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Bifurcation from interval at infinity for discrete eigenvalue problems which are not linearizable

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Abstract
In this paper, we are concerned with the bifurcation from infinity for a class of discrete eigenvalue problems with nonlinear boundary conditions which are not linearizable and give a description of the behavior of the bifurcation components.
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Keywords: difference equation; nonlinear boundary condition; bifurcation from interval; positive solutions

1 Introduction

For over a decade, there has been significant interest in positive solutions and multiple positive solutions for boundary value problems for finite difference equations; see, for example, [1–11]. Much of this interest has been spurred on by the applicability of the topological method such as the upper and lower solutions technique [2], a number of new fixed point theorems and multiple fixed point theorems [3–7] as applied to certain discrete boundary value problems. Quite recently, Rodríguez [9] and Ma [10] have given a topological proof and used a bifurcation theorem to study the structure of positive solutions of a difference equation. In this paper, we demonstrate a bifurcation technique that takes advantage of dealing with the discrete eigenvalue problem with nonlinear boundary conditions

$$\begin{aligned} &-\Delta\bigl[p(k-1)\Delta y(k-1)\bigr]+q(k)y(k)=\lambda a(k)f\bigl(y(k)\bigr), \quad k\in I,\\ &-\Delta y(0)+\alpha g\bigl(y(0)\bigr)=0, \qquad \Delta y(N)+\beta g\bigl(y(N+1)\bigr)=0, \end{aligned} \tag{1.1}$$

which has different asymptotic linearizations at infinity. Here $I=\{1,2,\dots,N\}$, Δ is a forward difference operator with $\Delta y(k)=y(k+1)-y(k)$, $\alpha,\beta\geq 0$ are constants, $\lambda>0$ is a parameter, the functions $p:\{0,1,\dots,N\}\to(0,\infty)$, $q,a:I\to[0,\infty)$ with $a(k)>0$ on $k\in I$ and functions f,g satisfy $f\in C^1([0,\infty))$ and $g\in C^1([0,\infty))$.

Since problem (1.1) has different linearizations at infinity, the standard global bifurcation results [12] are not immediately applicable. However, Schmitt [13] and Peitgen and Schmitt [14] obtained a theorem on bifurcation from intervals at infinity. We can use this theorem to discuss the bifurcation from infinity for problem (1.1) and obtain further information on the location and behavior of the bifurcating sets of solutions.

We will make the following assumptions:

(H1) $p : \{0, 1, \dots, N\} \to (0, \infty)$, $q, a : I \to [0, \infty)$ with $a(k) > 0$ on $k \in I$, $q \geq 0$ and $q \not\equiv 0$.
(H2) $f \in C^1([0, \infty))$ and there exist constants $f_\infty, f^\infty \in (0, \infty)$ and functions $h_1, h_2 \in C^1([0, \infty))$ such that

$$f_\infty = \liminf_{s\to\infty} \frac{f(s)}{s}, \qquad f^\infty = \limsup_{s\to\infty} \frac{f(s)}{s}, \tag{1.2}$$

$$f_\infty s + h_1(s) \leq f(s) \leq f^\infty s + h_2(s), \quad s \in [0, \infty) \tag{1.3}$$

with

$$h_j(s) = o\bigl(|s|\bigr) \quad \text{as } s \to \infty, j = 1, 2.$$

(H3) $g \in C^1([0, \infty))$ and there exist constants $g_\infty, g^\infty \in (0, \infty)$ and functions $\gamma_1, \gamma_2 \in C^1([0, \infty))$ such that

$$g_\infty = \liminf_{s\to\infty} \frac{g(s)}{s}, \qquad g^\infty = \limsup_{s\to\infty} \frac{g(s)}{s},$$

$$g_\infty s + \gamma_1(s) \leq g(s) \leq g^\infty s + \gamma_2(s), \quad s \in [0, \infty) \tag{1.4}$$

with

$$\gamma_j(s) = o\bigl(|s|\bigr) \quad \text{as } s \to \infty, j = 1, 2.$$

Let $\hat{I} = \{0, 1, \dots, N+1\}$ and $X = \{y | y : \hat{I} \to \mathbb{R}\}$ be the space of all real-valued functions on $\hat{I}$. Then it is a Banach space with the norm $\|y\| = \max\{|y(k)| | k \in \hat{I}\}$.

