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Nonlinear nonhomogeneous Robin problems with dependence on the gradient

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Abstract

We consider a nonlinear elliptic equation driven by a nonhomogeneous partial differential operator with Robin boundary condition and a convection term. Using a topological approach based on the Leray–Schauder alternative principle, together with truncation and comparison techniques, we show the existence of a smooth positive solution without imposing any global growth condition on the reaction term.

MSC: 35J92; 35P30

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1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$, and let $1 < p < +\infty$. In this paper we study the following nonlinear nonhomogeneous Robin problem with convection:

$$\begin{cases} -\operatorname{div} a(Du(z)) = f(z,u(z),Du(z)) & \text{in } \Omega, \\ \frac{\partial u}{\partial n_a} + \beta(z)u(z)^{p-1} = 0 & \text{on } \partial\Omega, u > 0. \end{cases} \tag{1.1}$$

In this problem, $a\colon \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is a continuous and strictly monotone map which satisfies certain regularity and growth conditions listed in hypotheses $H(a)$ below. These hypotheses are mild and incorporate in our framework many differential operators of interest such as the p-Laplacian and the (p,q)-Laplacian (that is, the sum of a p-Laplacian and a q-Laplacian with $1 < q < p < \infty$). The forcing term has the form of a convection term, that is, it depends also on the gradient of the unknown function. This dependence on the gradient prevents the use of variational methods directly on equation (1.1). In the boundary condition, $\frac{\partial u}{\partial n_a}$ denotes the conormal derivative of u and is defined by extension of the map

$$C^1(\overline{\Omega}) \ni u \longmapsto \big(a(Du), n\big)_{\mathbb{R}^N}$$

to all $u \in W^{1,p}(\Omega)$, with n being the outward unit normal on $\partial\Omega$. This generalized normal derivative is dictated by the nonlinear Green's identity (see, e.g., Gasiński and Papageorgiou [1, Theorem 2.4.53, p. 210]) and was used also by Lieberman [2, 3].

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Problems with convection were studied in the past using a variety of methods. We mention the works of de Figueiredo et al. [4], Girardi and Matzeu [5] for semilinear equations driven by the Dirichlet Laplacian; the works of Faraci et al. [6], Huy et al. [7], Iturriaga et al. [8] and Ruiz [9] for nonlinear equations driven by the Dirichlet p-Laplacian; and the works of Averna et al. [10], Faria et al. [11] and Tanaka [12] for equations driven by the Dirichlet (p,q)-Laplacian. Finally, we mention also the recent work of Gasiński and Papageorgiou [13] for Neumann problems driven by a differential operator of the form $\operatorname{div}(a(u)Du)$.

In this paper, in contrast to the aforementioned works, we do not impose any global growth condition on the convection term. Instead we assume that $f(z,\cdot,y)$ admits a positive root (zero) and all the other conditions refer to the behavior of the function $x \longmapsto f(z,x,y)$ near zero locally in $y \in \mathbb{R}^N$. Our approach is topological based on the Leray–Schauder alternative principle.

2 Mathematical background—hypotheses

In the analysis of problem (1.1) we will use the following spaces:

$$W^{1,p}(\Omega) \quad (1 < p < \infty), \qquad C^1(\overline{\Omega}) \text{ and } L^q(\partial\Omega) \quad (1 \leq q \leq \infty).$$

By $\|\cdot\|$ we denote the norm of the Sobolev space $W^{1,p}(\Omega)$ defined by

$$\|u\| = \left(\|u\|_p^p + \|Du\|_p^p\right)^{\frac{1}{p}} \quad \forall u \in W^{1,p}(\Omega).$$

The Banach space $C^1(\overline{\Omega})$ is an ordered Banach space with positive (order) cone given by

$$C_+ = \left\{u \in C^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\right\}.$$

This cone has a nonempty interior

$$\operatorname{int} C_+ = \left\{u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega, \frac{\partial u}{\partial n}\Big|_{\partial\Omega \cap u^{-1}(0)} < 0 \text{ if } \partial\Omega \cap u^{-1}(0) \neq \emptyset\right\}$$

which contains the set

$$D_+ = \left\{u \in C_+ : u(z) > 0 \text{ for all } z \in \overline{\Omega}\right\}.$$

In fact D_+ is the interior of C_+ when $C^1(\overline{\Omega})$ is equipped with the relative $C(\overline{\Omega})$-norm topology.

On $\partial\Omega$ we consider the $(N-1)$-dimensional Hausdorff (surface) measure $\sigma(\cdot)$. Using this measure, we can define the boundary Lebesgue spaces $L^q(\partial\Omega)$ $(1 \leq q \leq \infty)$ in the usual way. We have that there exists a unique continuous linear map $\gamma_0 \colon W^{1,p}(\Omega) \longrightarrow L^p(\partial\Omega)$ known as the *trace map* such that

$$\gamma_0(u) = u|_{\partial\Omega} \quad \forall u \in W^{1,p}(\Omega) \cap C(\overline{\Omega}).$$

So, the trace map γ_0 extends the notion of boundary values to any Sobolev function. We have

$$\operatorname{im} \gamma_0 = W^{\frac{1}{p'},p}(\partial\Omega) \qquad \left(\frac{1}{p} + \frac{1}{p'} = 1\right) \quad \text{and} \quad \ker \gamma_0 = W_0^{1,p}(\Omega).$$

The trace map γ_0 is compact into $L^q(\partial\Omega)$ for all $q \in [1, \frac{(N-1)p}{N-p})$ if $p < N$ and into $L^q(\partial\Omega)$ for all $q \in [1,\infty)$ if $p \geq N$. In what follows, for the sake of notational simplicity, we drop the use of the trace map γ_0. The restrictions of all Sobolev functions on $\partial\Omega$ are understood in the sense of traces.

Now we introduce the conditions on the map $a(y)$. So, let $\vartheta \in C^1(0,\infty)$ and assume that

$$0 < \widehat{c} \leq \frac{\vartheta'(t)t}{\vartheta(t)} \leq c_0 \quad \text{and} \quad c_1 t^{p-1} \leq \vartheta(t) \leq c_2\bigl(t^{\tau-1} + t^{p-1}\bigr) \quad \forall t > 0 \tag{2.1}$$

for some $1 \leq \tau < p$, $c_1, c_2 > 0$.

The hypotheses on the map $a(y)$ are the following:

$H(a)$: $a(y) = a_0(|y|)y$ for all $y \in \mathbb{R}^N$ with $a_0(t) > 0$ for all $t > 0$ and

(i) $a_0 \in C^1(0,\infty)$, $t \longmapsto a_0(t)t$ is strictly increasing on $(0,\infty)$ and

$$\lim_{t\to 0^+} a_0(t)t = 0 \quad \text{and} \quad \lim_{t\to 0^+} \frac{a_0'(t)t}{a_0(t)} = c > -1;$$

(ii) there exists $c_3 > 0$ such that

$$\bigl|\nabla a(y)\bigr| \leq c_3 \frac{\vartheta(|y|)}{|y|} \quad \forall y \in \mathbb{R}^N \setminus \{0\};$$

(iii) we have

$$\bigl(\nabla a(y)\xi, \xi\bigr)_{\mathbb{R}^N} \geq \frac{\vartheta(|y|)}{|y|}|\xi|^2 \quad \forall y \in \mathbb{R}^N \setminus \{0\}, \xi \in \mathbb{R}^N;$$

(iv) if $G_0(t) = \int_0^t a_0(s)s\,ds$, then there exists $q \in (1,p)$ such that

$$t \longmapsto G_0\bigl(t^{\frac{1}{q}}\bigr) \quad \text{is convex on } \mathbb{R}_+ = [0,+\infty),$$

$$\lim_{t\to 0^+} \frac{qG_0(t)}{t^q} = c^* > 0$$

and

$$0 \leq pG_0(t) - a_0(t)t^2 \quad \forall t > 0.$$

Remark 2.1 Hypotheses $H(a)$(i), (ii) and (iii) are dictated by the nonlinear regularity theory of Lieberman [3] and the nonlinear strong maximum principle of Pucci and Serrin [14]. Hypothesis $H(a)$(iv) serves the needs of our problem. The examples given below show that hypothesis $H(f)$(iv) is mild and it is satisfied in all cases of interest. Note that hypotheses $H(a)$ imply that G_0 is strictly increasing and strictly convex. We set

$$G(y) = G_0\bigl(|y|\bigr) \quad \forall y \in \mathbb{R}^N.$$

We have

$$\nabla G(y) = G_0'\bigl(|y|\bigr)\frac{y}{|y|} = a_0\bigl(|y|\bigr)y = a(y) \quad \forall y \in \mathbb{R}^N \setminus \{0\}.$$

So, $G(\cdot)$ is the primitive of $a(\cdot)$ and $y \longmapsto G(y)$ is convex with $G(0) = 0$. Hence

$$G(y) \le \big(a(y), y\big)_{\mathbb{R}^N} \quad \forall y \in \mathbb{R}^N. \tag{2.2}$$

Such hypotheses were also used in the works of Gasiński et al. [15] and Papageorgiou and Rădulescu [16–18].

