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Almost sure exponential stability of an explicit stochastic orthogonal Runge-Kutta-Chebyshev method for stochastic delay differential equations

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Abstract
Compared with Euler-Maruyama type schemes, there is a lack of studies on the stability of Runge-Kutta type methods applied to stochastic delay differential equations (SDDEs). This paper is concerned with filling this imbalance. The focus is on the almost sure exponential stability of an explicit stochastic Runge-Kutta-Chebyshev (S-ROCK) method for an Itô-type linear test equation, which is analyzed by applying the techniques based on a discrete semimartingale convergence theorem.

MSC: 65C30; 60H10; 65L06

Keywords: stochastic delay differential equations; discrete semimartingale convergence theorem; almost sure stability; Chebyshev method; Runge-Kutta method; explicit schemes

1 Introduction

Stability analysis of numerical methods for stochastic differential equations (SDEs) has recently attracted an increasing interest. Most researchers are concerned with two kinds of stability, *i.e.*, almost sure stability [1–7] and moment stability [8–13], of the numerical solutions to SDEs as well as SDDEs. Generally, almost sure stability is less restrictive than moment stability, and almost sure stability results are more difficult to establish if deriving from the moment stability by the Chebyshev inequality and the Borel-Cantelli lemma. The situation has been improved since the martingale techniques were introduced to investigate the almost sure stability. By the discrete semimartingale convergence theorem (*cf.* [1]), the numerical stability of SDDEs has been examined, for example, by [4, 5].

To the best knowledge of authors, there is no similar result about almost sure stability of Runge-Kutta type methods for SDDEs, and nearly all existing results concerned with Euler-Maruyama type schemes. Recently, stabilized explicit Runge-Kutta schemes have proved successful for solving SDEs, which are called S-ROCK (stochastic orthogonal Runge-Kutta-Chebyshev) method; see, for example, [14, 15]. In this paper, we investigate the almost sure stability of the S-ROCK method applied to SDDEs. Consider Itô SDDEs of the form

$$dy = f\big(y(t), y(t-\tau)\big)\,dt + g\big(y(t), y(t-\tau)\big)\,dw(t) \tag{1.1}$$

for every $t \geq 0$. Here time delay $\tau > 0$. The initial function $y(t) = \psi(t)$ when $t \in [-\tau, 0]$. We further assume that the initial data is independent of Wiener measure driving the equation and $w(t)$ is a scalar Brownian motion on the complete probability space $(\Omega, \mathcal{F}, \mathcal{F}_{t\geq 0}, \mathbb{P})$ with a filtration satisfying the usual conditions. Moreover, $f, g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are Borel-measurable functions.

The rest of this paper is organized as follows. In the next section, we propose the S-ROCK method for SDDEs. Our main stability results will be derived in Section 3.

2 The S-ROCK method and preliminary results

In the following, we employ an equidistant step points $\mathcal{I}_{\Delta t} = \{t_0, t_1, \ldots, t_N\}$ where the time step size is a submultiple of the delay τ, *i.e.*, $\Delta t = \tau/m$ for a given positive integer m, and the nth step point is denoted by $t_n = n\Delta t$ for $0 \leq n \leq N$. The numerical approximation of $y(t)$ at t_n is denoted by Y_n, and we denote the increment $w(t_{n+1}) - w(t_n)$ by J_n. Next we introduce the S-ROCK method for solving SDDEs (1.1), which is given by

$$Y_{n+1} = \frac{2\Delta t}{\nu^2} f\big(K_n^{(\nu)}, Z_n^{(\nu)}\big) + 2K_n^{(\nu)} - K_n^{(\nu-1)} + J_n(1-2\alpha) g\big(K_n^{(\nu-1)}, Z_n^{(\nu-1)}\big), \tag{2.1}$$