Definition 1.1 ([13, p.450], [14]) A solution set $\mathcal{S}$ of (1.1) is said to bifurcate from infinity in the interval $[a, b]$ if

(i) the solutions of (1.1) are *a priori* bounded in X for $\lambda = a$ and $\lambda = b$.
(ii) there exists $\{(\mu_n, y_n)\} \subset \mathcal{S}$ such that $\{\mu_n\} \subset [a, b]$ and $\|y_n\| \to \infty$.

Let λ_1 and μ_1 be the first eigenvalues of

$$\begin{aligned} &-\Delta\bigl[p(k-1)\Delta\varphi(k-1)\bigr] + q(k)\varphi(k) = \lambda a(k)\varphi(k), \quad k \in I, \\ &-\Delta\varphi(0) + \alpha g^\infty\varphi(0) = 0, \qquad \Delta\varphi(N) + \beta g^\infty\varphi(N+1) = 0, \end{aligned} \tag{1.5}$$

and

$$\begin{aligned} &-\Delta\bigl[p(k-1)\Delta\psi(k-1)\bigr] + q(k)\psi(k) = \mu a(k)\psi(k), \quad k \in I, \\ &-\Delta\psi(0) + \alpha g_\infty\psi(0) = 0, \qquad \Delta\psi(N) + \beta g_\infty\psi(N+1) = 0, \end{aligned} \tag{1.6}$$

respectively, φ_1 and ψ_1 be the corresponding eigenfunctions of λ_1 and μ_1, respectively, and $\varphi_1, \psi_1 \in X$ be positive and normalized as $\|\varphi_1\| = 1$ and $\|\psi_1\| = 1$.

The main results are the following.

Theorem 1.1 *Assume that* (H1)-(H3) *hold. Then, for any* $\sigma \in (0, \frac{\mu_1}{f^\infty})$, *the interval* $[\frac{\mu_1}{f^\infty} - \sigma, \frac{\lambda_1}{f_\infty} + \sigma]$ *is a bifurcation interval from infinity of* (1.1), *and there exists no bifurcation interval from infinity of* (1.1) *in the set* $(0,\infty) \setminus [\frac{\mu_1}{f^\infty} - \sigma, \frac{\lambda_1}{f_\infty} + \sigma]$. *More precisely, there exists an unbounded, closed and connected component in* $(0,\infty) \times X$, *consisting of positive solutions of* (1.1) *and bifurcating from* $[\frac{\mu_1}{f^\infty} - \sigma, \frac{\lambda_1}{f_\infty} + \sigma] \times \{\infty\}$.

Theorem 1.2 *Assume that* (H1)-(H3) *hold.*

(i) *If*

$$\liminf_{u\to\infty} h_1(u) > \frac{f_\infty}{g^\infty} \limsup_{u\to\infty} \gamma_2(u), \tag{1.7}$$

then the component obtained by Theorem 1.1 *bifurcates into the region* $\lambda < \frac{\lambda_1}{f_\infty}$.

(ii) *If*

$$\limsup_{u\to\infty} h_2(u) < \frac{f^\infty}{g_\infty} \liminf_{u\to\infty} \gamma_1(u), \tag{1.8}$$

then the component obtained by Theorem 1.1 *bifurcates into the region* $\lambda > \frac{\mu_1}{f^\infty}$.

Remark 1.1 Notice that $\mu_1 \le \lambda_1$. Indeed, let (λ_1, φ_1) and (μ_1, ψ_1) satisfy (1.5) and (1.6), respectively. Multiplying (1.5) by ψ_1 and (1.6) by φ_1, summing from $k = 1$ to $k = N$ and subtracting, we have that

$$\begin{aligned}
&(\lambda_1 - \mu_1) \sum_{k=1}^{N} a(k)\varphi_1(k)\psi_1(k) \\
&\quad = \sum_{k=1}^{N} \bigl[\Delta\bigl[p(k-1)\Delta\psi_1(k-1)\bigr]\varphi_1(k) - \Delta\bigl[p(k-1)\Delta\varphi_1(k-1)\bigr]\psi_1(k)\bigr] \\
&\quad = \sum_{k=1}^{N} \bigl[p(k)\bigl[\psi_1(k+1)\varphi_1(k) - \psi_1(k)\varphi_1(k+1)\bigr] \\
&\qquad + p(k-1)\bigl[\psi_1(k-1)\varphi_1(k) - \psi_1(k)\varphi_1(k-1)\bigr]\bigr] \\
&\quad = p(N)\bigl[\psi_1(N+1)\varphi_1(N) - \psi_1(N)\varphi_1(N+1)\bigr] + p(0)\bigl[\psi_1(0)\varphi_1(1) - \psi_1(1)\varphi_1(0)\bigr] \\
&\quad = \bigl[\alpha p(0)\varphi_1(0)\psi_1(0) + \beta p(N)\varphi_1(N+1)\psi_1(N+1)\bigr]\bigl(g^\infty - g_\infty\bigr) \ge 0.
\end{aligned}$$

This together with $\sum_{k=1}^{N} a(k)\varphi_1(k)\psi_1(k) > 0$ implies $\mu_1 \le \lambda_1$.