The next lemma is an easy consequence of hypotheses $H(a)$ which summarizes the basic properties of the map a.

Lemma 2.2 *If hypotheses* $H(a)$(i), (ii) *and* (iii) *hold, then*

(a) $y \longmapsto a(y)$ *is continuous and strictly monotone* (*hence maximal monotone, too*);
(b) $|a(y)| \le c_4(1 + |y|^{p-1})$ *for all* $y \in \mathbb{R}^N$, *for some* $c_4 > 0$;
(c) $(a(y), y)_{\mathbb{R}^N} \ge \frac{c_1}{p-1}|y|^p$ *for all* $y \in \mathbb{R}^N$.

Using this lemma together with (2.1) and (2.2), we have the following bilateral growth restrictions on the primitive G.

Corollary 2.3 *If hypotheses* $H(a)$(i), (ii) *and* (iii) *hold, then*

$$\frac{c_1}{p(p-1)}|y|^p \le G(y) \le c_5\big(1 + |y|^p\big) \quad \forall y \in \mathbb{R}^N$$

for some $c_5 > 0$.

Example 2.4 The following maps a satisfy hypotheses $H(a)$ (see Papageorgiou and Rădulescu [16]).

(a) $a(y) = |y|^{p-2}y$ with $1 < p < \infty$;
The map corresponds to the p-Laplace differential operator

$$\Delta_p u = \operatorname{div}\big(|Du|^{p-2}Du\big) \quad \forall u \in W^{1,p}(\Omega).$$

(b) $a(y) = |y|^{p-2}y + |y|^{q-2}y$ with $1 < q < p < \infty$.
This map corresponds to the (p,q)-Laplace differential operator

$$\Delta_p u + \Delta_q u \quad \forall u \in W^{1,p}(\Omega).$$

Such operators arise in problems of mathematical physics (see Cherfils and Il'yasov [19]).

(c) $a(y) = (1 + |y|^2)^{\frac{p-2}{2}} y$ with $1 < p < \infty$.
This operator corresponds to the generalized p-mean curvature differential operator

$$\operatorname{div}\Big(\big(1 + |Du|^2\big)^{\frac{p-2}{2}} Du\Big) \quad \forall u \in W^{1,p}(\Omega).$$

(d) $a(y) = |y|^{p-2}y(1 + \frac{1}{1+|y|^2})$ with $1 < p < \infty$.

In what follows, by $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the dual pair $(W^{1,p}(\Omega)^*, W^{1,p}(\Omega))$. Let $A\colon W^{1,p}(\Omega) \longrightarrow W^{1,p}(\Omega)^*$ be the nonlinear map defined by

$$\langle A(u),h\rangle = \int_\Omega \bigl(a(Du),Dh\bigr)_{\mathbb{R}^N}\,dz \quad \forall u,h \in W^{1,p}(\Omega).$$

The next proposition is a special case of a more general result of Gasiński and Papageorgiou [20].

Proposition 2.5 *If hypotheses* $H(a)$(i), (ii) *and* (iii) *hold, then the map* $A\colon W^{1,p}(\Omega) \longrightarrow W^{1,p}(\Omega)^*$ *is bounded* (*that is, maps bounded sets to bounded sets*), *continuous, monotone* (*hence maximal monotone, too*) *and of type* $(S)_+$, *that is,*

"*if* $u_n \xrightarrow{w} u$ *in* $W^{1,p}(\Omega)$ *and* $\limsup_{n\to+\infty}\langle A(u_n),u_n-u\rangle \le 0$, *then* $u_n \longrightarrow u$ *in* $W^{1,p}(\Omega)$."

The hypotheses on the boundary coefficient β are the following:
$H(\beta)$: $\beta \in C^{0,\alpha}(\partial\Omega)$ with $\alpha \in (0,1)$ and $\beta(z) \ge 0$ for all $z \in \partial\Omega$.

Remark 2.6 When $\beta \equiv 0$, we recover the Neumann problem.

Let $\vartheta_q\colon W^{1,q}(\Omega) \longrightarrow \mathbb{R}$ be the C^1-functional defined by

$$\vartheta_q(u) = \|Du\|_q^q + \int_{\partial\Omega} \beta(z)|u|^q\,d\sigma \quad \forall u \in W^{1,q}(\Omega).$$

Also, we consider the following nonlinear eigenvalue problem:

$$\begin{cases} -\Delta_q u(z) = \widehat{\lambda}|u(z)|^{q-2}u(z) & \text{in } \Omega,\\ \frac{\partial u}{\partial n_q} + \beta(z)|u|^{q-2}u = 0 & \text{on } \partial\Omega. \end{cases}$$

Here $1 < q < +\infty$ is as in hypothesis $H(a)$(iv) and $\frac{\partial u}{\partial n_q} = |Du|^{q-2}(Du,n)_{\mathbb{R}^N}$. If the above Robin problem admits a nontrivial solution, then we say that $\widehat{\lambda}$ is an eigenvalue of $-\Delta_q$ with Robin boundary condition and the nontrivial solution $\widehat{u}$ is an eigenfunction corresponding to $\widehat{\lambda}$. From Papageorgiou and Rădulescu [17], we know that $\widehat{u} \in L^\infty(\Omega)$, and then from Theorem 2 of Lieberman [2] (see also Lieberman [3]) we have that $\widehat{u} \in C^1(\overline{\Omega})$.

From Papageorgiou and Rădulescu [21], we know that there exists a smallest eigenvalue $\widehat{\lambda}_1(q)$ such that:

- $\widehat{\lambda}_1(q) \ge 0$ and it is isolated in the spectrum $\widehat{\sigma}(q)$ (that is, we can find $\varepsilon > 0$ such that $(\widehat{\lambda}_1(q), \widehat{\lambda}_1(q)+\varepsilon) \cap \widehat{\sigma}(q) = \emptyset$) and if $\beta \equiv 0$ (Neumann problem), then $\widehat{\lambda}_1(q) = 0$, while if $\beta \not\equiv 0$, then $\widehat{\lambda}_1(q) > 0$.
- $\widehat{\lambda}_1(q)$ is simple (that is, if $\widehat{u},\widehat{v}$ are eigenfunctions corresponding to $\widehat{\lambda}_1(q)$, then $\widehat{u} = \xi\widehat{v}$ for some $\xi \in \mathbb{R}\setminus\{0\}$).
- we have

$$\widehat{\lambda}_1(q) = \inf\left\{\frac{\vartheta_q(u)}{\|u\|_q^q} : u \in W^{1,q}(\Omega), u \ne 0\right\}. \tag{2.3}$$

The infimum in (2.3) is realized on the one-dimensional eigenspace corresponding to $\widehat{\lambda}_1(q)$. It follows that the elements of this eigenspace have constant sign. By $\widehat{u}_1(q)$ we denote the L^q-normalized (that is, $\|\widehat{u}_1(q)\|_q = 1$) positive eigenfunction corresponding to $\widehat{\lambda}_1(q)$. We have $\widehat{u}_1(q) \in C_+$ and, using the nonlinear strong maximum principle (see, e.g., Gasiński and Papageorgiou [1, p. 738]), we have $\widehat{u}_1(q) \in D_+$. An eigenfunction $\widehat{u}$ corresponding to an eigenvalue $\widehat{\lambda} \neq \widehat{\lambda}_1(q)$ is necessarily nodal. Sometimes, in order to emphasize the dependence on β, we write $\widehat{\lambda}_1(q,\beta) \geq 0$.

Recall that a function $f\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$ is *Carathéodory*, if

- for all $(x,y) \in \mathbb{R} \times \mathbb{R}^N$, $z \longmapsto f(z,x,y)$ is measurable;
- for a.a. $z \in \Omega$, $(x,y) \longmapsto f(z,x,y)$ is continuous.

Such a function is automatically jointly measurable (see Hu and Papageorgiou [22, p. 142]).