where $\{K_n^{(i)}\}$ and $\{Z_n^{(i)}\}$ are the stage values defined by

$$\begin{aligned}
K_n^{(1)} &= Y_n, \\
K_n^{(2)} &= Y_n + \frac{\Delta t}{\nu^2} f\big(K_n^{(1)}, Z_n^{(1)}\big), \\
K_n^{(i)} &= \frac{2\Delta t}{\nu^2} f\big(K_n^{(i-1)}, Z_n^{(i-1)}\big) + 2K_n^{(i-1)} - K_n^{(i-2)} \quad \text{for } i = 3, \ldots, \nu - 1, \\
K_n^{(\nu)} &= \frac{2\Delta t}{\nu^2} f\big(K_n^{(\nu-1)}, Z_n^{(\nu-1)}\big) + 2K_n^{(\nu-1)} - K_n^{(\nu-2)} + J_n \alpha g\big(K_n^{(\nu-1)}, Z_n^{(\nu-1)}\big),
\end{aligned} \tag{2.2}$$

and

$$Z_n^{(i)} = \begin{cases} \psi(t_n + \beta_i \Delta t - \tau), & t_n + \beta_i \Delta t - \tau \leq 0, \\ K_{n-m}^{(i)}, & t_n + \beta_i \Delta t - \tau > 0. \end{cases} \tag{2.3}$$

Here parameters $\alpha \in [0, 1/2]$ and $\beta_i = (i-1)^2/\nu^2$ for $i = 1, \ldots, \nu$.

Let $C([-\tau, 0]; \mathbb{R})$ be the family of continuous functions φ from $[-\tau, 0]$ to $\mathbb{R}$, equipped with the supremum norm $\|\varphi\| = \sup_{-\tau\leq\theta\leq 0} |\varphi(\theta)|$. Also, denote by $C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R})$ the family of bounded, $\mathcal{F}_0$-measurable, $C([-\tau, 0]; \mathbb{R})$-valued random variables.

Now we give some definitions on the almost sure exponential stability of exact and numerical solutions to SDDEs (*cf.* [16]).

Definition 2.1 The solution $y(t, \psi)$ to SDDEs (1.1) is said to be almost surely exponentially stable if there exists a constant $\eta > 0$ such that

$$\limsup_{t\to\infty} \frac{1}{t} \log\big|y(t, \psi)\big| \leq -\eta \quad \text{a.s.} \tag{2.4}$$

for any initial data $\psi \in C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R})$.

Definition 2.2 The solution Y_n to numerical scheme (2.1) is said to be almost surely exponentially stable if there exists a constant $\gamma > 0$ such that

$$\limsup_{n\to\infty} \frac{1}{n\Delta t} \log|Y_n| \le -\gamma \quad \text{a.s.} \tag{2.5}$$

for any bounded variables $\psi(\vartheta\Delta t)$ when $\vartheta\Delta t \in [-\tau, 0]$.

For the purpose of stability, we assume that $f(0,0) = g(0,0) = 0$, which implies that (1.1) admits the equilibrium solution $y(t) = 0$ corresponding to the initial condition $\psi(t) = 0$ for $t \in [-\tau, 0]$. As a standing hypothesis, we shall impose the following local Lipschitz condition (*cf.* [7]) on the coefficients f and g.

(A1) For each integer D, there exists a positive constant K_D such that, for all $y_1, y_2, z_1, z_2 \in R$ with $|y_1| \vee |y_2| \vee |z_1| \vee |z_2| \le D$, $|f(y_1,z_1) - f(y_2,z_2)|^2 \vee |g(y_1,z_1) - g(y_2,z_2)|^2 \le K_D(|y_1 - y_2|^2 + |z_1 - z_2|^2)$, where $\vee$ is the maximal operator.

In what follows we introduce the result of almost sure stability of SDDEs (1.1). The proof of the following lemma can be found in [4].

Lemma 2.3 *Let Assumptions* (A1) *hold. Assume that there are four nonnegative constants* $\lambda_1, \dots, \lambda_4$ *such that*

$$2yf(y,0) \le -\lambda_1|y|^2,$$
$$\bigl|f(y,z) - f(y,0)\bigr| \le \lambda_2|z|,$$
$$\bigl|g(y,z)\bigr|^2 \le \lambda_3|y|^2 + \lambda_4|z|^2$$

for $y, z \in \mathbb{R}$. *If*

$$\lambda_1 > 2\lambda_2 + \lambda_3 + \lambda_4,$$

then the trivial solution of (1.1) *is almost surely exponentially stable*.