2 Bifurcation theorem and reduction to a compact operator equation

Our main tools in the proof of Theorems 1.1-1.2 are topological arguments [15] and the global bifurcation theorem for mappings which are not necessary smooth [13, 14].

Let V be a real Banach space and $F : \mathbb{R} \times V \to V$ be completely continuous. Let us consider the equation

$$u = F(\lambda, u). \tag{2.1}$$

Lemma 2.1 ([13, Theorem 1.3.3]) *Let* V *be a Banach space*, $F : \mathbb{R} \times V \to V$ *be completely continuous and* $a, b \in \mathbb{R}$ $(a < b)$ *be such that the solutions of* (2.1) *are a priori bounded in*

V for $\lambda = a$ and $\lambda = b$, i.e., there exists $R > 0$ such that

$$u \neq F(a,u), \qquad u \neq F(b,u) \quad \text{for all } u : \|u\| \geq R.$$

Furthermore, assume that

$$d\big(I - F(a,\cdot), B_R(0), 0\big) \neq d\big(I - F(b,\cdot), B_R(0), 0\big)$$

for $R > 0$ large. Then there exists a closed connected set $\mathcal{C}$ of solutions of (2.1) *that is unbounded in $[a,b] \times V$, and either*

(i) *$\mathcal{C}$ is unbounded in λ direction, or else*
(ii) *there exists an interval $[c,d]$ such that $(a,b) \cap (c,d) = \emptyset$ and $\mathcal{C}$ bifurcates from infinity in $[c,d] \times V$.*

To establish Theorem 1.1, we begin with the reduction of (1.1) to a suitable equation for a compact operator and give some preliminary results.

Let $u_{\bar{\alpha}}(k)$, $v_{\bar{\beta}}(k)$ be the solutions of the initial value problems

$$-\Delta\big[p(k-1)\Delta u_{\bar{\alpha}}(k-1)\big] + q(k)u_{\bar{\alpha}}(k) = 0 \quad \text{for } k \in I,$$
$$u_{\bar{\alpha}}(0) = 1, \qquad \Delta u_{\bar{\alpha}}(0) = \bar{\alpha},$$

and

$$-\Delta\big[p(k-1)\Delta v_{\bar{\beta}}(k-1)\big] + q(k)v_{\bar{\beta}}(k) = 0 \quad \text{for } k \in I,$$
$$v_{\bar{\beta}}(N+1) = 1, \qquad \Delta v_{\bar{\beta}}(N) = -\bar{\beta},$$

respectively, here $\bar{\alpha}, \bar{\beta} \in [0,\infty)$. It is easy to compute and show that

(i) $u_{\bar{\alpha}}(k) = 1 + \bar{\alpha}\sum_{s=0}^{k-1}\frac{p(0)}{p(s)} + \sum_{s=1}^{k-1}(\sum_{j=s}^{k-1}\frac{1}{p(j)})q(s)u_{\bar{\alpha}}(s) > 0$, and u is increasing on $\hat{I}$;
(ii) $v_{\bar{\beta}}(k) = 1 + \bar{\beta}\sum_{s=k}^{N}\frac{p(N)}{p(s)} + \sum_{s=k+1}^{N}(\sum_{j=k+1}^{N-1}\frac{1}{p(j)})q(s)v_{\bar{\beta}}(s) > 0$, and v is decreasing on $\hat{I}$.

Lemma 2.2 *Let $h : I \to \mathbb{R}$. Then the linear boundary value problem*

$$\begin{aligned} &-\Delta\big[p(k-1)\Delta y(k-1)\big] + q(k)y(k) = h(k), \quad k \in I, \\ &-\Delta y(0) + \bar{\alpha}y(0) = 0, \qquad \Delta y(N) + \bar{\beta}y(N+1) = 0 \end{aligned} \tag{2.2}$$

has a solution

$$y(k) = \sum_{s=1}^{N} G_{\bar{\alpha},\bar{\beta}}(k,s)h(s), \quad k \in \hat{I}, \tag{2.3}$$

where

$$G_{\bar{\alpha},\bar{\beta}}(k,s) = \begin{cases} u_{\bar{\alpha}}(s)v_{\bar{\beta}}(k), & 1 \leq s \leq k \leq N+1, \\ u_{\bar{\alpha}}(k)v_{\bar{\beta}}(s), & 0 \leq k \leq s \leq N. \end{cases}$$

Moreover, if $h(k) \geq 0$ and $h \not\equiv 0$ on I, then $y(k) > 0$ on $\hat{I}$.