The hypotheses on the convection term f in problem (1.1) are the following:

$H(f)$: $f\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$ is a Carathéodory function such that $f(z,0,y) = 0$ for a.a. $z \in \Omega$, all $y \in \mathbb{R}^N$ and

(i) there exists $\eta > 0$ such that

$$f(z,\eta,y) = 0 \quad \text{for a.a. } z \in \Omega, \text{ all } y \in \mathbb{R}^N,$$
$$f(z,x,y) \geq 0 \quad \text{for a.a. } z \in \Omega, \text{ all } 0 \leq x \leq \eta, \text{ all } y \in \mathbb{R}^N,$$
$$f(z,x,y) \leq \widetilde{c}_1 + \widetilde{c}_2|y|^p \quad \text{for a.a. } z \in \Omega, \text{ all } 0 \leq x \leq \eta, \text{ all } y \in \mathbb{R}^N,$$

with $\widetilde{c}_1 > 0, \widetilde{c}_2 < \frac{c_1}{p-1}$;

(ii) for every $M > 0$, there exists $\eta_M \in L^\infty(\Omega)$ such that

$$\eta_M(z) \geq c^*\widehat{\lambda}_1(q) \quad \text{for a.a. } z \in \Omega, \eta_M \not\equiv c^*\widehat{\lambda}_1(q),$$
$$\liminf_{x\to 0^+} \frac{f(z,x,y)}{x^{q-1}} \geq \eta_M(z) \quad \text{uniformly for a.a. } z \in \Omega, \text{ all } |y| \leq M$$

(here $q \in (1,p)$ is as in hypothesis $H(a)$(iv));

(iii) there exists $\xi_\eta > 0$ such that, for a.a. $z \in \Omega$, all $y \in \mathbb{R}^N$, the function

$$x \longmapsto f(z,x,y) + \xi_\eta x^{p-1}$$

is nondecreasing on $[0,\eta]$, for a.a. $z \in \Omega$, all $y \in \mathbb{R}^N$ and

$$\lambda^{p-1} f\left(z, \frac{1}{\lambda}x, y\right) \leq f(z,x,y) \tag{2.4}$$

and

$$f(z,x,y) \leq \lambda^p f\left(z, x, \frac{1}{\lambda}y\right)$$

for a.a. $z \in \Omega$, all $0 \leq x \leq \eta$, all $y \in \mathbb{R}^N$ and all $\lambda \in (0,1)$.

Remark 2.7 Since we look for positive solutions and all the above hypotheses are for $x \geq 0$, without any loss of generality, we assume that

$$f(z,x,y) = 0 \quad \text{for a.a. } z \in \Omega, \text{ all } x \leq 0, \text{ all } y \in \mathbb{R}^N.$$

Note that (2.4) is satisfied if, for example, for a.a. $z \in \Omega$, all $y \in \mathbb{R}^N$, the function $x \longmapsto \frac{f(z,x,y)}{x^{p-1}}$ is nonincreasing on $(0,+\infty)$.

Example 2.8 The following function satisfies hypotheses $H(f)$. For the sake of simplicity, we drop the z-dependence:

$$f(x,y) = \begin{cases} \eta(x^{p-1} - x^{r-1}) + c(x^{p-1} - x^{\mu-1})|y|^p & \text{if } 0 \leq x \leq 1, \\ (x^{\tau-1} \ln x)|y|^p & \text{if } 1 < x, \end{cases}$$

with $\eta > c^*\widehat{\lambda}_1(q) \geq 0$, $p < \min\{r,\mu\}$, $c < \frac{c_1}{2(p-1)}$, $1 < \tau < \infty$.

As we have already mentioned, our approach is topological based on the Leray–Schauder alternative principle, which we recall here (see, e.g., Gasiński and Papageorgiou [1, p. 827]).

Theorem 2.9 *If X is a Banach space, $C \subseteq X$ is nonempty convex and $\vartheta : C \longrightarrow C$ is a compact map, then exactly one of the following two statements is true*:

(a) *ϑ has a fixed point*;
(b) *the set $S(\vartheta) = \{u \in C : u = \lambda\vartheta(u), \lambda \in (0,1)\}$ is unbounded.*

Finally, let us fix our notation. For $x \in \mathbb{R}$, we set $x^{\pm} = \max\{\pm x, 0\}$. Then, given $u \in W^{1,p}(\Omega)$, we define $u^{\pm}(\cdot) = u(\cdot)^{\pm}$. We know that

$$u^{\pm} \in W^{1,p}(\Omega), \qquad u = u^+ - u^-, \qquad |u| = u^+ + u^-.$$

Also, if $u \in W^{1,p}(\Omega)$, then

$$[0,u] = \{h \in W^{1,p}(\Omega) : 0 \leq h(z) \leq u(z) \text{ for a.a. } z \in \Omega\}.$$

3 Positive solutions

Consider the following truncation-perturbation of the convection term $f(z,\cdot,y)$:

$$\widehat{f}(z,x,y) = \begin{cases} f(z,x,y) + \xi_\eta (x^+)^{p-1} & \text{if } x \leq \eta, \\ f(z,\eta,y) + \xi_\eta \eta^{p-1} & \text{if } \eta < x. \end{cases} \tag{3.1}$$

Evidently $\widehat{f}$ is a Carathéodoty function.

Given $v \in C^1(\overline{\Omega})$, we consider the following auxiliary Robin problem:

$$\begin{cases} -\operatorname{div} a(Du(z)) + \xi_\eta u(z)^{p-1} = \widehat{f}(z,u(z),Dv(z)) & \text{in } \Omega, \\ \frac{\partial u}{\partial n_a} + \beta(z)u(z)^{p-1} = 0 & \text{on } \partial\Omega, u \geq 0. \end{cases} \tag{3.2}$$

Proposition 3.1 *If hypotheses* $H(a)$, $H(\beta)$ *and* $H(f)$ *hold, then problem* (3.2) *admits a positive solution* $u_\nu \in [0,\eta] \cap D_+$.

Proof Let

$$\widehat{F}_\nu(z,x) = \int_0^x \widehat{f}\bigl(z,s,D\nu(z)\bigr)\,ds$$

and consider the C^1-functional $\widehat{\varphi}_\nu\colon W^{1,p}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\widehat{\varphi}_\nu(u) = \int_\Omega G(Du)\,dz + \frac{\xi_\eta}{p}\|u\|_p^p + \frac{1}{p}\int_{\partial\Omega}\beta(z)|u|^p\,d\sigma - \int_\Omega \widehat{F}_\nu(z,u)\,dz$$

for all $u \in W^{1,p}(\Omega)$. From (3.1), Corollary 2.3 and hypothesis $H(\beta)$, we see that $\widehat{\varphi}_\nu$ is coercive. Also, using the Sobolev embedding theorem, the compactness of the trace map and the convexity of G, we see that $\widehat{\varphi}_\nu$ is sequentially weakly lower semicontinuous. So, by the Weierstrass–Tonelli theorem, we can find $u_\nu \in W^{1,p}(\Omega)$ such that

$$\widehat{\varphi}_\nu(u_\nu) = \inf_{u\in W^{1,p}(\Omega)} \widehat{\varphi}_\nu(u). \tag{3.3}$$

Let $M > \|\nu\|_{C^1(\overline{\Omega})}$. Hypothesis $H(f)$(ii) implies that given $\varepsilon > 0$, we can find $\delta \in (0,\eta]$ such that

$$f(z,x,y) \geq \bigl(\eta_M(z) - \varepsilon\bigr)x^{q-1} \quad \text{for a.a. } z \in \Omega, \text{ all } 0 \leq x \leq \delta, \text{ all } |y| \leq M,$$

so

$$\widehat{f}\bigl(z,x,D\nu(z)\bigr) \geq \bigl(\eta_M(z) - \varepsilon\bigr)x^{q-1} + \xi_\eta x^{p-1} \quad \text{for a.a. } z \in \Omega, \text{ all } 0 \leq x \leq \delta$$

(see (3.1)) and thus

$$\widehat{F}_\nu(z,x) \geq \frac{1}{q}\bigl(\eta_M(z) - \varepsilon\bigr)x^q + \frac{\xi_\eta}{p}x^p \quad \text{for a.a. } z \in \Omega, \text{ all } 0 \leq x \leq \delta. \tag{3.4}$$

Hypothesis $H(a)$(iv) implies that

$$G(y) \leq \frac{c^* + \varepsilon}{q}|y|^q \quad \text{for all } |y| \leq \delta. \tag{3.5}$$