To explain our idea, we cite the discrete semimartingale convergence theorem as follows (see also [4]).

Theorem 2.4 *Let* $\{A_j\}$, $\{U_j\}$ *be two sequences of nonnegative random variables such that both* A_j *and* U_j *are* $\mathcal{F}_{j-1}$*-measurable for* $j = 1, 2, \dots,$ *and* $A_0 = U_0 = 0$ *a.s. Let* $\mathcal{M}_j$ *be a real-value local martingale with* $\mathcal{M}_0 = 0$ *a.s. Let* ζ *be a nonnegative* $\mathcal{F}_0$*-measurable random variable. Assume that* $\{X_j\}$ *is a nonnegative semimartingale with the Doob-Mayer decomposition*

$$X_j = \zeta + A_j - U_j + \mathcal{M}_j.$$

If $\lim_{j\to+\infty} A_j < +\infty$ *a.s. then for almost all* $\omega \in \Omega$,

$$\lim_{j\to+\infty} X_j < +\infty \quad \textit{and} \quad \lim_{j\to+\infty} U_j < +\infty.$$

3 Almost sure asymptotic exponential stability of numerical solution

Consider the linear SDDE

$$\mathrm{d}y(t) = \big(ay(t) + dy(t-\tau)\big)\,\mathrm{d}t + \big(by(t) + cy(t-\tau)\big)\,\mathrm{d}w(t). \tag{3.1}$$

It seems that the stability of approximate solutions to (3.1) using Runge-Kutta type methods is still an open problem. Here we consider the almost sure stability of the linear equation

$$\mathrm{d}y(t) = ay(t)\,\mathrm{d}t + \big(by(t) + cy(t-\tau)\big)\,\mathrm{d}w(t), \tag{3.2}$$

which is the same test model as [17]. In this section, our aim is to examine how the S-ROCK method can reproduce the almost sure exponential stability of the exact solution of (3.2). By applying Lemma 2.3, the exact solution of (3.2) is almost surely exponentially stable when $a < -(b^2 + c^2)$. Now we give a main result of the almost sure stability of the approximate solution (2.1).

Theorem 3.1 *Suppose that the conditions of Lemma* 2.3 *are satisfied. Then the approximate solution* (2.1) *applied to test model* (3.2) *is almost surely exponentially stable if the step size* Δt *satisfies*

$$\left[T_\nu\left(1+\frac{a\Delta t}{\nu^2}\right)\right]^2 + 2\Delta t\big(b^2+c^2\big)\left(1+2\alpha\frac{a\Delta t}{\nu^2}\right)^2\left[T_{\nu-2}\left(1+\frac{a\Delta t}{\nu^2}\right)\right]^2 < 1, \tag{3.3}$$

where $T_\nu(x)$ *is defined as a Chebyshev polynomial of the first kind of degree* ν.

Proof Applying (2.2) to test model (3.2), we have $K_n^{(1)} = T_0(1+\frac{a\Delta t}{\nu^2})Y_n$ and $K_n^{(2)} = T_1(1+\frac{a\Delta t}{\nu^2})Y_n$. Next, by the three-term recurrence relation for Chebyshev polynomials, it is easy to prove that

