Isaev *Complex Analysis and its Synergies* 2015, **1**:1
http://www.casjournal.com/content/1/1/1

Complex Analysis and its Synergies
a SpringerOpen Journal

RESEARCH **Open Access**

A complex-analytic proof of a criterion for isomorphism of Artinian Gorenstein algebras

Alexander Isaev

Correspondence:
alexander.isaev@anu.edu.au
Department of Mathematics, The Australian National University, Canberra, ACT 0200, Australia

Abstract

Let A be an Artinian Gorenstein algebra over a field k of characteristic zero with $\dim_k A > 1$. To every such algebra and a linear projection π on its maximal ideal $\mathfrak{m}$ with range equal to the socle $\mathrm{Soc}(A)$ of A, one can associate a certain algebraic hypersurface $S_\pi \subset \mathfrak{m}$, which is the graph of a polynomial map $P_\pi : \ker \pi \to \mathrm{Soc}(A) \simeq k$. Recently, the following surprising criterion has been obtained: two Artinian Gorenstein algebras A and $\widetilde{A}$ are isomorphic if and only if any two hypersurfaces S_π and $S_{\widetilde{\pi}}$ arising from A and $\widetilde{A}$, respectively, are affinely equivalent. In the present paper, we focus on the cases $k = \mathbb{R}$ and $k = \mathbb{C}$ and explain how in these situations the above criterion can be derived by complex-analytic methods. The complex-analytic proof for $k = \mathbb{R}$ has not previously appeared in print but is foundational for the general result. The purpose of our paper is to present this proof and compare it with that for $k = \mathbb{C}$, thus highlighting a curious connection between complex analysis and commutative algebra.

Mathematics Subject Classification: 13H10; 32V40

Keywords: Artinian Gorenstein algebras; CR-automorphisms of real quadrics in complex space

1 Introduction

This paper concerns a result in commutative algebra but is intended mainly for experts in complex analysis and CR-geometry. The background required for understanding the algebraic content is rather minimal and is given in Section 2.

We consider Artinian local commutative associative algebras over a field k. Such an algebra A is Gorenstein if and only if the socle $\mathrm{Soc}(A)$ of A is a one-dimensional vector space over k. Gorenstein algebras frequently occur in various areas of mathematics and its applications to physics (see, e.g., [1],[12]). In the case when the field k has characteristic zero, in articles [6],[11], a surprising criterion for isomorphism of Artinian Gorenstein algebras was found. The criterion is given in terms of a certain algebraic hypersurface S_π in the maximal ideal $\mathfrak{m}$ of A associated to a linear projection π on $\mathfrak{m}$ with range $\mathrm{Soc}(A)$, where we assume that $\dim_k A > 1$. The hypersurface S_π passes through the origin and is the graph of a polynomial map $P_\pi : \ker \pi \to \mathrm{Soc}(A) \simeq k$. It is shown in [6],[11] that two Artinian Gorenstein algebras A and $\widetilde{A}$ are isomorphic if and only if any two hypersurfaces S_π and $S_{\widetilde{\pi}}$ arising from A and $\widetilde{A}$, respectively, are affinely equivalent, that is, there exists a bijective affine map $f : \mathfrak{m} \to \widetilde{\mathfrak{m}}$ such that $f(S_\pi) = S_{\widetilde{\pi}}$.

Currently, there are two different proofs of the above criterion. The one given in [11] is purely algebraic, whereas the one proposed in [6] reduces the case of an arbitrary field to that of $k = \mathbb{C}$. A proof of the criterion in the latter case is contained in our earlier article [7] and, quite surprisingly, is based on a complex-analytic argument. It turns out that one can give an independent complex-analytic proof of the criterion for $k = \mathbb{R}$ as well. This proof has not been previously published and is derived from ideas developed in paper [5], which never appeared in print (see Remark 4.1). However, the argument utilized in the case $k = \mathbb{R}$ is rather important as it inspired our proof for the case $k = \mathbb{C}$ in [7] and therefore laid the foundation of the general result for an arbitrary field. Thus, the purpose of the present article is to provide a complex-analytic proof of the criterion for $k = \mathbb{R}$ and to compare it with that for $k = \mathbb{C}$ given in [7]. These proofs emphasize an intriguing connection between complex analysis and commutative algebra. In each of the two cases, the idea is to consider certain tube submanifolds in complex space associated to the algebras in question and utilize their CR-automorphisms. In fact, as stated in Remark 5.1, there is a deep relationship between Artinian Gorenstein algebras for $k = \mathbb{R}, \mathbb{C}$ and tube hypersurfaces locally CR-equivalent to Levi non-degenerate hyperquadrics. We believe that this relationship has much to offer if fully understood.

The paper is organized as follows. Section 2 contains algebraic preliminaries and the precise statement of the criterion in Theorem 2.1. The proof of the necessity implication of Theorem 2.1 is given in Section 3. This part of the proof has no relation to complex analysis and is only included for the completeness of our exposition. Further, Sections 4 and 5 contain the complex-analytic proofs of the sufficiency implications for $k = \mathbb{R}$ and $k = \mathbb{C}$, respectively. Finally, in Section 6, we demonstrate how powerful our criterion can be in applications. Namely, we apply it to a one-parameter family A_t of 15-dimensional Gorenstein algebras. While directly finding all pairwise isomorphic algebras in the family A_t seems to be quite hard, this problem is easily solved with the help of Theorem 2.1.