Since $\widehat{u}_1(q) \in D_+$, we can find $t \in (0,1)$ small such that

$$t\widehat{u}_1(q)(z) \in (0,\delta], \qquad t\bigl|D\widehat{u}_1(q)(z)\bigr| \leq \delta \quad \forall z \in \overline{\Omega}. \tag{3.6}$$

Then we have

$$\begin{aligned}\widehat{\varphi}_\nu\bigl(t\widehat{u}_1(q)\bigr) &\leq \frac{c^* + \varepsilon}{q}t^q\widehat{\lambda}_1(q) - \frac{t^q}{q}\int_\Omega \bigl(\eta_M(z) - \varepsilon\bigr)\widehat{u}_1(q)^q\,dz \\ &\leq \frac{t^q}{q}\biggl(\int_\Omega \bigl(c^*\widehat{\lambda}_1(q) - \eta_M(z)\bigr)\widehat{u}_1(q)^q\,dz + \varepsilon\widehat{\lambda}_1(q)\biggr)\end{aligned} \tag{3.7}$$

(recall that $\|\widehat{u}_1(q)\|_q = 1$). Using hypothesis $H(f)$(ii) and the fact that $\widehat{u}_1(q) \in D_+$, we have

$$r_0 = \int_\Omega \big(\eta_M(z) - c^*\widehat{\lambda}_1(q)\big)\widehat{u}_1(q)^q\,dz > 0.$$

Then from (3.7) we have

$$\widehat{\varphi}_\nu\big(t\widehat{u}_1(q)\big) \le \frac{t^q}{q}\big(-r_0 + \varepsilon\widehat{\lambda}_1(q)\big).$$

Choosing $\varepsilon \in (0, \frac{r_0}{\widehat{\lambda}_1(q)})$, we see that

$$\widehat{\varphi}_\nu\big(t\widehat{u}_1(q)\big) < 0,$$

so

$$\widehat{\varphi}_\nu(u_\nu) < 0 = \widehat{\varphi}_\nu(0),$$

thus

$$u_\nu \neq 0.$$

From (3.3) we have

$$\widehat{\varphi}_\nu'(u_\nu) = 0,$$

so

$$\begin{aligned}&\big\langle A(u_\nu), h\big\rangle + \xi_\eta \int_\Omega |u_\nu|^{p-2}u_\nu h\,dz + \int_{\partial\Omega} \beta(z)|u_\nu|^{p-2}u_\nu h\,d\sigma \\ &\quad = \int_\Omega \widehat{f}(z, u_\nu, D\nu)h\,dz \quad \forall h \in W^{1,p}(\Omega).\end{aligned} \tag{3.8}$$

In (3.8) we choose $h = -u_\nu^- \in W^{1,p}(\Omega)$. Using Lemma 2.2 and (3.1), we have

$$\frac{c_1}{p-1}\big\|Du_\nu^-\big\|_p^p + \xi_\eta\big\|u_\nu^-\big\|_p^p \le 0,$$

so

$$u_\nu \ge 0, \qquad u_\nu \neq 0.$$

Next in (3.8) we choose $h = (u_\nu - \eta)^+ \in W^{1,p}(\Omega)$. Then

$$\begin{aligned}&\big\langle A(u_\nu), (u_\nu - \eta)^+\big\rangle + \xi_\eta \int_\Omega u_\nu^{p-1}(u_\nu - \eta)^+\,dz + \int_{\partial\Omega} \beta(z)u_\nu^{p-1}(u_\nu - \eta)^+\,d\sigma \\ &\quad = \int_\Omega \big(f(z, \eta, D\nu) + \xi_\eta\eta^{p-1}\big)(u_\nu - \eta)^+\,dz + \int_\Omega \xi_\eta\eta^{p-1}(u_\nu - \eta)^+\,dz\end{aligned}$$

(see (3.1) and hypothesis $H(f)$(i)), so

$$\langle A(u_\nu) - A(\eta), (u_\nu - \eta)^+ \rangle + \xi_\eta \int_\Omega \big(u_\nu^{p-1} - \eta^{p-1}\big)(u_\nu - \eta)^+ \, dz \le 0$$

(see hypothesis $H(\beta)$ and note that $A(\eta) = 0$), thus

$$u_\nu \le \eta.$$

So, we have proved that

$$u_\nu \in [0, \eta]. \tag{3.9}$$

Then, from (3.1), (3.8) and (3.9), we have

$$\langle A(u_\nu), h \rangle + \int_{\partial\Omega} \beta(z) u_\nu^{p-1} h \, d\sigma = \int_\Omega f(z, u_\nu, Dv) h \, dz \quad \forall h \in W^{1,p}(\Omega),$$

so

$$\begin{cases} -\operatorname{div} a(Du_\nu(z)) = f(z, u_\nu(z), Dv(z)) & \text{for a.a. } z \in \Omega, \\ \frac{\partial u_\nu}{\partial n_a} + \beta(z) u_\nu(z)^{p-1} = 0 & \text{on } \partial\Omega \end{cases} \tag{3.10}$$

(see Papageorgiou and Rădulescu [21]). From (3.10) and Papageorgiou and Rădulescu [17], we have

$$u_\nu \in L^\infty(\Omega).$$

Then from Lieberman [3] (see also Fukagai and Narukawa [23]), we have

$$u_\nu \in C_+ \setminus \{0\}.$$

Hypothesis $H(f)$(iii) implies that

$$f(z, x, y) + \xi_\eta x^{p-1} \ge 0 \quad \text{for a.a. } z \in \Omega, \text{ all } 0 \le x \le \eta, \text{ all } y \in \mathbb{R}^N.$$

Then from (3.10) we have

$$\operatorname{div} a\big(Du_\nu(z)\big) \le \xi_\eta u_\nu(z)^{p-1} \quad \text{for a.a. } z \in \Omega. \tag{3.11}$$

From (3.11), the strong maximum principle (see Pucci and Serrin [14, p. 111]) and the boundary point lemma (see Pucci and Serrin [14, p. 120]), we have $u_\nu \in D_+$. □

Next we show that problem (3.2) has a smallest positive solution in the order interval $[0, \eta]$. So, let

$$S_\nu = \big\{u \in W^{1,p}(\Omega) : u \ne 0, u \in [0, \eta] \text{ is a solution of (3.2)}\big\}.$$

From Proposition 3.1 we know that

$$\emptyset \neq S_v \subseteq [0,\eta] \cap D_+.$$

Given $\varepsilon > 0$ and $r \in (p, p^*)$, where

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{if } N \leq p \end{cases}$$

(the critical Sobolev exponent corresponding to p), hypotheses $H(f)$(i) and (ii) imply that we can find $c_6 = c_6(\varepsilon, r, M) > 0$ (recall that $M > \|v\|_{C^1(\overline{\Omega})}$) such that

$$f\bigl(z, x, Dv(z)\bigr) \geq \bigl(\eta_M(z) - \varepsilon\bigr)x^{q-1} - c_6 x^{r-1} \quad \text{for a.a. } z \in \Omega, \text{ all } 0 \leq x \leq \eta. \tag{3.12}$$

This unilateral growth restriction on $f(z, \cdot, Dv(z))$ leads to the following auxiliary Robin problem:

$$\begin{cases} -\operatorname{div} a(Du(z)) = (\eta_M(z) - \varepsilon)u(z)^{q-1} - c_6 u(z)^{r-1} & \text{in } \Omega, \\ \frac{\partial u}{\partial n_a} + \beta(z)u(z)^{p-1} = 0 & \text{on } \partial\Omega, u \geq 0. \end{cases} \tag{3.13}$$

Proposition 3.2 *If hypotheses $H(a)$ and $H(\beta)$ hold, then for all $\varepsilon > 0$ small problem* (3.13) *admits a unique positive solution $u^* \in D_+$.*

Proof First we show the existence of a positive solution for problem (3.13). To this end, let $\psi : W^{1,p}(\Omega) \longrightarrow \mathbb{R}$ be the C^1-functional defined by

$$\begin{aligned} \psi(u) = {} & \int_\Omega G(Du)\,dz + \frac{1}{p}\bigl\|u^-\bigr\|_p^p + \frac{1}{p}\int_{\partial\Omega} \beta(z)|u|^p\,d\sigma \\ & - \frac{1}{q}\int_\Omega \bigl(\eta_M(z) - \varepsilon\bigr)\bigl(u^+\bigr)^q\,dz + \frac{c_6}{r}\bigl\|u^+\bigr\|_p^p \quad \forall u \in W^{1,p}(\Omega). \end{aligned}$$