$$K_n^{(i)} = T_{i-1}\left(1+\frac{a\Delta t}{\nu^2}\right)Y_n \tag{3.4}$$

for $i = 1,\ldots,\nu-1$. Then, from (2.2), we have

$$\begin{aligned}
K_n^{(\nu)} &= 2\frac{\Delta t}{\nu^2}aK_n^{(\nu-1)} + 2K_n^{(\nu-1)} - K_n^{(\nu-2)} + J_n\alpha\big[bK_n^{(\nu-1)} + cZ_n^{(\nu-1)}\big]\\
&= 2\left(1+\frac{a\Delta t}{\nu^2}\right)T_{\nu-2}\left(1+\frac{a\Delta t}{\nu^2}\right)Y_n - T_{\nu-3}\left(1+\frac{a\Delta t}{\nu^2}\right)Y_n + J_n\alpha bT_{\nu-2}\left(1+\frac{a\Delta t}{\nu^2}\right)Y_n\\
&\quad + J_n\alpha cZ_n^{(\nu-1)}\\
&= T_{\nu-1}\left(1+\frac{a\Delta t}{\nu^2}\right)Y_n + J_n\alpha\left[bT_{\nu-2}\left(1+\frac{a\Delta t}{\nu^2}\right)Y_n + cZ_n^{(\nu-1)}\right].
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
Y_{n+1} &= 2\left(1+\frac{a\Delta t}{\nu^2}\right)K_n^{(\nu)} - K_n^{(\nu-1)} + J_n(1-2\alpha)\big[bK_n^{(\nu-1)} + cZ_n^{(\nu-1)}\big]\\
&= T_\nu\left(1+\frac{a\Delta t}{\nu^2}\right)Y_n + J_n\left(1+2\frac{a\Delta t}{\nu^2}\alpha\right)\left[bT_{\nu-2}\left(1+\frac{a\Delta t}{\nu^2}\right)Y_n + cZ_n^{(\nu-1)}\right].
\end{aligned}$$

Note that

$$\begin{aligned}
|Y_{n+1}|^2 &= |Y_n|^2 + \left\{\left[T_\nu\left(1+\frac{a\Delta t}{\nu^2}\right)\right]^2 - 1\right\}|Y_n|^2 \\
&\quad + \left(J_n^2 - \Delta t\right)\left(1+2\frac{a\Delta t}{\nu^2}\alpha\right)^2\left[bT_{\nu-2}\left(1+\frac{a\Delta t}{\nu^2}\right)Y_n + cZ_n^{(\nu-1)}\right]^2 \\
&\quad + \Delta t\left(1+2\frac{a\Delta t}{\nu^2}\alpha\right)^2\left[bT_{\nu-2}\left(1+\frac{a\Delta t}{\nu^2}\right)Y_n + cZ_n^{(\nu-1)}\right]^2 \\
&\quad + 2T_\nu\left(1+\frac{a\Delta t}{\nu^2}\right)\left(1+2\frac{a\Delta t}{\nu^2}\alpha\right)\left[bT_{\nu-2}\left(1+\frac{a\Delta t}{\nu^2}\right)Y_n + cZ_n^{(\nu-1)}\right]Y_nJ_n \\
&\le |Y_n|^2 + \left\{\left[T_\nu\left(1+\frac{a\Delta t}{\nu^2}\right)\right]^2 - 1\right\}|Y_n|^2 + M_n \\
&\quad + 2\Delta t\left(1+2\frac{a\Delta t}{\nu^2}\alpha\right)^2\left\{\left[bT_{\nu-2}\left(1+\frac{a\Delta t}{\nu^2}\right)Y_n\right]^2 + \left[cZ_n^{(\nu-1)}\right]^2\right\},
\end{aligned} \tag{3.5}$$

where $M_n = (J_n^2 - \Delta t)(1 + 2\frac{a\Delta t}{\nu^2}\alpha)^2[bT_{\nu-2}(1 + \frac{a\Delta t}{\nu^2})Y_n + cZ_n^{(\nu-1)}]^2 + 2T_\nu(1 + \frac{a\Delta t}{\nu^2})(1 + 2\frac{a\Delta t}{\nu^2}\alpha) \times [bT_{\nu-2}(1 + \frac{a\Delta t}{\nu^2})Y_n + cZ_n^{(\nu-1)}]Y_nJ_n$.

