}\left(\frac{[2][n+p]x+(2\alpha+1)b_n}{2([n+1]+\beta)}-u\right)g''(u)\,du\right| \\
&\quad \le \|g''\|\left(\left(\frac{[2][n+p]}{2([n+1]+\beta)}-1\right)x+\frac{(2\alpha+1)b_n}{2([n+1]+\beta)}\right)^2,
\end{aligned}$$

we get

$$\begin{aligned}
&\left|K_{n,p}^{*}(g;q;x)-g(x)\right| \\
&\quad \le \|g''\|K_{n,p}^{(\alpha,\beta)}\left((y-x)^2;q;x\right)+\|g''\|\left(\left(\frac{[2][n+p]}{2([n+1]+\beta)}-1\right)x+\frac{(2\alpha+1)b_n}{2([n+1]+\beta)}\right)^2.
\end{aligned}$$

Hence Lemma 2.1 implies that

$$\begin{aligned}
&\left|K_{n,p}^{*}(g;q;x)-g(x)\right| \\
&\quad \le \|g''\|\left[\left(\frac{[3][n+p-1][n+p]q}{3([n+1]+\beta)^2}-\frac{[2][n+p]}{([n+1]+\beta)}+1\right)x^2\right. \\
&\qquad +\left(\frac{(q^2+3q+2+3[2]\alpha)[n+p]}{3([n+1]+\beta)^2}-\frac{(2\alpha+1)}{([n+1]+\beta)}\right)b_nx+\frac{(3\alpha^2+3\alpha+1)b_n^2}{3([n+1]+\beta)^2} \\
&\qquad \left.+\left(\left(\frac{[2][n+p]}{2([n+1]+\beta)}-1\right)x+\frac{(2\alpha+1)b_n}{2([n+1]+\beta)}\right)^2\right]. \qquad (4.8)
\end{aligned}$$

Since $\|K_{n,p}^{*}(f;q;\cdot)\| \le 3\|f\|$, considering (4.4) and (4.5), for all $f \in C_B[0,\infty)$ and $g \in C_B^2[0,\infty)$, we may write from (4.8) that

$$\begin{aligned}
&\left|K_{n,p}^{(\alpha,\beta)}(f;q;x)-f(x)\right| \\
&\quad \le \left|K_{n,p}^{*}(f-g;q;x)-(f-g)(x)\right|
\end{aligned}$$

$$\begin{aligned}&+\left|K_{n,p}^{*}(g;q;x)-g(x)\right|+\left|f\left(\frac{[2][n+p]x+(2\alpha+1)b_n}{2([n+1]+\beta)}\right)-f(x)\right|\\&\leq 4\|f-g\|+\alpha_{n,q}(x)\|g''\|+\left|f\left(\frac{[2][n+p]x+(2\alpha+1)b_n}{2([n+1]+\beta)}\right)-f(x)\right|\\&\leq 4\bigl(\|f-g\|+\alpha_{n,q}(x)\|g''\|\bigr)+\omega\bigl(f,\beta_{n,q}(x)\bigr),\end{aligned}$$

which yields that

$$\begin{aligned}\left|K_{n,p}^{(\alpha,\beta)}(f;q;x)-f(x)\right|&\leq 4K\bigl(f,\alpha_{n,q}(x)\bigr)+\omega\bigl(f,\beta_{n,q}(x)\bigr)\\&\leq C\omega_2\bigl(f,\sqrt{\alpha_{n,q}(x)}\bigr)+\omega\bigl(f,\beta_{n,q}(x)\bigr),\end{aligned}$$

where

$$\begin{aligned}\alpha_{n,q}(x):=\Biggl[&\left(\left(\frac{[3]}{3}+\frac{[2]^2}{4}\right)\frac{[n+p]^2}{([n+1]+\beta)^2}-2\frac{[2][n+p]}{[n+1]+\beta}+2\right)x^2\\&+\left(\frac{q^2+3q+2+3[2]\alpha}{3([n+1]+\beta)^2}+\frac{(2\alpha+1)[2][n+p]}{2([n+1]+\beta)}-\frac{2(2\alpha+1)}{[n+1]+\beta}\right)b_nx\\&+\left(\frac{24\alpha^2+24\alpha+7}{12}\right)\frac{b_n^2}{([n+1]+\beta)^2}\Biggr]\end{aligned}$$

and

$$\beta_{n,q}(x):=\left|\frac{[2][n+p]}{2([n+1]+\beta)}-1\right|x+\frac{(2\alpha+1)b_n}{2([n+1]+\beta)}.$$