Using Corollary 2.3, we obtain

$$\begin{aligned} \psi(u) \geq {} & \frac{c_1}{p(p-1)}\bigl\|Du^+\bigr\|_p^p + \frac{c_6}{r}\bigl\|u^+\bigr\|_r^r + \frac{c_1}{p(p-1)}\bigl\|Du^-\bigr\|_p^p + \frac{1}{p}\bigl\|u^-\bigr\|_p^p \\ & - \frac{1}{q}\int_\Omega \bigl(\eta_M(z) - \varepsilon\bigr)\bigl(u^+\bigr)^q\,dz, \end{aligned}$$

so

$$\psi(u) \geq c_7\|u\|^p - c_8\bigl(\|u\|^q + 1\bigr)$$

for some $c_7, c_8 > 0$. Since $q < p$, it follows that ψ is coercive. Also, from the Sobolev embedding theorem, the compactness of the trace map and the convexity of G, we have that ψ is sequentially weakly lower semicontinuous. Invoking the Weierstrass–Tonelli theorem, we can find $u^* \in W^{1,p}(\Omega)$ such that

$$\psi\bigl(u^*\bigr) = \inf_{u \in W^{1,p}(\Omega)} \psi(u). \tag{3.14}$$

As in the proof of Proposition 3.1, using the condition on η_M (see hypothesis $H(f)$(ii)), we show that, for $t \in (0,1)$ and $\varepsilon > 0$ small, we have

$$\psi\big(t\widehat{u}_1(q)\big) < 0,$$

so

$$\psi\big(u^*\big) < 0 = \psi(0)$$

(see (3.14)), thus

$$u^* \neq 0.$$

From (3.14) we have

$$\psi'\big(u^*\big) = 0,$$

so, for all $h \in W^{1,p}(\Omega)$, we have

$$\begin{aligned}
&\big\langle A\big(u^*\big), h\big\rangle - \int_\Omega \big(\big(u^*\big)^-\big)^{p-1} h\,dz + \int_{\partial\Omega} \beta(z)\big|u^*\big|^{p-2} u^* h\,d\sigma \\
&\quad = \int_\Omega \big(\eta_M(z) - \varepsilon\big)\big(\big(u^*\big)^+\big)^{q-1} h\,dz - c_6 \int_\Omega \big(\big(u^*\big)^+\big)^{r-1} h\,dz.
\end{aligned} \tag{3.15}$$

In (3.15) we choose $h = -(u^*)^- \in W^{1,p}(\Omega)$. Then

$$\frac{c_1}{p-1}\big\|D\big(u^*\big)^-\big\|_p^p + \big\|\big(u^*\big)^-\big\|_p^p \leq 0$$

(see Lemma 2.2 and hypothesis $H(\beta)$), so

$$u^* \geq 0, \qquad u^* \neq 0.$$

Hence (3.15) becomes

$$\big\langle A\big(u^*\big), h\big\rangle + \int_{\partial\Omega} \beta(z)\big(u^*\big)^{p-1} h\,d\sigma = \int_\Omega \big(\eta_M(z) - \varepsilon\big)\big(u^*\big)^{q-1} h\,dz - c_6 \int_\Omega \big(u^*\big)^{r-1} h\,dz$$

for all $h \in W^{1,p}(\Omega)$, thus

$$\begin{cases} -\operatorname{div} a(Du^*(z)) = (\eta_M - \varepsilon)(u^*)(z)^{q-1} - c_6(u^*)(z)^{r-1} & \text{for a.a. } z \in \Omega, \\ \frac{\partial u^*}{\partial n_a} + \beta(z)(u^*)^{p-1} = 0 & \text{on } \partial\Omega, u \geq 0 \end{cases} \tag{3.16}$$

(see Papageorgiou and Rădulescu [21]). As before, via the nonlinear regularity theory, we have

$$u^* \in C_+ \setminus \{0\}.$$

From (3.16) we have

$$\operatorname{div} a\big(Du^*(z)\big) \le c_6 \|u^*\|_\infty^{r-p} u^*(z)^{p-1} \quad \text{for a.a. } z \in \Omega$$

(recall $r > p$), so

$$u^* \in D_+$$

(see Pucci and Serrin [14, pp. 111, 120]).

Next we show that this positive solution is unique. For this purpose, we introduce the integral functional $j\colon L^1(\Omega) \longrightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ defined by

$$j(u) = \begin{cases} \int_\Omega G(Du^{\frac{1}{q}})\,dz + \frac{1}{p}\int_{\partial\Omega} \beta(z) u^{\frac{p}{q}}\,d\sigma & \text{if } u \ge 0, u^{\frac{1}{q}} \in W^{1,p}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Let $\operatorname{dom} j = \{u \in L^1(\Omega) : j(u) < +\infty\}$ (the effective domain of the functional j) and consider $u_1, u_2 \in \operatorname{dom} j$. We set $u = (1-t)u_1 + tu_2$ with $t \in [0,1]$. Using Lemma 1 of Díaz and Saá [24], we have

$$\big|Du(z)^{\frac{1}{q}}\big| \le \big((1-t)\big|Du_1(z)^{\frac{1}{q}}\big|^q + t\big|Du_2(z)^{\frac{1}{q}}\big|^q\big)^{\frac{1}{q}} \quad \text{for a.a. } z \in \Omega.$$

Recalling that G_0 is increasing, we have

$$\begin{aligned} G_0\big(\big|Du(z)^{\frac{1}{q}}\big|\big) &\le G_0\big(\big((1-t)\big|Du_1(z)^{\frac{1}{q}}\big|^q + t\big|Du_2(z)^{\frac{1}{q}}\big|^q\big)^{\frac{1}{q}}\big) \\ &\le (1-t)G_0\big(\big|Du_1(z)^{\frac{1}{q}}\big|\big) + tG_0\big(\big|Du_2(z)^{\frac{1}{q}}\big|\big) \end{aligned}$$

(see hypothesis $H(a)$(iv)), so

$$G\big(Du(z)^{\frac{1}{q}}\big) \le (1-t)G\big(Du_1(z)^{\frac{1}{q}}\big) + tG\big(Du_2(z)^{\frac{1}{q}}\big) \quad \text{for a.a. } z \in \Omega,$$

thus the map $\operatorname{dom} j \ni u \longmapsto \int_\Omega G(Du^{\frac{1}{q}})\,dz$ is convex.

Since $q < p$ and $\beta \ge 0$, it follows that the map $\operatorname{dom} j \ni u \longmapsto \frac{1}{p}\int_{\partial\Omega} \beta(z) u^{\frac{p}{q}}\,d\sigma$ is convex.

Therefore the integral functional j is convex.

Suppose that $\widetilde{u}^*$ is another positive solution of (3.13). As we did for u^*, we can show that

$$\widetilde{u}^* \in D_+.$$

Hence, given $h \in C^1(\overline{\Omega})$ for $|t|$ small, we have

$$u^* + th \in \operatorname{dom} j \quad \text{and} \quad \widetilde{u}^* + th \in \operatorname{dom} j.$$

Using the convexity of j, we can easily see that j is Gâteaux differentiable at u^* and at $\widetilde{u}^*$ in the direction h. Using the chain rule and the nonlinear Green's identity (see Gasiński and Papageorgiou [1, p. 210]), we have

$$j'\big(u^*\big)(h) = \frac{1}{q}\int_\Omega \frac{-\operatorname{div} a(Du^*)}{(u^*)^{q-1}} h\,dz \quad \forall h \in C^1(\overline{\Omega})$$

and

$$j'(\widetilde{u}^*)(h) = \frac{1}{q}\int_\Omega \frac{-\operatorname{div} a(D\widetilde{u}^*)}{(\widetilde{u}^*)^{q-1}} h\,dz \quad \forall h \in C^1(\overline{\Omega}).$$

The convexity of j implies the monotonicity of j'. Therefore

$$\begin{aligned} 0 &\le \frac{1}{q}\int_\Omega \left(\frac{-\operatorname{div}(Du^*)}{(u^*)^{q-1}} - \frac{-\operatorname{div} a(D\widetilde{u}^*)}{(\widetilde{u}^*)^{q-1}} \right)\left((u^*)^q - (\widetilde{u}^*)^q\right) dz \\ &= \frac{c_6}{q}\int_\Omega \left((\widetilde{u}^*)^{r-q} - (u^*)^{r-q}\right)\left((u^*)^q - (\widetilde{u}^*)^q\right) dz \end{aligned}$$

(see (3.13)), so

$$u^* = \widetilde{u}^*$$

(since $q < p < r$). This proves the uniqueness of the positive solution $u^* \in D_+$. □