For any positive constant $C > 1$, we have

$$C^{(\ell+1)\Delta t}|Y_{\ell+1}|^2 - C^{\ell\Delta t}|Y_\ell|^2 = C^{(\ell+1)\Delta t}\left(|Y_{\ell+1}|^2 - |Y_\ell|^2\right) + \left(C^{(\ell+1)\Delta t} - C^{\ell\Delta t}\right)|Y_\ell|^2. \tag{3.6}$$

Therefore, by (3.5) and (3.6), we obtain

$$\begin{aligned}
&C^{(\ell+1)\Delta t}|Y_{\ell+1}|^2 - C^{\ell\Delta t}|Y_\ell|^2 \\
&\quad \le C^{(\ell+1)\Delta t}\left\{-C^{-\Delta t} + \left[T_\nu\left(1+\frac{a\Delta t}{\nu^2}\right)\right]^2\right\}|Y_\ell|^2 + C^{(\ell+1)\Delta t}M_\ell \\
&\qquad + 2C^{(\ell+1)\Delta t}b^2\Delta t\left(1+2\frac{a\Delta t}{\nu^2}\alpha\right)^2\left[T_{\nu-2}\left(1+\frac{a\Delta t}{\nu^2}\right)\right]^2|Y_\ell|^2 \\
&\qquad + 2C^{(\ell+1)\Delta t}c^2\Delta t\left(1+2\frac{a\Delta t}{\nu^2}\alpha\right)^2\left|Z_\ell^{(\nu-1)}\right|^2
\end{aligned} \tag{3.7}$$

for any nonnegative integer ℓ. Summing up both sides of inequality (3.7) from $\ell = 0$ to $n - 1$ ($n \ge 1$), we have

$$\begin{aligned}
C^{n\Delta t}|Y_n|^2 &\le |Y_0|^2 + \sum_{\ell=0}^{n-1} C^{(\ell+1)\Delta t}M_\ell + \left\{-C^{-\Delta t} + \left[T_\nu\left(1+\frac{a\Delta t}{\nu^2}\right)\right]^2\right. \\
&\qquad \left. + 2b^2\Delta t\left(1+2\frac{a\Delta t}{\nu^2}\alpha\right)^2\left[T_{\nu-2}\left(1+\frac{a\Delta t}{\nu^2}\right)\right]^2\right\}\sum_{\ell=0}^{n-1} C^{(\ell+1)\Delta t}|Y_\ell|^2 \\
&\qquad + 2c^2\Delta t\left(1+2\frac{a\Delta t}{\nu^2}\alpha\right)^2\sum_{\ell=0}^{n-1} C^{(\ell+1)\Delta t}\left|Z_\ell^{(\nu-1)}\right|^2.
\end{aligned} \tag{3.8}$$

Let $\mathcal{M}_n = \sum_{\ell=0}^{n-1} C^{(\ell+1)\Delta t} M_\ell$. Note that the expectation values $E(J_{n-1}^2 - \Delta t) = 0$, $E(J_{n-1}) = 0$, moreover, Y_{n-1} and $Z_{n-1}^{(\nu-1)}$ are $\mathcal{F}_{(n-1)\Delta t}$-measurable, then we have

$$\begin{aligned} E[\mathcal{M}_n|\mathcal{F}_{(n-1)\Delta t}] &= \mathcal{M}_{n-1} + C^{n\Delta t} E[M_{n-1}|\mathcal{F}_{(n-1)\Delta t}] \\ &= \mathcal{M}_{n-1}, \end{aligned}$$

which implies that $\mathcal{M}_n$ is a martingale with $\mathcal{M}_0 = 0$.

When $t_n \le \tau$, from (2.3) and (3.8), we have

$$\begin{aligned} C^{n\Delta t}|Y_n|^2 \le{}& |Y_0|^2 + \mathcal{M}_n + \biggl\{ -C^{-\Delta t} + \biggl[T_\nu \biggl(1 + \frac{a\Delta t}{\nu^2} \biggr) \biggr]^2 \\ &+ 2b^2 \Delta t \biggl(1 + 2\frac{a\Delta t}{\nu^2}\alpha \biggr)^2 \biggl[T_{\nu-2} \biggl(1 + \frac{a\Delta t}{\nu^2} \biggr) \biggr]^2 \biggr\} \sum_{\ell=0}^{n-1} C^{(\ell+1)\Delta t} |Y_\ell|^2 \\ &+ 2c^2 \Delta t \biggl(1 + 2\frac{a\Delta t}{\nu^2}\alpha \biggr)^2 \sum_{\ell=0}^{n-1} C^{(\ell+1)\Delta t} \psi^2(t_\ell + \beta_{\nu-1}\Delta t - \tau). \end{aligned} \tag{3.9}$$