2 Preliminaries

Let A be a commutative associative algebra over a field k. We assume that A is unital and denote by $\mathbf{1}$ its multiplicative identity element. Further, we assume that A is *local*, that is, (i) A has a unique maximal ideal (which we denote by $\mathfrak{m}$), and (ii) the natural injective homomorphism $k \to A/\mathfrak{m}$ is in fact an isomorphism. In this case, one has the vector space decomposition $A = \langle \mathbf{1} \rangle \oplus \mathfrak{m}$, where $\langle \cdot \rangle$ denotes linear span. Furthermore, A is isomorphic to the *unital extension* of its maximal ideal $\mathfrak{m}$, which is the direct sum $k \oplus \mathfrak{m}$ endowed with an operation of multiplication in the obvious way. For example, the algebra $\mathcal{O}_0^2$ of germs of holomorphic functions at the origin in $\mathbb{C}^2$ is a complex local algebra whose maximal ideal consists of all germs vanishing at 0. Clearly, every element of $\mathcal{O}_0^2$ is the sum of the germ of a constant function and a germ vanishing at the origin.

Next, we suppose that $\dim_k A > 1$ and that A is *Artinian*, i.e., $\dim_k A < \infty$. In addition, we let A to be *Gorenstein*, which means that the *socle* of A, defined as $\mathrm{Soc}(A) := \{x \in \mathfrak{m} : x\,\mathfrak{m} = 0\}$, is a one-dimensional vector space over k (see, e.g., [9]). For example, if I is the ideal in $\mathcal{O}_0^2$ generated by the germs of z_1^2, z_2^2, then $\mathcal{A} := \mathcal{O}_0^2/I$ is a complex Artinian Gorenstein algebra with $\dim_{\mathbb{C}} \mathcal{A} = 4$ and $\mathrm{Soc}(\mathcal{A})$ spanned by the element of $\mathcal{A}$ represented by the germ of $z_1 z_2$.

We now assume that the field k has characteristic zero and consider the *exponential map* $\exp : \mathfrak{m} \to \mathbf{1} + \mathfrak{m}$

$$\exp(x) := \sum_{m=0}^{\infty} \frac{1}{m!} x^m,$$

where $x^0 := \mathbf{1}$. By Nakayama's lemma (see, e.g., Theorem 2.2 on p. 8 in [13]), $\mathfrak{m}$ is a nilpotent algebra, and therefore the above sum is in fact finite, with the highest-order term corresponding to $m = \nu$, where $\nu \geq 1$ is the *socle degree* of A, i.e., the largest among all integers μ for which $\mathfrak{m}^\mu \neq 0$. Observe that $\mathrm{Soc}(A) = \mathfrak{m}^\nu$. The map exp is bijective with the inverse given by the polynomial transformation

$$\log(\mathbf{1} + x) := \sum_{m=1}^{\nu} \frac{(-1)^{m+1}}{m} x^m, \quad x \in \mathfrak{m}. \tag{2.1}$$

Fix a linear projection π on A with range $\mathrm{Soc}(A)$ and kernel containing $\mathbf{1}$ (we call such projections *admissible*). Set $\mathcal{K} := \ker \pi \cap \mathfrak{m}$ and let S_π be the hypersurface in $\mathfrak{m}$ given as the graph of the polynomial map $P_\pi : \mathcal{K} \to \mathrm{Soc}(A)$ of degree ν defined as follows:

$$P_\pi(x) := \pi(\exp(x)) = \pi\left(\sum_{m=2}^{\nu} \frac{1}{m!} x^m\right), \quad x \in \mathcal{K} \tag{2.2}$$

(note that for $\dim_k A = 2$, one has $P_\pi = 0$). Observe that the $\mathrm{Soc}(A)$-valued quadratic part of P_π is non-degenerate on $\mathcal{K}$ since the $\mathrm{Soc}(A)$-valued bilinear form

$$b_\pi(a, c) := \pi(ac), \quad a, c \in A \tag{2.3}$$

is non-degenerate on A (see, e.g., p. 11 in [8]). Numerous examples of hypersurfaces S_π explicitly computed for particular algebras can be found in [2],[6],[7] (see also Section 6 below).

We will now state the criterion for isomorphism of Artinian Gorenstein algebras obtained in [6],[11].

Theorem 2.1. *Let A and $\widetilde{A}$ be Gorenstein algebras of finite vector space dimension greater than 1 over a field of characteristic zero and π and $\widetilde{\pi}$ admissible projections on A and $\widetilde{A}$, respectively. Then A and $\widetilde{A}$ are isomorphic if and only if the hypersurfaces S_π and $S_{\widetilde{\pi}}$ are affinely equivalent.*

Remark 2.2. For every hypersurface S_π, we let $\mathcal{S}_\pi$ be the graph over $\mathcal{K}$ of the polynomial map $-P_\pi$ (see (2.2)). Observe that

$$\mathcal{S}_\pi = \{x \in \mathfrak{m} : \pi(\exp(x)) = 0\}. \tag{2.4}$$

Clearly, S_π and $S_{\widetilde{\pi}}$ are affinely equivalent if and only if $\mathcal{S}_\pi$ and $\mathcal{S}_{\widetilde{\pi}}$ are affinely equivalent. Therefore, in order to prove Theorem 2.1, it is sufficient to obtain its statement with $\mathcal{S}_\pi$ and $\mathcal{S}_{\widetilde{\pi}}$ in place of S_π and $S_{\widetilde{\pi}}$, respectively. The hypersurfaces $\mathcal{S}_\pi$ and $\mathcal{S}_{\widetilde{\pi}}$ are easier to work with, and we utilize them instead of S_π and $S_{\widetilde{\pi}}$ in our proofs below.

3 Proof of the necessity implication

First of all, we explain how the necessity implication of Theorem 2.1 is derived. As stated in the introduction, this part of the proof has no relation to complex analysis and is only included in the paper for the completeness of our exposition. The proof below works for any field of characteristic zero. The idea is to show that if π_1 and π_2 are admissible projections on A, then $\mathcal{S}_{\pi_1} = \mathcal{S}_{\pi_2} + x_0$ for some $x_0 \in \mathfrak{m}$. Clearly, the necessity implication is a consequence of this fact (cf. [7]).