Proposition 3.3 *If hypotheses $H(a)$, $H(\beta)$, $H(f)$ hold and $u \in S_\nu$, then $u^* \le u$.*

Proof We consider the Carathéodory function $e\colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$e(z,x) = \begin{cases} (\eta_M(z) - \varepsilon)(x^+)^{q-1} - c_6(x^+)^{r-1} + \xi_\eta(x^+)^{p-1} & \text{if } x \le u(z), \\ (\eta_M(z) - \varepsilon)u(z)^{q-1} - c_6 u(z)^{r-1} + \xi_\eta u(z)^{p-1} & \text{if } u(z) < x. \end{cases} \tag{3.17}$$

We set

$$E(z,x) = \int_0^x e(z,s)\,ds$$

and consider the C^1-functional $\tau\colon W^{1,p}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\tau(u) = \int_\Omega G(Du)\,dz + \frac{\xi_\eta}{p}\|u\|_p^p + \frac{1}{p}\int_{\partial\Omega} \beta(z)|u|^p\,d\sigma - \int_\Omega E(z,u)\,dz \quad \forall u \in W^{1,p}(\Omega).$$

From (3.17) it is clear that τ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\widetilde{u}^* \in W^{1,p}(\Omega)$ such that

$$\tau(\widetilde{u}^*) = \inf_{h \in W^{1,p}(\Omega)} \tau(h). \tag{3.18}$$

As before, since $q < p < r$, we have

$$\tau(\widetilde{u}^*) < 0 = \tau(0),$$

so

$$\widetilde{u}^* \ne 0.$$

From (3.18) we have

$$\tau'(\widetilde{u}^*) = 0,$$

so

$$\langle A(\widetilde{u}^*), h\rangle + \xi_\eta \int_\Omega |\widetilde{u}^*|^{p-2}\widetilde{u}^* h\,dz + \int_{\partial\Omega} \beta(z)|\widetilde{u}^*|^{p-2}\widetilde{u}^* h\,d\sigma$$
$$= \int_\Omega e(z,\widetilde{u}^*)h\,dz \quad \forall h \in W^{1,p}(\Omega). \tag{3.19}$$

In (3.19) first we choose $h = -(\widetilde{u}^*)^- \in W^{1,p}(\Omega)$. Then

$$\frac{c_1}{p-1}\|D(\widetilde{u}^*)^-\|_p^p + \xi_\eta \|(\widetilde{u}^*)^-\|_p^p + \int_{\partial\Omega} \beta(z)((\widetilde{u}^*)^-)^p\,d\sigma = 0$$

(see (3.17)), so

$$\widetilde{u}^* \geq 0, \qquad \widetilde{u}^* \neq 0$$

(see hypothesis $H(\beta)$).

Next in (3.19) we choose $h = (\widetilde{u}^* - u)^+ \in W^{1,p}(\Omega)$. Then

$$\begin{aligned}
&\langle A(\widetilde{u}^*), (\widetilde{u}^* - u)^+\rangle + \xi_\eta \int_\Omega (\widetilde{u}^*)^{p-1}(\widetilde{u} - u)^+\,dz \\
&\quad + \int_{\partial\Omega} \beta(z)(\widetilde{u}^*)^{p-1}(\widetilde{u}^* - u)^+\,d\sigma \\
&= \int_\Omega ((\eta_M(z) - \varepsilon)u^{q-1} - c_6 u^{r-1})(\widetilde{u}^* - u)^+\,dz \\
&\leq \int_\Omega f(z,u,Dv)(\widetilde{u}^* - u)^+\,dz \\
&= \langle A(u), (\widetilde{u}^* - u)^+\rangle + \xi_\eta \int_\Omega u^{p-1}(\widetilde{u}^* - u)^+\,dz \\
&\quad + \int_{\partial\Omega} \beta(z)u^{p-1}(\widetilde{u}^* - u)^+\,d\sigma
\end{aligned}$$

(see (3.17), (3.12) and recall that $u \in S_v$), so

$$\langle A(\widetilde{u}^*) - A(u), (\widetilde{u}^* - u)^+\rangle + \xi_\eta \int_\Omega ((\widetilde{u}^*)^{p-1} - u^{p-1})(\widetilde{u}^* - u)\,dz \leq 0$$

(see hypothesis $H(\beta)$), thus

$$\widetilde{u}^* \leq u.$$

We have proved that

$$\widetilde{u}^* \in [0,u] \setminus \{0\}. \tag{3.20}$$

Then, from (3.17) and (3.20), equation (3.19) becomes

$$\begin{aligned}&\langle A(\widetilde{u}^*), h\rangle + \int_{\partial\Omega} \beta(z)(\widetilde{u}^*)^{p-1} h\,d\sigma \\ &\quad= \int_\Omega \bigl((\eta_M(z)-\varepsilon)(\widetilde{u}^*)^{q-1} - c_6(\widetilde{u}^*)^{r-1}\bigr)h\,dz \quad \forall h \in W^{1,p}(\Omega),\end{aligned}$$

so $\widetilde{u}^* = u^*$ (see Proposition 3.2), thus

$$u^* \le u. \qquad \square$$

Using this proposition, we can show that problem (3.2) admits a smallest positive solution $\widehat{u}_\nu \in D_+$ on $[0,\eta]$.

Proposition 3.4 *If hypotheses $H(a)$, $H(\beta)$, $H(f)$ hold, then problem* (3.2) *admits a smallest positive solution $\widehat{u}_\nu \in D_+$.*

Proof Invoking Lemma 3.10 of Hu and Papageorgiou [22, p. 178], we can find a decreasing sequence $\{u_n\}_{n\ge1} \subseteq S_\nu$ such that

$$\inf S_\nu = \inf_{n\ge1} u_n. \tag{3.21}$$

For all $n \ge 1$, we have

$$\langle A(u_n), h\rangle + \int_{\partial\Omega} \beta(z) u_n^{p-1} h\,d\sigma = \int_\Omega f(z,u_n,D\nu)h\,dz \quad \forall h \in W^{1,p}(\Omega), \tag{3.22}$$

so

$$u^* \le u_n \le \eta. \tag{3.23}$$

Then, on account of hypotheses $H(f)$(i), $H(\beta)$ and Lemma 2.2, we have that the sequence $\{u_n\}_{n\ge1} \subseteq W^{1,p}(\Omega)$ is bounded. Passing to a subsequence, we may assume that

$$u_n \xrightarrow{w} \widehat{u}_\nu \quad \text{in } W^{1,p}(\Omega) \quad \text{and} \quad u_n \longrightarrow \widehat{u}_\nu \quad \text{in } L^p(\Omega) \text{ and in } L^p(\partial\Omega). \tag{3.24}$$

In (3.22) we choose $h = u_n - \widehat{u}_\nu \in W^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (3.24). Then

$$\lim_{n\to+\infty} \langle A(u_n), u_n - \widehat{u}_\nu\rangle = 0,$$

so

$$u_n \longrightarrow \widehat{u}_\nu \quad \text{in } W^{1,p}(\Omega) \tag{3.25}$$

(see Proposition 2.5). If in (3.22) we pass to the limit as $n \to +\infty$ and use (3.25), then

$$\langle A(\widehat{u}_\nu), h\rangle + \int_{\partial\Omega} \beta(z)\widehat{u}_\nu^{p-1} h\,d\sigma = \int_\Omega f(z,\widehat{u}_\nu,D\nu)h\,dz \quad \forall h \in W^{1,p}(\Omega),$$

so $u^* \le \widehat{u}_\nu$ (see (3.23)).

From the above it follows that

$$\widehat{u}_\nu \in S_\nu \quad \text{and} \quad \widehat{u}_\nu = \inf S_\nu. \qquad \square$$

Let

$$C = \{u \in C^1(\overline{\Omega}) : 0 \leq u(z) \leq \eta \text{ for all } z \in \overline{\Omega}\},$$

and let $\vartheta : C \longrightarrow C$ be the map defined by

$$\vartheta(\nu) = \widehat{u}_\nu.$$

A fixed point of this map is clearly a positive solution of problem (1.1). We will produce a fixed point for ϑ using the Leray–Schauder alternative principle (see Theorem 2.9). To this end, we will need the following lemma.