Proof It is a consequence of Atici [5, Section 2], so we omit it. □

Let $\omega_1, \omega_2 \in (0, \infty)$. Then the linear boundary value problem

$$\begin{aligned} &-\Delta\bigl[p(k-1)\Delta y(k-1)\bigr] + q(k)y(k) = 0, \quad k \in I, \\ &-\Delta y(0) + \bar{\alpha} y(0) = w_1, \qquad \Delta y(N) + \bar{\beta} y(N+1) = w_2 \end{aligned} \tag{2.4}$$

has a solution

$$y(k) = \frac{\omega_2 u_{\bar{\alpha}}(k)}{(1+\bar{\beta})u_{\bar{\alpha}}(N+1) - u_{\bar{\alpha}}(N)} + \frac{\omega_1 v_{\bar{\beta}}(k)}{(1+\bar{\alpha})v_{\bar{\beta}}(0) - v_{\bar{\beta}}(1)}, \quad k \in \hat{I}.$$

From the properties of $u_{\bar{\alpha}}(k)$, $v_{\bar{\beta}}(k)$, it follows that

$$y(k) > 0, \quad k \in \hat{I}.$$

Define the operator $T : X \to X$ as follows:

$$T[h](k) = \sum_{s=1}^{N} G_{\bar{\alpha},\bar{\beta}}(k,s)h(s), \quad k \in \hat{I}.$$

By a standard argument of compact operator, it is easy to show that T is a compact operator and it is strong positive, meaning that $Th > 0$ on $\hat{I}$ for any $h \in X$ with the condition that $h \geq 0$ and $h \not\equiv 0$ on I; see [2, 4, 5].

Let $\mathscr{R}[\omega_1, \omega_2] : \mathbb{R}^2 \to X$ be defined as

$$\mathscr{R}[\omega_1, \omega_2](k) = \frac{\omega_2 u_{\bar{\alpha}}(k)}{(1+\bar{\beta})u_{\bar{\alpha}}(N+1) - u_{\bar{\alpha}}(N)} + \frac{\omega_1 v_{\bar{\beta}}(k)}{(1+\bar{\alpha})v_{\bar{\beta}}(0) - v_{\bar{\beta}}(1)}, \quad k \in \hat{I}. \tag{2.5}$$

Then $\mathscr{R}[\omega_1, \omega_2]$ is a linear bounded function.

From Lemma 2.2, let $T_\infty, T^\infty : X \to X$ denote the resolvent of the linear boundary value problems

$$\begin{aligned} &-\Delta\bigl[p(k-1)y(k-1)\bigr] + q(k)y(k) = h(k), \quad k \in I, \\ &-\Delta y(0) + \alpha g_\infty y(0) = 0, \qquad \Delta y(N) + \beta g_\infty y(N+1) = 0, \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} &-\Delta\bigl[p(k-1)y(k-1)\bigr] + q(k)y(k) = h(k), \quad k \in I, \\ &-\Delta y(0) + \alpha g^\infty y(0) = 0, \qquad \Delta y(N) + \beta g^\infty y(N+1) = 0, \end{aligned}$$

respectively. Taking into account $\bar{\alpha} = \alpha g_\infty$ ($\bar{\alpha} = \alpha g^\infty$), $\bar{\beta} = \beta g_\infty$ ($\bar{\beta} = \beta g^\infty$), one can repeat the argument of the operator T with some obvious changes. It follows that T_∞, T^∞ are linear mappings of X compactly into X and they are strong positive.