For every $y \in \mathfrak{m}$, let M_y be the multiplication operator from A to $\mathfrak{m}$ defined by $a \mapsto ya$ and set $\mathcal{K}_1 := \ker \pi_1 \cap \mathfrak{m}$. The correspondence $y \mapsto \pi_1 \circ M_y|_{\mathcal{K}_1}$ defines a linear map $\mathcal{L}$ from $\mathcal{K}_1$ into the space $L(\mathcal{K}_1, \mathrm{Soc}(A))$ of linear maps from $\mathcal{K}_1$ to $\mathrm{Soc}(A)$. Since for every admissible projection π the form b_π defined in (2.3) is non-degenerate on A and since $\dim_k L(\mathcal{K}_1, \mathrm{Soc}(A)) = \dim_k \mathcal{K}_1$, it follows that $\mathcal{L}$ is an isomorphism.

Next, let $\lambda := \pi_2 - \pi_1$ and observe that $\lambda(\mathbf{1}) = 0$, $\lambda(\mathrm{Soc}(A)) = 0$. Clearly, $\lambda|_{\mathcal{K}_1}$ lies in $L(\mathcal{K}_1, \mathrm{Soc}(A))$, and therefore there exists $y_0 \in \mathcal{K}_1$ such that $\lambda|_{\mathcal{K}_1} = \pi_1 \circ M_{y_0}|_{\mathcal{K}_1}$. We then have $\lambda = \pi_1 \circ M_{y_0}$ everywhere on A, hence

$$\pi_2(\exp(x)) = \pi_1\left((\mathbf{1} + y_0)\exp(x)\right)) = \pi_1(\exp(x + x_0))$$

for $x_0 := \log(\mathbf{1} + y_0)$, which implies $\mathcal{S}_{\pi_1} = \mathcal{S}_{\pi_2} + x_0$ as claimed. □

4 Proof of the sufficiency implication for $k = \mathbb{R}$

By assumption, $\dim_{\mathbb{R}} A = \dim_{\mathbb{R}} \widetilde{A}$, and we denote this common dimension by N. If $N = 2$, then A and $\widetilde{A}$ are clearly isomorphic, and thus, from now on, we suppose that $N > 2$.

Let $A^{\mathbb{C}} = A \oplus iA$ be the complexification of the real algebra A. Then $\dim_{\mathbb{C}} A^{\mathbb{C}} = N$, and $A^{\mathbb{C}}$ is a complex Artinian Gorenstein algebra with maximal ideal $\mathfrak{m}^{\mathbb{C}} = \mathfrak{m} \oplus i\mathfrak{m}$. Next, we denote by $\pi^{\mathbb{C}}$ the complex-linear extension of π to $A^{\mathbb{C}}$ and by $z \mapsto \bar{z} := x - iy$ the conjugation on $A^{\mathbb{C}}$ defining the real form A, for all $z = x + iy \in A^{\mathbb{C}}$, with $x, y \in A$. Then $h(z, z') := \pi^{\mathbb{C}}(z\bar{z}')$ is a $\mathrm{Soc}(A^{\mathbb{C}})$-valued Hermitian form on $A^{\mathbb{C}}$ that coincides with b_π on A (see (2.3)). Since the bilinear form b_π is non-degenerate on A, the Hermitian form h is non-degenerate on $A^{\mathbb{C}}$.

Consider the following real Levi non-degenerate hyperquadric in the complex projective space $\mathbb{P}(A^{\mathbb{C}})$:

$$\mathcal{Q} := \left\{ [z] \in \mathbb{P}\left(A^{\mathbb{C}}\right) : h(z, z) = 0 \right\},$$

where $[z]$ denotes the point of $\mathbb{P}(A^{\mathbb{C}})$ represented by $z \in A^{\mathbb{C}}$. We think of $\mathbf{1} + \mathfrak{m}^{\mathbb{C}}$ as the affine part of $\mathbb{P}(A^{\mathbb{C}})$ and of

$$\mathcal{Q}' := \mathcal{Q} \cap \left(\mathbf{1} + \mathfrak{m}^{\mathbb{C}}\right)$$

as the affine part of the hyperquadric $\mathcal{Q}$. One can choose complex coordinates $w_1, \ldots, w_{N-1}$ in $\mathfrak{m}^{\mathbb{C}}$ so that, upon identification of $\mathbf{1} + \mathfrak{m}^{\mathbb{C}}$ with $\mathfrak{m}^{\mathbb{C}}$, the affine quadric $\mathcal{Q}'$ is given by the equation

$$\mathrm{Re} w_{N-1} = H(w, w), \tag{4.1}$$

where $w := (w_1, \ldots, w_{N-2})$ and H is a non-degenerate Hermitian form on $\mathbb{C}^{N-2}$.

Next, consider the following real tube hypersurface in $\mathfrak{m}^{\mathbb{C}}$:

$$T := \mathcal{S}_\pi + i\mathfrak{m}. \tag{4.2}$$

Let $\exp^{\mathbb{C}} : \mathfrak{m}^{\mathbb{C}} \to \mathbf{1} + \mathfrak{m}^{\mathbb{C}}$ be the exponential map associated to $A^{\mathbb{C}}$. It is straightforward to check that the biholomorphic transformation from $\mathfrak{m}^{\mathbb{C}}$ to $\mathbf{1} + \mathfrak{m}^{\mathbb{C}}$ given by

$$z \mapsto \exp^{\mathbb{C}}\left(\frac{z}{2}\right)$$

maps T onto $\mathcal{Q}'$.