Lemma 3.5 *If hypotheses $H(a), H(\beta), H(f)$ hold, $\{\nu_n\}_{n\geq 1} \subseteq C$, $\nu_n \to \nu$ in $C^1(\overline{\Omega})$ and $u \in S_\nu$, then we can find $u_n \in S_{\nu_n}$ for $n \geq 1$ such that $u_n \longrightarrow u$ in $C^1(\overline{\Omega})$.*

Proof Consider the following nonlinear Robin problem:

$$\begin{cases} -\operatorname{div} a(Dw(z)) + \xi_\eta |w(z)|^{p-2} w(z) = \widehat{f}(z, u(z), D\nu_n(z)) & \text{in } \Omega, \\ \frac{\partial w}{\partial n_a} + \beta(z)|w|^{p-2} w = 0 & \text{on } \partial\Omega, n \geq 1. \end{cases} \tag{3.26}$$

Since $u \in S_\nu \subseteq [0, \eta] \cap D_+$, we see that

$$\widehat{f}(\cdot, u(\cdot), D\nu_n(\cdot)) \not\equiv 0 \quad \forall n \geq 1$$

(see (3.1)) and

$$\widehat{f}(z, u(z), D\nu_n(z)) \geq 0 \quad \text{for a.a. } z \in \Omega, \text{ all } n \geq 1$$

(see hypothesis $H(f)$(i)). Therefore problem (3.26) has a unique nontrivial solution $u_n^0 \in D_+$. Also we have

$$\begin{aligned} &\langle A(u_n^0), (u_n^0 - \eta)^+ \rangle + \xi_\eta \int_\Omega (u_n^0)^{p-1} (u_n^0 - \eta)^+ \, dz \\ &\qquad + \int_{\partial\Omega} \beta(z) (u_n^0)^{p-1} (u_n^0 - \eta)^+ \, d\sigma \\ &\quad = \int_\Omega (f(z, u, D\nu_n) + \xi_\eta u^{p-1}) (u_n^0 - \eta)^+ \, dz \\ &\quad \leq \int_\Omega (f(z, \eta, D\nu_n) + \xi_\eta \eta^{p-1}) (u_n^0 - \eta)^+ \, dz \\ &\quad = \int_\Omega \xi_\eta \eta^{p-1} (u_n^0 - \eta)^+ \, dz \end{aligned}$$

(see (3.1), hypotheses $H(f)$(iii) and (i) and recall that $u \in S_v \subseteq [0,\eta] \cap D_+$), so

$$\langle A(u_n^0) - A(\eta), (u_n^0 - \eta)^+ \rangle + \xi_\eta \int_\Omega \big((u_n^0)^{p-1} - \eta^{p-1}\big)(u_n^0 - \eta)^+ \, dz \le 0$$

(see hypothesis $H(\beta)$ and note that $A(\eta) = 0$), thus

$$u_n^0 \le \eta.$$

So, we have that

$$u_n^0 \in [0,\eta] \setminus \{0\} \quad \forall n \ge 1.$$

Moreover, the nonlinear regularity theory (see Lieberman [3]) and the nonlinear maximum principle (see Pucci and Serrin [14]) imply that

$$u_n^0 \in [0,\eta] \cap D_+ \quad \forall n \ge 1. \tag{3.27}$$

We have

$$\begin{cases} -\operatorname{div} a(Du_n^0(z)) = f(z, u(z), Dv_n(z)) & \text{for a.a. } z \in \Omega, \\ \frac{\partial u_n^0}{\partial n_a} + \beta(z)(u_n^0)^{p-1} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.28}$$

Then $\{u_n^0\}_{n\ge1} \subseteq W^{1,p}(\Omega)$ is bounded (see (3.27), (3.28), Lemma 2.2 and hypothesis $H(f)$(i)). So, on account of the nonlinear regularity theory of Lieberman [3], we can find $\mu \in (0,1)$ and $c_9 > 0$ such that

$$u_n^0 \in C^{1,\mu}(\overline{\Omega}) \quad \text{and} \quad \|u_n^0\|_{C^{1,\mu}(\overline{\Omega})} \le c_9 \quad \forall n \ge 1.$$

The compactness of the embedding $C^{1,\mu}(\overline{\Omega}) \subseteq C^1(\overline{\Omega})$ implies that we can find a subsequence $\{u_{n_k}^0\}_{k\ge1}$ of the sequence $\{u_n^0\}_{n\ge1}$ such that

$$u_{n_k}^0 \longrightarrow \widetilde{u}_0 \quad \text{in } C^1(\overline{\Omega}) \text{ as } k \to +\infty.$$

Note that

$$\begin{cases} -\operatorname{div} a(D\widetilde{u}^0(z)) = f(z, u(z), Dv(z)) & \text{for a.a. } z \in \Omega, \\ \frac{\partial \widetilde{u}^0}{\partial n_a} + \beta(z)(\widetilde{u}^0)^{p-1} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.29}$$

Since $u \in S_v$ solves (3.29) which has a unique solution, we infer that

$$\widetilde{u}^0 = u \in S_v.$$

Hence, for the original sequence $\{u_n^0\}_{n\ge1}$, we have

$$u_n^0 \longrightarrow u \quad \text{in } C^1(\overline{\Omega}) \text{ as } n \to +\infty.$$

Next consider the following nonlinear Robin problem:

$$\begin{cases} -\operatorname{div} a(Dw(z)) + \xi_\eta |w(z)|^{p-2}w(z) = \widehat{f}(z, u_n^0(z), Dv_n(z)) & \text{in } \Omega, \\ \frac{\partial w}{\partial n_a} + \beta(z)|w|^{p-2}w = 0 & \text{on } \partial\Omega, n \geq 1. \end{cases}$$

As above, we establish that this problem has a unique solution

$$u_n^1 \in [0, \eta] \cap D_+ \quad \forall n \geq 1.$$

Again we have

$$u_n^1 \longrightarrow u \quad \text{in } C^1(\overline{\Omega}) \text{ as } n \to +\infty.$$

Continuing this way, we generate a sequence $\{u_n^k\}_{k,n\geq 1}$ such that

$$\begin{cases} -\operatorname{div} a(Du_n^k(z)) + \xi_\eta u_n^k(z)^{p-1} = \widehat{f}(z, u_n^{k-1}(z), Dv_n(z)) & \text{in } \Omega, \\ \frac{\partial u_n^k}{\partial n_a} + \beta(z)(u_n^k)^{p-1} = 0 & \text{on } \partial\Omega, n, k \geq 1, \end{cases} \tag{3.30}$$

$$u_n^k \in [0, \eta] \cap D_+ \quad \forall n, k \geq 1 \tag{3.31}$$

and

$$u_n^k \longrightarrow u \quad \text{in } C^1(\overline{\Omega}) \text{ as } n \to +\infty \ \forall k \geq 1. \tag{3.32}$$

Fix $n \geq 1$. As before we have that the sequence $\{u_n^k\}_{k\geq 1} \subseteq C^1(\overline{\Omega})$ is relatively compact. So, we can find a subsequence $\{u_n^{k_m}\}_{m\geq 1}$ of the sequence $\{u_n^k\}_{k\geq 1}$ such that

$$u_n^{k_m} \longrightarrow \widetilde{u}_n \quad \text{in } C^1(\overline{\Omega}) \text{ as } m \to +\infty,$$

so

$$\begin{cases} -\operatorname{div} a(D\widetilde{u}_n(z)) + \xi_\eta \widetilde{u}_n(z)^{p-1} = \widehat{f}(z, \widetilde{u}_n(z), Dv_n(z)) & \text{for a.a. } z \in \Omega, \\ \frac{\partial \widetilde{u}_n}{\partial n_a} + \beta(z)\widetilde{u}_n^{p-1} = 0 & \text{on } \partial\Omega, n \geq 1 \end{cases} \tag{3.33}$$

(see (3.30)). Using the nonlinear regularity theory of Lieberman [3], (3.32) and the double limit lemma (see Aubin and Ekeland [25] and Gasiński and Papageorgiou [26, p. 61]), we have

$$\widetilde{u}_n \longrightarrow u \quad \text{in } C^1(\overline{\Omega}),$$

so

$$\widetilde{u}_n \in [0, \eta] \cap D_+ \quad \forall n \geq n_0,$$

and thus

$$\widetilde{u}_n \in S_{v_n} \quad \forall n \geq n_0 \quad \text{and} \quad \widetilde{u}_n \longrightarrow u \quad \text{in } C^1(\overline{\Omega}). \qquad \square$$

Using this lemma, we can show that the map $\vartheta : C \longrightarrow C$ defined earlier is compact.

Proposition 3.6 *If hypotheses $H(a), H(\beta), H(f)$ hold, then the map $\vartheta : C \longrightarrow C$ is compact.*

Proof First we show that ϑ is continuous.