Hence we get the result. □

5 Generalization of the Kantorovich-Stancu type generalization of q-Bernstein-Chlodowsky operators

In this section, we introduce a generalization of Chlodowsky-type q-Bernstein-Stancu-Kantorovich operators. For $x\geq 0$, consider any continuous function $\omega(x)\geq 1$ and define

$$G_f(t)=f(t)\frac{1+t^2}{\omega(t)}.$$

Let us consider the generalization of $K_{n,p}^{(\alpha,\beta)}(f;q;x)$ as follows:

$$\begin{aligned}L_{n,p}^{\alpha,\beta}(f;q;x)&=\frac{\omega(x)}{1+x^2}\sum_{k=0}^{n+p}\begin{bmatrix}n+p\\k\end{bmatrix}\left(\frac{x}{b_n}\right)^k\prod_{s=0}^{n+p-k-1}\left(1-q^s\frac{x}{b_n}\right)\\&\quad\times\int_0^1 G_f\left(\frac{[k]+\alpha}{[n+1]+\beta}b_n+\frac{1+(q-1)[k]}{[n+1]+\beta}tb_n\right)dt,\end{aligned}$$

where $0\leq x\leq b_n$ and (b_n) has the same properties of Chlodowsky variant of q-Bernstein-Schurer-Stancu-Kantorovich operators.

Notice that this kind of generalization was considered earlier for the Bernstein-Chlodowsky polynomials [9], q-Bernstein-Chlodowsky polynomials [5] and Chlodowsky variant of q-Bernstein-Schurer-Stancu operators [28].

Now we have the following approximation theorem.

Theorem 5.1 *For the continuous functions satisfying*

$$\lim_{x\to\infty}\frac{f(x)}{\omega(x)}=K_f<\infty,$$

we have

$$\lim_{n\to\infty}\sup_{0\le x\le b_n}\frac{|L_{n,p}^{\alpha,\beta}(f;q;x)-f(x)|}{\omega(x)}=0$$

provided that $q:=(q_n)$ *with* $0<q_n<1$, $\lim_{n\to\infty}q_n=1$ *and* $\lim_{n\to\infty}\frac{b_n}{[n]}=0$ *as* $n\to\infty$.

Proof Obviously,

$$L_{n,p}^{\alpha,\beta}(f;q;x)-f(x)=\frac{\omega(x)}{1+x^2}\left(\sum_{k=0}^{n+p}\begin{bmatrix}n+p\\k\end{bmatrix}\left(\frac{x}{b_n}\right)^k\prod_{s=0}^{n+p-k-1}\left(1-q^s\frac{x}{b_n}\right)\right.$$
$$\left.\times\int_0^1 G_f\left(\frac{[k]+\alpha}{[n+1]+\beta}b_n+\frac{1+(q-1)[k]}{[n+1]+\beta}tb_n\right)dt-G_f(x)\right),$$

hence

$$\sup_{0\le x\le b_n}\frac{|L_{n,p}^{\alpha,\beta}(f;q;x)-f(x)|}{\omega(x)}=\sup_{0\le x\le b_n}\frac{|K_{n,p}^{(\alpha,\beta)}(G_f;q;x)-G_f(x)|}{1+x^2}.$$

From $|f(x)|\le M_f\omega(x)$ and the continuity of the function f, we have $|G_f(x)|\le M_f(1+x^2)$ for $x\ge 0$ and $G_f(x)$ is a continuous function on $[0,\infty)$. Using Theorem 3.3, we get the desired result. □

Finally note that taking $\omega(x)=1+x^2$, the operator $L_{n,p}^{\alpha,\beta}(f;q;x)$ reduces to $K_{n,p}^{(\alpha,\beta)}(f;q;x)$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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