Analogously, for the algebra $\widetilde{A}$, we obtain a Hermitian $\mathrm{Soc}\left(\widetilde{A}^{\mathbb{C}}\right)$-valued form $\widetilde{h}$ on $\widetilde{A}^{\mathbb{C}}$, a real Levi non-degenerate hyperquadric $\widetilde{\mathcal{Q}}$ in $\mathbb{P}\left(\widetilde{A}^{\mathbb{C}}\right)$, the corresponding affine hyperquadric $\widetilde{\mathcal{Q}}'$ in $\widetilde{\mathbf{1}}+\widetilde{\mathfrak{m}}^{\mathbb{C}}$, and a tube hypersurface $\widetilde{T}$ in $\widetilde{\mathfrak{m}}^{\mathbb{C}}$.

Now, let $f:\mathfrak{m}\to\widetilde{\mathfrak{m}}$ be a bijective affine map that establishes equivalence between $\mathcal{S}_\pi$ and $\mathcal{S}_{\widetilde{\pi}}$. We extend it to a complex affine map $f^{\mathbb{C}}:\mathfrak{m}^{\mathbb{C}}\to\widetilde{\mathfrak{m}}^{\mathbb{C}}$. Clearly, $f^{\mathbb{C}}$ transforms T into $\widetilde{T}$. Consider the biholomorphism from $\mathbf{1}+\mathfrak{m}^{\mathbb{C}}$ to $\widetilde{\mathbf{1}}+\widetilde{\mathfrak{m}}^{\mathbb{C}}$ defined as follows:

$$\Phi:=\widetilde{\exp}^{\mathbb{C}}\circ\left(z\mapsto\frac{z}{2}\right)\circ f^{\mathbb{C}}\circ(z\mapsto 2z)\circ\log^{\mathbb{C}}, \tag{4.3}$$

where $\log^{\mathbb{C}}:=\left(\exp^{\mathbb{C}}\right)^{-1}$ and $\widetilde{\exp}^{\mathbb{C}}$ is the exponential map associated to $\widetilde{A}^{\mathbb{C}}$. Observe that Φ maps $\mathcal{Q}'$ onto $\widetilde{\mathcal{Q}}'$.

We will now show that, upon identification of $\mathbf{1}+\mathfrak{m}^{\mathbb{C}}$ with $\mathfrak{m}^{\mathbb{C}}$ and $\widetilde{\mathbf{1}}+\widetilde{\mathfrak{m}}^{\mathbb{C}}$ with $\widetilde{\mathfrak{m}}^{\mathbb{C}}$, the map Φ is affine. Indeed, since $\mathcal{Q}'$ and $\widetilde{\mathcal{Q}}'$ are biholomorphically equivalent, the signatures of their Levi forms coincide. Therefore, one can choose complex coordinates $\widetilde{w}_1,\dots,\widetilde{w}_{N-1}$ in $\widetilde{\mathfrak{m}}^{\mathbb{C}}$ so that, upon identification of $\widetilde{\mathbf{1}}+\widetilde{\mathfrak{m}}^{\mathbb{C}}$ with $\widetilde{\mathfrak{m}}^{\mathbb{C}}$, the affine quadric $\widetilde{\mathcal{Q}}'$ is given by the equation

$$\mathrm{Re}\widetilde{w}_{N-1}=H(\widetilde{w},\widetilde{w}),$$

where $\widetilde{w}:=(\widetilde{w}_1,\dots,\widetilde{w}_{N-2})$ (cf. (4.1)). Thus, when written in the coordinates (w,w_{N-1}), $(\widetilde{w},\widetilde{w}_{N-1})$, the map Φ becomes an automorphism of $\mathbb{C}^{N-1}$ preserving quadric (4.1). It is well-known that every such transformation has the form

$$\begin{aligned}\widetilde{w}\quad &=\lambda Uw+a,\\ \widetilde{w}_{N-1}&=\sigma\lambda^2 w_{N-1}+2\lambda H(Uw,a)+H(a,a)+ib,\end{aligned}$$

where $U\in\mathrm{GL}(N-2,\mathbb{C})$ satisfies $H(Uw,Uw)\equiv\sigma H(w,w)$, $a\in\mathbb{C}^{N-2}$, $b\in\mathbb{R}^{N-2}$, $\lambda>0$, $\sigma=\pm1$, and σ may be equal to -1 only if the numbers of positive and negative eigenvalues of H are equal. In particular, Φ is an affine map as claimed.

By formulas (2.1) and (4.3), for $x\in\mathfrak{m}$ we have

$$\begin{aligned}\Phi(\mathbf{1}+x)\;=\;&\widetilde{\exp}\left(\frac{x_0}{2}\right)\Big(\widetilde{\mathbf{1}}+g(x)+\frac{1}{2}\left(g(x)^2-g(x^2)\right)+\\ &\text{higher-order terms}\Big),\end{aligned} \tag{4.4}$$

where $x_0:=f(0)$, $g:=f-x_0$ is the linear part of f, and $\widetilde{\exp}$ is the exponential map associated to $\widetilde{A}$. Since Φ is affine, formula (4.4) implies $g(x)^2=g(x^2)$ for all $x\in\mathfrak{m}$, which is equivalent to the statement that $g:\mathfrak{m}\to\widetilde{\mathfrak{m}}$ is an algebra isomorphism. Therefore, $\mathfrak{m}$ and $\widetilde{\mathfrak{m}}$ are isomorphic, hence A and $\widetilde{A}$ are also isomorphic as required. □

Remark 4.1. The method utilized in the above proof can be extracted, in principle, from ideas contained in paper [5] (which should be read in conjunction with [4]), but is by no means explicitly articulated there.

5 Proof of the sufficiency implication for $k=\mathbb{C}$

The argument that follows is contained in [7] and is included in this paper for comparison with the proof in the case $k=\mathbb{R}$ given in Section 4.