So, suppose that $v_n \longrightarrow v$ in $C^1(\overline{\Omega})$, $\{v_n\}_{n\geq 1} \subseteq C$, $v \in C$, and let $\widehat{u}_n = \vartheta(v_n)$ for $n \geq 1$. We have

$$\begin{cases} -\operatorname{div} a(D\widehat{u}_n(z)) = f(z, \widehat{u}_n(z), Dv_n(z)) & \text{for a.a. } z \in \Omega, \\ \frac{\partial \widehat{u}_n}{\partial n_a} + \beta(z)\widehat{u}_n(z)^{p-1} = 0 & \text{on } \partial\Omega, \widehat{u}_n \in [0, \eta], n \geq 1. \end{cases} \tag{3.34}$$

From (3.34) we see that $\{\widehat{u}_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega)$ is bounded and so, according to Lieberman [3], we can find $\tau \in (0,1)$ and $c_{10} > 0$ such that

$$\widehat{u}_n \in C^{1,\tau}(\overline{\Omega}) \quad \text{and} \quad \|\widehat{u}_n\|_{C^{1,\tau}(\overline{\Omega})} \leq c_{10} \quad \forall n \geq 1.$$

So, we may assume that

$$\widehat{u}_n \longrightarrow \widehat{u} \quad \text{in } C^1(\overline{\Omega}) \text{ as } n \to +\infty. \tag{3.35}$$

In (3.34) we pass to the limit as $n \to \infty$ and use (3.35). Then

$$\begin{cases} -\operatorname{div} a(D\widehat{u}(z)) = f(z, \widehat{u}(z), Dv(z)) & \text{for a.a. } z \in \Omega, \\ \frac{\partial \widehat{u}}{\partial n_a} + \beta(z)\widehat{u}(z)^{p-1} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.36}$$

From Proposition 3.3 we have

$$u^* \leq \widehat{u}_n \quad \forall n \geq 1$$

(in this case $M > \sup_{n\geq 1} \|v_n\|_{C^1(\overline{\Omega})}$), so

$$u^* \leq \widehat{u}$$

(see (3.35)), thus

$$\widehat{u} \in S_v. \tag{3.37}$$

We claim that $\widehat{u} = \vartheta(v)$. According to Lemma 3.5, we can find $u_n \in S_{v_n}$, $n \geq 1$, such that

$$u_n \longrightarrow \vartheta(v) \quad \text{in } C^1(\overline{\Omega}) \text{ as } n \to +\infty. \tag{3.38}$$

We have

$$\widehat{u}_n = \vartheta(v_n) \leq u_n \quad \forall n \geq 1,$$

so

$$\widehat{u} \le \vartheta(v)$$

(see (3.35) and (3.38)), thus

$$\widehat{u} = \vartheta(v)$$

(see (3.37)), and hence ϑ is continuous.

Next we show that ϑ maps bounded sets in C to relatively compact subsets of C. So, let $B \subseteq C$ be bounded in $C^1(\overline{\Omega})$. As above, we have that the set $\vartheta(B) \subseteq W^{1,p}(\Omega)$ is bounded. But then the nonlinear regularity theory of Lieberman [3] and the compactness of the embedding $C^{1,s}(\overline{\Omega}) \subseteq C^1(\overline{\Omega})$ (with $0 < s < 1$) imply that the set $\vartheta(B) \subseteq C^1(\overline{\Omega})$ is relatively compact, thus ϑ is compact. □

Now we are ready for the existence theorem.

Theorem 3.7 *If hypotheses* $H(a)$, $H(\beta)$, $H(f)$ *hold, then problem* (1.1) *admits a solution* $\widehat{u} \in [0, \eta] \cap D_+$.

Proof We consider the set

$$S(\vartheta) = \{u \in C : u = \lambda\vartheta(u), 0 < \lambda < 1\}.$$

If $u \in S(\vartheta)$, then

$$\frac{1}{\lambda}u = \vartheta(u),$$

so

$$\left\langle A\left(\frac{1}{\lambda}u\right), h\right\rangle + \int_{\partial\Omega} \beta(z)\left(\frac{u}{\lambda}\right)^{p-1} h\,d\sigma = \int_\Omega f\left(z, \frac{u}{\lambda}, Du\right) h\,dz \quad \forall h \in W^{1,p}(\Omega). \tag{3.39}$$

In (3.39) we choose $h = \frac{u}{\lambda} \in W^{1,p}(\Omega)$. Using Lemma 2.2 and hypothesis $H(\beta)$, we have

$$\begin{aligned} \frac{c_1}{p-1}\left\|D\left(\frac{u}{\lambda}\right)\right\|_p^p &\le \int_\Omega f\left(z, \frac{u}{\lambda}, Du\right)\frac{u}{\lambda}\,dz \le \int_\Omega f(z, u, Du)\frac{u}{\lambda^p}\,dz \\ &\le \int_\Omega f\left(z, u, D\left(\frac{u}{\lambda}\right)\right) u\,dz \le \int_\Omega \left(\widetilde{c}_1 + \widetilde{c}_2\left|D\left(\frac{u}{\lambda}\right)\right|^p\right) dz \end{aligned}$$

(see (2.4), hypotheses $H(f)$(iii) and (i)). Recalling that $\widetilde{c}_2 < \frac{\widetilde{c}_1}{p-1}$ (see hypothesis $H(f)$(i)), we have

$$\left\|D\left(\frac{u}{\lambda}\right)\right\|_p \le c_{11} \quad \forall \lambda \in (0, 1),$$

for some $c_{11} > 0$, thus

$$\left\{D\left(\frac{u}{\lambda}\right)\right\}_{u \in S(\vartheta)} \subseteq L^p(\Omega; \mathbb{R}^N) \quad \text{is bounded.} \tag{3.40}$$

As above, from (3.39) with $h = \frac{u}{\lambda} \in W^{1,p}(\Omega)$, using hypotheses $H(f)$(i), (iii) and (3.40), we obtain

$$\frac{c_1}{p-1}\left\| D\left(\frac{u}{\lambda}\right)\right\|_p^p + \int_{\partial\Omega} \beta(z)\left(\frac{u}{\lambda}\right)^p dz \leq c_{12} \quad \forall \lambda \in (0,1),$$

for some $c_{12} > 0$, so

$$\frac{c_1}{p-1}\widehat{\lambda}_1(p,\widehat{\beta})\left\|\frac{u}{\lambda}\right\|_p^p \leq c_{12},$$

where $\widehat{\beta} = \frac{p-1}{c_1}\beta$ (see (2.3)), thus

$$\left\{\frac{u}{\lambda}\right\}_{u\in S(\vartheta)} \subseteq L^p(\Omega) \quad \text{is bounded,}$$

hence

$$\left\{\frac{u}{\lambda}\right\}_{u\in S(\vartheta)} \subseteq W^{1,p}(\Omega) \quad \text{is bounded} \tag{3.41}$$

(see (3.40)). From (3.39) we have

$$\begin{cases} -\operatorname{div} a(D(\frac{u}{\lambda})(z)) = f(z, \frac{u}{\lambda}(z), Du(z)) & \text{for a.a. } z \in \Omega, \\ \frac{\partial(\frac{u}{\lambda})}{\partial n_a} + \beta(z)(\frac{u}{\lambda})^{p-1} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.42}$$

Hypothesis $H(f)$(iii) implies that

$$f\left(z, \frac{u}{\lambda}, Du\right) \leq \lambda^p f\left(z, \frac{u}{\lambda}, D\left(\frac{u}{\lambda}\right)\right) \quad \text{for a.a. } z \in \Omega. \tag{3.43}$$

Then, from (3.41), (3.42), (3.43) and the nonlinear regularity theory of Lieberman [3], we have

$$\left\|\frac{u}{\lambda}\right\|_{C^1(\overline{\Omega})} \leq c_{13} \quad \forall u \in S(\vartheta),$$

for some $c_{13} > 0$, thus $S(\vartheta) \subseteq C^1(\overline{\Omega})$ is bounded.

Since ϑ is compact (see Proposition 3.6), we can use the Leray–Schauder alternative theorem (see Theorem 2.9) and find $\widehat{u} \in C$ such that

$$\widehat{u} = \vartheta(\widehat{u}),$$

so $\widehat{u} \in [0,\eta] \cap D_+$ is a solution of (1.1). □

4 Conclusion

This is the first work producing positive smooth solutions for problems driven by a nonhomogeneous differential operator with Robin boundary condition where the forcing term has the form of a convection term, that is, it depends also on the gradient of the unknown function. In addition, in contrast to the previous works in the field, we do not impose any global growth condition on the convection term. Our formulation incorporates (p,q)-equations which are important in physical applications.

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