According to Theorem 2.4, we denote the right side of inequality (3.9) by X_n. Then, let $\zeta = |Y_0|^2 + 2c^2\Delta t(1 + 2\frac{a\Delta t}{\nu^2}\alpha)^2 \sum_{\ell=0}^{n-1} C^{(\ell+1)\Delta t}\psi^2(t_\ell + \beta_{\nu-1}\Delta t - \tau)$, $U_n = 0$, and

$$A_n = H_1(C) \sum_{\ell=0}^{n-1} C^{(\ell+1)\Delta t} |Y_\ell|^2,$$

where $H_1(C) = -C^{-\Delta t} + [T_\nu(1 + \frac{a\Delta t}{\nu^2})]^2 + 2b^2\Delta t(1 + 2\frac{a\Delta t}{\nu^2}\alpha)^2 [T_{\nu-2}(1 + \frac{a\Delta t}{\nu^2})]^2$. There exists a unique $C^* > 1$ such that $H_1(C^*) = 0$ if

$$\biggl[T_\nu \biggl(1 + \frac{a\Delta t}{\nu^2} \biggr) \biggr]^2 + 2b^2 \Delta t \biggl(1 + 2\frac{a\Delta t}{\nu^2}\alpha \biggr)^2 \biggl[T_{\nu-2} \biggl(1 + \frac{a\Delta t}{\nu^2} \biggr) \biggr]^2 < 1. \tag{3.10}$$

Applying Theorem 2.4, we therefore have $\lim_{n\to\infty} X_n < +\infty$, which means

$$\lim_{n\to\infty} \bigl(C^*\bigr)^{n\Delta t} |Y_n|^2 < +\infty. \tag{3.11}$$

When $t_n > \tau$, that is, $n > m$, we have

$$\begin{aligned} \sum_{\ell=0}^{n-1} C^{(\ell+1)\Delta t} \bigl|Z_\ell^{(\nu-1)}\bigr|^2 ={}& \sum_{\ell=0}^{m-1} C^{(\ell+1)\Delta t} \psi^2(t_\ell + \beta_{\nu-1}\Delta t - \tau) \\ &+ \sum_{\ell=0}^{n-1} C^{(\ell+m+1)\Delta t} \bigl|K_\ell^{(\nu-1)}\bigr|^2 - \sum_{\ell=n-m}^{n-1} C^{(\ell+m+1)\Delta t} \bigl|K_\ell^{(\nu-1)}\bigr|^2. \end{aligned}$$

Then, by using (3.4) and (3.8), we have

$$C^{n\Delta t}|Y_n|^2 + 2c^2 \Delta t \biggl(1 + 2\frac{a\Delta t}{\nu^2}\alpha \biggr)^2 \sum_{\ell=n-m}^{n-1} C^{(\ell+m+1)\Delta t} \bigl|K_\ell^{(\nu-1)}\bigr|^2 \le X_n \tag{3.12}$$

and

$$X_n = \zeta_0 + \mathcal{M}_n + H_2(C)\sum_{\ell=0}^{n-1} C^{(\ell+1)\Delta t}|Y_\ell|^2, \tag{3.13}$$

where $\zeta_0 = |Y_0|^2 + 2c^2\Delta t(1+2\frac{a\Delta t}{\nu^2}\alpha)^2\sum_{\ell=0}^{m-1} C^{(\ell+1)\Delta t}\psi^2(t_\ell+\beta_{\nu-1}\Delta t-\tau)$, $H_2(C) = H_1(C) + 2c^2\Delta t(1+2\frac{a\Delta t}{\nu^2}\alpha)^2 C^{m\Delta t}[T_{\nu-2}(1+\frac{a\Delta t}{\nu^2})]^2$.