By assumption, $\dim_{\mathbb{C}}A=\dim_{\mathbb{C}}\widetilde{A}$, and we denote this common dimension by N. If $N=2$, then A and $\widetilde{A}$ are isomorphic, and thus, from now on, we suppose that $N>2$.

Consider the maximal ideal $\mathfrak{m}$ of A. We will now forget the complex structure on $\mathfrak{m}$ and treat it as a real algebra. Let $\mathcal{A}$ be the (real) unital extension of $\mathfrak{m}$ (see Section 2) and *1* the multiplicative identity element in $\mathcal{A}$. Observe that $\mathcal{A}$ is not Gorenstein since

$\dim_{\mathbb{R}} \mathrm{Soc}(\mathcal{A}) = 2$. We now consider the complexification $\mathcal{A}^{\mathbb{C}} = \mathcal{A} \oplus i\mathcal{A}$ of $\mathcal{A}$. Then $\dim_{\mathbb{C}} \mathcal{A}^{\mathbb{C}} = 2N - 1$, and $\mathcal{A}^{\mathbb{C}}$ is a complex Artinian local algebra with maximal ideal $\mathfrak{m}^{\mathbb{C}} = \mathfrak{m} \oplus i\mathfrak{m}$. The complex algebra $\mathcal{A}^{\mathbb{C}}$ is not Gorenstein since $\dim_{\mathbb{C}} \mathrm{Soc}(\mathcal{A}^{\mathbb{C}}) = 2$.

Next, let $\hat{\pi}$ be the extension of $\pi|_{\mathfrak{m}}$ to $\mathcal{A}$ defined by the condition $\hat{\pi}(1) = 0$ and denote by $\pi^{\mathbb{C}}$ the complex-linear extension of $\hat{\pi}$ to $\mathcal{A}^{\mathbb{C}}$. Further, denote by $z \mapsto \bar{z} := x - iy$ the conjugation on $\mathcal{A}^{\mathbb{C}}$ defining the real form $\mathcal{A}$, for all $z = x + iy \in \mathcal{A}^{\mathbb{C}}$, with $x, y \in \mathcal{A}$. Then $h(z, z') := \pi^{\mathbb{C}}(z\bar{z}')$ is a $\mathrm{Soc}(\mathcal{A}^{\mathbb{C}})$-valued Hermitian form on $\mathcal{A}^{\mathbb{C}}$ that coincides on $\mathcal{A}$ with the $\mathrm{Soc}(\mathcal{A})$-valued bilinear form $b_{\hat{\pi}}$ defined analogously to (2.3).

Consider the following subset of the complex projective space $\mathbb{P}(\mathcal{A}^{\mathbb{C}})$:

$$\mathcal{Q} := \left\{ [z] \in \mathbb{P}(\mathcal{A}^{\mathbb{C}}) : h(z,z) = 0 \right\},$$

where $[z]$ denotes the point of $\mathbb{P}(\mathcal{A}^{\mathbb{C}})$ represented by $z \in \mathcal{A}^{\mathbb{C}}$. We think of $1 + \mathfrak{m}^{\mathbb{C}}$ as the affine part of $\mathbb{P}(\mathcal{A}^{\mathbb{C}})$ and of

$$\mathcal{Q}' := \mathcal{Q} \cap (1 + \mathfrak{m}^{\mathbb{C}})$$

as the affine part of $\mathcal{Q}$. Observe that $\mathcal{Q}'$ is a real-analytic Levi non-degenerate CR-submanifold of $1+\mathfrak{m}^{\mathbb{C}}$ of real codimension 2. In fact, one can choose complex coordinates $w_1, \dots, w_{2N-2}$ in $\mathfrak{m}^{\mathbb{C}}$ so that, upon identification of $1+\mathfrak{m}^{\mathbb{C}}$ with $\mathfrak{m}^{\mathbb{C}}$, the affine quadric $\mathcal{Q}'$ is given by the equations

$$\begin{aligned} \mathrm{Re} w_{2N-3} &= \sum_{j=1}^{N-2} \left(|w_j|^2 - |w_{j+N-2}|^2 \right), \\ \mathrm{Re} w_{2N-2} &= \sum_{j=1}^{N-2} \left(w_j \overline{w}_{j+N-2} + w_{j+N-2} \overline{w}_j \right). \end{aligned} \tag{5.1}$$

This can be seen by choosing coordinates in $\mathfrak{m}$ (regarded as a complex algebra) in which the restriction of the $\mathrm{Soc}(A)$-valued bilinear form b_π to $\mathcal{K} = \ker \pi \cap \mathfrak{m}$ is given by the identity matrix.

Next, consider the following real tube codimension 2 submanifold in $\mathfrak{m}^{\mathbb{C}}$:

$$T := \mathcal{S}_\pi + i\mathfrak{m}.$$

Let $\exp^{\mathbb{C}} : \mathfrak{m}^{\mathbb{C}} \to 1 + \mathfrak{m}^{\mathbb{C}}$ be the exponential map associated to $\mathcal{A}^{\mathbb{C}}$. It is straightforward to check that the biholomorphic transformation from $\mathfrak{m}^{\mathbb{C}}$ to $1 + \mathfrak{m}^{\mathbb{C}}$ given by

$$z \mapsto \exp^{\mathbb{C}} \left(\frac{z}{2} \right)$$

maps T onto $\mathcal{Q}'$.

Analogously, for the algebra $\widetilde{A}$, we obtain algebras $\widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{A}}^{\mathbb{C}}$, a Hermitian $\mathrm{Soc}\left(\widetilde{\mathcal{A}}^{\mathbb{C}}\right)$-valued form $\widetilde{h}$ on $\widetilde{\mathcal{A}}^{\mathbb{C}}$, a real Levi non-degenerate codimension 2 affine quadric $\widetilde{\mathcal{Q}}'$ in $\widetilde{1} + \widetilde{\mathfrak{m}}^{\mathbb{C}}$, the corresponding coordinates $\widetilde{w}_1, \dots, \widetilde{w}_{2N-2}$ in $\widetilde{\mathfrak{m}}^{\mathbb{C}}$, and a tube hypersurface $\widetilde{T}$ in $\widetilde{\mathfrak{m}}^{\mathbb{C}}$.

Now, let $f : \mathfrak{m} \to \widetilde{\mathfrak{m}}$ be a bijective affine map that establishes equivalence between $\mathcal{S}_\pi$ and $\mathcal{S}_{\widetilde{\pi}}$. We treat f as a real affine map and extend it to a complex affine map $f^{\mathbb{C}} : \mathfrak{m}^{\mathbb{C}} \to \widetilde{\mathfrak{m}}^{\mathbb{C}}$. Notice that $f^{\mathbb{C}}$ transforms T into $\widetilde{T}$. Consider the biholomorphism from $1 + \mathfrak{m}^{\mathbb{C}}$ to $\widetilde{1} + \widetilde{\mathfrak{m}}^{\mathbb{C}}$ defined as follows:

$$\Phi := \widetilde{\exp}^{\mathbb{C}} \circ \left(z \mapsto \frac{z}{2} \right) \circ f^{\mathbb{C}} \circ (z \mapsto 2z) \circ \log^{\mathbb{C}}, \tag{5.2}$$

where $\log^{\mathbb{C}} := \left(\exp^{\mathbb{C}}\right)^{-1}$ and $\widetilde{\exp}^{\mathbb{C}}$ is the exponential map associated to $\widetilde{\mathcal{A}}^{\mathbb{C}}$. Observe that Φ maps $\mathcal{Q}'$ onto $\widetilde{\mathcal{Q}}'$.

We claim that, upon identification of $\mathcal{I} + \mathfrak{m}^{\mathbb{C}}$ with $\mathfrak{m}^{\mathbb{C}}$ and $\widetilde{\mathcal{I}} + \widetilde{\mathfrak{m}}^{\mathbb{C}}$ with $\widetilde{\mathfrak{m}}^{\mathbb{C}}$, the map Φ is affine. Indeed, when written in the coordinates $w_1, \dots, w_{2N-2}, \widetilde{w}_1, \dots, \widetilde{w}_{2N-2}$, the map Φ becomes an automorphism of $\mathbb{C}^{2N-2}$ preserving quadric (5.1). The fact that Φ is affine now follows from a description of CR-automorphism of this quadric (see the elliptic case on pp. 37–38 in [3]).

By formulas (2.1) and (5.2), for $x \in \mathfrak{m}$ we have

$$\Phi(\mathcal{I} + x) = \widetilde{\exp}\left(\frac{x_0}{2}\right)\Big(\widetilde{\mathcal{I}} + g(x) + \frac{1}{2}\left(g(x)^2 - g(x^2)\right) + \text{higher-order terms}\Big), \tag{5.3}$$

where $x_0 := f(0)$, $g := f - x_0$ is the linear part of f, and $\widetilde{\exp}$ is the exponential map associated to $\widetilde{\mathcal{A}}$. Since Φ is affine, formula (5.3) implies $g(x)^2 = g(x^2)$ for all $x \in \mathfrak{m}$, i.e., $g : \mathfrak{m} \to \widetilde{\mathfrak{m}}$ is an algebra isomorphism. Therefore, $\mathfrak{m}$ and $\widetilde{\mathfrak{m}}$ are isomorphic, hence A and $\widetilde{A}$ are also isomorphic as required. □

Remark 5.1. The proofs of Theorem 2.1 for the cases $k = \mathbb{R}$, $\mathbb{C}$ presented in this paper are based on considering real tube submanifolds in complex space CR-equivalent to Levi non-degenerate affine quadrics. For $k = \mathbb{R}$, we utilized hypersurfaces, whereas for $k = \mathbb{C}$, codimension 2 submanifolds were required. The former are called *spherical tube hypersurfaces* (see [10]), and there is in fact an intriguing relationship between them and real and complex Artinian Gorenstein algebras. This relationship was outlined in [5] (see also Section 9.2 in [10] for a brief survey). It turns out that

- to every real Artinian Gorenstein algebra of dimension greater than 2 and to every complex Artinian Gorenstein algebra one can associate a (closed) spherical tube hypersurface, where in the real case one obtains exactly hypersurfaces with bases (2.4) as in (4.2);
- any two such hypersurfaces are affinely equivalent if and only if the corresponding algebras are isomorphic (one way to obtain the necessity implication in the real case is to proceed as in Section 4);
- in a certain sense, all spherical tube hypersurfaces can be obtained by combining hypersurfaces of these two types.

We note that no higher-codimensional analogues of spherical tube hypersurfaces were considered in [5]. However, as the proof of Theorem 2.1 for $k = \mathbb{C}$ suggests, analogues of this kind are related to Artinian Gorenstein algebras as well. We believe that the curious connection between complex analysis and commutative algebra manifested through tube submanifolds CR-equivalent to affine quadrics deserves further investigation.

6 Example of application of Theorem 2.1

Theorem 2.1 is particularly useful when at least one of the hypersurfaces S_π and $S_{\widetilde{\pi}}$ is affinely homogeneous (recall that a subset S of a vector space V is called affinely homogeneous if for every pair of points $p, q \in S$ there exists a bijective affine map g of V such that $g(S) = S$ and $g(p) = q$). In this case, the hypersurfaces S_π and $S_{\widetilde{\pi}}$ are affinely equivalent if and only if they are linearly equivalent. Indeed, if, for instance, S_π is affinely homogeneous and $f : \mathfrak{m} \to \widetilde{\mathfrak{m}}$ is an affine equivalence between S_π and $S_{\widetilde{\pi}}$, then $f \circ g$ is a linear equivalence between S_π and $S_{\tilde{\pi}}$, where g is an affine automorphism of S_π such that $g(0) = f^{-1}(0)$. Clearly, in this case, $S_{\tilde{\pi}}$ is affinely homogeneous as well.