Note that

$$H_2(1) = -1 + \left[T_\nu\left(1+\frac{a\Delta t}{\nu^2}\right)\right]^2 + 2\Delta t\left(b^2+c^2\right)\left(1+2\frac{a\Delta t}{\nu^2}\alpha\right)^2\left[T_{\nu-2}\left(1+\frac{a\Delta t}{\nu^2}\right)\right]^2,$$

$$H_2'(C) = \Delta t C^{(-\Delta t-1)} + 2c^2 m(\Delta t)^2\left(1+2\frac{a\Delta t}{\nu^2}\alpha\right)^2\left[T_{\nu-2}\left(1+\frac{a\Delta t}{\nu^2}\right)\right]^2 C^{(m\Delta t-1)} > 0$$

for any $C > 1$, and $H_2(\infty) > 0$.

Obviously, the condition (3.3) yields $H_2(1) < 0$, which implies that there exists a unique $C^* > 1$ such that $H_2(C^*) = 0$. We therefore have $\lim_{n\to\infty} X_n < +\infty$ with Theorem 2.4, which means

$$\lim_{n\to\infty}\left(C^*\right)^{n\Delta t}|Y_n|^2 \le \lim_{n\to\infty} X_n < +\infty \tag{3.14}$$

by (3.12). Choose the $\gamma > 0$, such that $C^* = e^\gamma$ and hence

$$\lim_{n\to\infty} e^{\gamma n\Delta t}|Y_n|^2 < +\infty. \tag{3.15}$$

We therefore obtain

$$\lim_{n\to\infty}\frac{1}{n\Delta t}\log|Y_n| \le -\frac{\gamma}{2}, \quad \text{a.s.} \tag{3.16}$$

as required.

Finally, (3.3) also implies (3.10). This completes the proof of Theorem 3.1. □

Next, we state how to choose a parameter α and the stage number ν to obtain almost surely stable numerical solution based on Theorem 3.1.

Corollary 3.2 *Suppose that conditions of Lemma* 2.3 *are satisfied. The approximate solution* (2.1) *applied to test model* (3.2) *is almost surely exponentially stable if we choose the parameter* $\alpha = -\frac{\nu^2}{2a\Delta t}$ *and* $a\Delta t$ *satisfies*

$$-2\nu^2 < a\Delta t \le -\nu^2 \quad \textit{and} \quad T_\nu\left(1+\frac{a\Delta t}{\nu^2}\right) \neq \pm 1,$$

where stage number $\nu \ge 2$.

Proof The inequality $a\Delta t \le -\nu^2$ guarantees that $-\frac{\nu^2}{2a\Delta t} \in (0,1/2]$, hence choosing $\alpha = -\frac{\nu^2}{2a\Delta t}$ satisfies the definition of the S-ROCK method and also simplifies the left hand side of (3.3) into $[T_\nu(1+\frac{a\Delta t}{\nu^2})]^2$. Finally, $[T_\nu(1+\frac{a\Delta t}{\nu^2})]^2 < 1$ if $-2\nu^2 < a\Delta t < 0$ and $T_\nu(1+\frac{a\Delta t}{\nu^2}) \neq \pm 1$

such that the inequality (3.3) is valid. This completes the proof of Corollary 3.2 by using Theorem 3.1. □

Now we consider the case $-4 < a\Delta t < 0$.

To guarantee the sufficient condition (3.3), $(1 + 2\frac{a\Delta t}{\nu^2}\alpha)^2$ should be as small as possible such that (3.3) is valid. Therefore, it is a good choice to set $\alpha = 1/2$ and a small ν because $(1 + 2\frac{a\Delta t}{\nu^2}\alpha)^2$ is a monotonically decreasing and continuous function of the parameter α on $[0, 1/2]$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The main idea of this paper was proposed by QG and QG wrote the paper. JZ participated in the proof of Theorem 3.1 and helped to draft the manuscript. All authors read and approved the final manuscript.

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