The proof of Theorem 2.1 in [11] shows that every linear equivalence f between S_π and $S_{\tilde{\pi}}$ has the block-diagonal form with respect to the decompositions $\mathfrak{m} = \mathcal{K} \oplus \mathrm{Soc}(A)$ and $\tilde{\mathfrak{m}} = \tilde{\mathcal{K}} \oplus \mathrm{Soc}(\tilde{A})$, where $\tilde{\mathcal{K}} := \ker \tilde{\pi} \cap \tilde{\mathfrak{m}}$, that is, there exist linear isomorphisms $f_1 : \mathcal{K} \to \tilde{\mathcal{K}}$ and $f_2 : \mathrm{Soc}(A) \to \mathrm{Soc}(\tilde{A})$ such that $f(x+y) = f_1(x) + f_2(y)$, with $x \in \mathcal{K}$, $y \in \mathrm{Soc}(A)$. Therefore, for the corresponding polynomial maps P_π and $P_{\tilde{\pi}}$ (see (2.2)), we have

$$f_2 \circ P_\pi^{[m]} = P_{\tilde{\pi}}^{[m]} \circ f_1 \quad \text{for all } m \geq 2, \tag{6.1}$$

where $P_\pi^{[m]}$ and $P_{\tilde{\pi}}^{[m]}$ are the homogeneous components of degree m of P_π and $P_{\tilde{\pi}}$, respectively.

Thus, Theorem 2.1 yields the following corollary (cf. Theorem 2.11 in [7]) .

Corollary 6.1. *Let A and $\tilde{A}$ be Gorenstein algebras of finite vector space dimension greater than 1 over a field of characteristic zero and π and $\tilde{\pi}$ admissible projections on A and $\tilde{A}$, respectively.*

(i) If A and $\tilde{A}$ are isomorphic and at least one of S_π and $S_{\tilde{\pi}}$ is affinely homogeneous, then for some linear isomorphisms $f_1 : \mathcal{K} \to \tilde{\mathcal{K}}$ and $f_2 : \mathrm{Soc}(A) \to \mathrm{Soc}(\tilde{A})$ identity (6.1) holds. In this case, both S_π and $S_{\tilde{\pi}}$ are affinely homogeneous.

(ii) If for some linear isomorphisms $f_1 : \mathcal{K} \to \tilde{\mathcal{K}}$ and $f_2 : \mathrm{Soc}(A) \to \mathrm{Soc}(\tilde{A})$ identity (6.1) holds, then the hypersurfaces S_π and $S_{\tilde{\pi}}$ are linearly equivalent, and therefore the algebras A and $\tilde{A}$ are isomorphic.

In [11] (see also Corollary 4.10 in [6]), we found a criterion for the affine homogeneity of some (hence every) hypersurface S_π arising from an Artinian Gorenstein algebra A. Namely, S_π is affinely homogeneous if and only if the action of the automorphism group of the algebra $\mathfrak{m}$ on the set of all hyperplanes in $\mathfrak{m}$ complementary to $\mathrm{Soc}(A)$ is transitive. Furthermore, we showed (see also Corollary 4.11 in [6]) that this condition is satisfied if A is non-negatively graded in the sense that it can be represented as a direct sum

$$A = \bigoplus_{j \geq 0} A^j, \quad A^j A^\ell \subset A^{j+\ell},$$

where A^j are linear subspaces of A, with $A^0 = k$ (in this case $\mathfrak{m} = \oplus_{j>0} A^j$ and $\mathrm{Soc}(A) = A^d$ for $d := \max\{j : A^j \neq 0\}$). It then follows that part (i) of Corollary 6.1 applies in the situation when one (hence the other) of the algebras A and $\tilde{A}$ is non-negatively graded (see also [7] for the case $k = \mathbb{C}$). Note, however, that the existence of a non-negative grading on A is not a necessary condition for the affine homogeneity of S_π (see, e.g., Remark 2.6 in [7]). Also, as shown in Section 8.2 in [6], the hypersurface S_π need not be affinely homogeneous in general.

To demonstrate how our method works, we will now apply Corollary 6.1 to a one-parameter family of non-negatively graded Artinian Gorenstein algebras. As before, let k be a field of characteristic zero. For $t \in k$, $t \neq \pm 2$, define

$$A_t := k[x, y] \, / \left(2x^3 + txy^3, tx^2y^2 + 2y^5\right).$$

It is straightforward to verify that every A_t is a Gorenstein algebra of dimension 15. We will prove the following proposition.

Proposition 6.2. *A_r and A_s are isomorphic if and only if $r = \pm s$.*

Proof. The sufficiency implication is trivial (just replace y by $-y$). For the converse implication, consider the following monomials in $k[x, y]$:

$$1,\ x,\ y,\ x^2,\ xy,\ y^2,\ x^2y,\ xy^2,\ y^3,\ xy^3,\ x^2y^2,\ y^4,\ x^2y^3,\ xy^4,\ x^2y^4.$$

Let $e_0 = \mathbf{1}, e_1, \ldots, e_{14}$, respectively, be the vectors in A_t arising from these monomials (to simplify the notation, we do not indicate the dependence of e_j on t). They form a basis of A_t. Define

$$A_t^0 := \langle e_0\rangle,\ A_t^1 := 0,\ A_t^2 := \langle e_2\rangle,\ A_t^3 := \langle e_1\rangle,\ A_t^4 := \langle e_5\rangle,$$

$$A_t^5 := \langle e_4\rangle,\ A_t^6 := \langle e_3, e_8\rangle,\ A_t^7 := \langle e_7\rangle,\ A_t^8 := \langle e_6, e_{11}\rangle,\ A_t^9 := \langle e_9\rangle,$$

$$A_t^{10} := \langle e_{10}\rangle,\ A_t^{11} := \langle e_{13}\rangle,\ A_t^{12} := \langle e_{12}\rangle,\ A_t^{13} := 0,\ A_t^{14} := \langle e_{14}\rangle,$$

$$A_t^j = 0 \quad \text{for } j \geq 15,$$

where, as before, $\langle\,\cdot\,\rangle$ denotes linear span. It is straightforward to check that the subspaces A_t^j form a non-negative grading on A_t.

Next, denote by $\mathfrak{m}_t$ the maximal ideal of A_t and let π_t be the projection on A_t with range $\mathrm{Soc}(A_t) = A_t^{14}$ and kernel $\langle e_0, \ldots, e_{13}\rangle$. Denote by $w_1, \ldots, w_{14}$ the coordinates in $\mathfrak{m}_t$ with respect to the basis $e_1, \ldots, e_{14}$. In these coordinates the corresponding polynomial map $P_t := P_{\pi_t}$ is written as

$$\begin{aligned} P_t = &-\frac{t}{10080}w_2^7 + \frac{1}{48}w_2^4\left(w_1^2 - \frac{t}{5}w_2w_5\right) - \frac{t}{48}w_1^4w_2 + \frac{1}{4}w_1^2w_2^2w_5 + \\ &\frac{1}{6}w_1w_2^3w_4 - \frac{t}{24}w_2^3w_5^2 - \frac{t}{48}w_2^4w_8 + \frac{1}{24}w_2^4w_3 + \text{terms of degree} \leq 4\,. \end{aligned}$$

Suppose that for some $r \neq s$ the algebras A_r and A_s are isomorphic. By part (i) of Corollary 6.1, there exist $C \in \mathrm{GL}(13, k)$ and $c \in k^*$ such that

$$c\,P_r(w) \equiv P_s(Cw), \tag{6.2}$$

where $w := (w_1, \ldots, w_{13})$. Since 0 is the only value of t for which P_t has degree 7, we have $r, s \neq 0$. Comparing the terms of order 7 in identity (6.2), we obtain that the second row in the matrix C has the form $(0, \mu, 0, \ldots, 0)$, and

$$c = \frac{s}{r}\mu^7\,. \tag{6.3}$$

Next, comparing the terms of order 6 in (6.2), we see that the first row in the matrix C has the form $(\sigma, \rho, 0, \ldots, 0)$, and

$$c = \mu^4\sigma^2\,. \tag{6.4}$$

Further, comparing the terms of order 5 in (6.2) that do not involve w_2^2, we obtain

$$c = \frac{s}{r}\mu\sigma^4\,. \tag{6.5}$$

Now, (6.3), (6.4), and (6.5) yield $r^2 = s^2$ as required. □

Acknowledgements
This work is supported by the Australian Research Council.

Received: 5 March 2014 Accepted: 9 March 2014
Published: 3 June 2015

References

1. Bass, H: On the ubiquity of Gorenstein rings. Math. Z. **82**, 8–28 (1963)
2. Eastwood, MG, Isaev, AV: Extracting invariants of isolated hypersurface singularities from their moduli algebras. Math. Ann. **356**, 73–98 (2013)
3. Ezhov, V, Schmalz, G: A matrix Poincaré formula for holomorphic automorphisms of quadrics of higher codimension. Real associative quadrics. J. Geom. Anal. **8**, 27–41 (1998)
4. Fels, G, Kaup, W: Local tube realizations of CR-manifolds and maximal Abelian subalgebras. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5). **10**, 99–128 (2011)
5. Fels, G, Kaup, W: Classification of commutative algebras and tube realizations of hyperquadrics. preprint, available from http://arxiv.org/pdf/0906.5549v2
6. Fels, G, Kaup, W: Nilpotent algebras and affinely homogeneous surfaces. Math. Ann. **353**, 1315–1350 (2012)
7. Fels, G, Isaev, A, Kaup, W, Kruzhilin, N: Isolated hypersurface singularities and special polynomial realizations of affine quadrics. J. Geom. Analysis. **21**, 767–782 (2011)
8. Hertling, C: Frobenius Manifolds and Moduli Spaces for Singularities. Cambridge Tracts in Mathematics 151, Cambridge University Press, Cambridge (2002)
9. Huneke, C: Hyman Bass and ubiquity: Gorenstein rings. In: Algebra, K-theory, Groups, and Education (New York, 1997), Contemp. Math. 243, Amer. Math. Soc., pp. 55–78. Providence, RI (1999)
10. Isaev, AV: Spherical Tube Hypersurfaces. Lecture Notes in Mathematics 2020. Springer, Berlin (2011)
11. Isaev, AV: On the affine homogeneity of algebraic hypersurfaces arising from Gorenstein algebras. Asian J. Math. **15**, 631–640 (2011)
12. Lurie, J: On the classification of topological field theories. In: Current Developments in Mathematics, 2008, pp. 129–280. Int. Press, Somerville, MA, (2009)
13. Matsumura, H: Commutative Ring Theory. Cambridge Studies in Advanced Mathematics 8, Cambridge University Press, Cambridge (1986)

doi:10.1186/2197-120X-1-1
Cite this article as: Isaev: A complex-analytic proof of a criterion for isomorphism of Artinian Gorenstein algebras. *Complex Analysis and its Synergies* 2015 **1**:1.