Let $\mathscr{R}_\infty, \mathscr{R}^\infty$ be the solutions of linear boundary value problem (2.4) with $\bar{\alpha}$, $\bar{\beta}$ in place of $\bar{\alpha} = \alpha g_\infty$, $\bar{\beta} = \beta g_\infty$ and $\bar{\alpha} = \alpha g^\infty$, $\bar{\beta} = \beta g^\infty$, respectively. Repeating the argument of

$\mathscr{R}[\omega_1,\omega_2]$ with some minor changes, we have that $\mathscr{R}_\infty, \mathscr{R}^\infty : \mathbb{R}^2 \to X$ is a linear, bounded mapping and

$$\mathscr{R}^\infty[\omega_1,\omega_2](k) = \frac{\omega_2 u_{\alpha g^\infty}(k)}{(1+\beta g^\infty)u_{\alpha g^\infty}(N+1) - u_{\alpha g^\infty}(N)} + \frac{\omega_1 v_{\beta g^\infty}(k)}{(1+\alpha g^\infty)v_{\beta g^\infty}(0) - v_{\beta g^\infty}(1)},$$
$$\mathscr{R}_\infty[\omega_1,\omega_2](k) = \frac{\omega_2 u_{\alpha g_\infty}(k)}{(1+\beta g_\infty)u_{\alpha g_\infty}(N+1) - u_{\alpha g_\infty}(N)} + \frac{\omega_1 v_{\beta g_\infty}(k)}{(1+\alpha g_\infty)v_{\beta g_\infty}(0) - v_{\beta g_\infty}(1)}.$$

Lemma 2.3 *Let* (H1)-(H3) *hold. If* $[\alpha,\beta]$ *is a bifurcation interval from infinity of the set of nonnegative solutions of* (1.1), *then we have* $(\alpha,\beta) \supset [\frac{\mu_1}{f^\infty}, \frac{\lambda_1}{f_\infty}]$. *Moreover, there exist constants* $\epsilon > 0$ *small enough and* $M > 0$ *large enough such that any nonnegative solution* u *of* (1.1) *is positive on* $\hat{I}$ *whenever* $\operatorname{dist}(\lambda, [\frac{\mu_1}{f^\infty}, \frac{\lambda_1}{f_\infty}]) < \epsilon$ *and* $\|u\| \geq M$.

Proof Let (μ_j, y_j) be a nonnegative solution of (1.1) with $\lambda = \mu_j \in [\alpha,\beta]$ such that

$$\|y_j\| \to \infty, \quad \text{and} \quad \mu_j \to \hat{\mu} \quad \text{as } j \to \infty. \tag{2.7}$$

If

$$w_j = \frac{y_j}{\|y_j\|}, \tag{2.8}$$

then we have

$$w_j = \mu_j T^\infty\left[a\frac{f(y_j)}{\|y_j\|}\right] + \mathscr{R}^\infty\left[\tau\left(g^\infty w_j - \frac{g(y_j)}{\|y_j\|}\right)\right] \quad \text{in } X, \tag{2.9}$$

here $\tau : \{y(0), y(1), \dots, y(N+1)\} \to \{y(0), y(N+1)\}$ is a linear operator and

$$\begin{aligned}\mathscr{R}^\infty\left[\tau\left(g^\infty w_j - \frac{g(y_j)}{\|y_j\|}\right)\right](k) &= \frac{g^\infty w_j(N+1) - g(y_j(N+1))/\|y_j\|}{(1+\beta g^\infty)u_{\alpha g^\infty}(N+1) - u_{\alpha g^\infty}(N)} u_{\alpha g^\infty}(k) \\ &\quad + \frac{g^\infty w_j(0) - g(y_j(0))/\|y_j\|}{(1+\alpha g^\infty)v_{\beta g^\infty}(0) - v_{\beta g^\infty}(1)} v_{\beta g^\infty}(k).\end{aligned}$$

From conditions (1.3) and (1.4), for any $\epsilon > 0$, there exist constants $d_\epsilon, c_\epsilon > 0$ such that

$$\begin{aligned} f(s) - f^\infty s &\leq \epsilon s + d_\epsilon, \quad \forall u \geq 0, \\ f(s) - f_\infty s &\geq -\epsilon s - d_\epsilon, \quad \forall u \geq 0, \end{aligned} \tag{2.10}$$

$$\begin{aligned} g(s) - g^\infty s &\leq \epsilon s + c_\epsilon, \quad \forall u \geq 0, \\ g(s) - g_\infty s &\geq -\epsilon s - c_\epsilon, \quad \forall u \geq 0. \end{aligned} \tag{2.11}$$

Those imply that both $f(y_j)/\|y_j\|$ and $g(y_j)/\|y_j\|$ are bounded. By the compactness of T^∞ and $\mathscr{R}^\infty$, it follows from (2.9) that there exist a function $w_0 \in X$ and a subsequence of $\{w_j\}$, still denoted by $\{w_j\}$, such that

$$w_j \to w_0 \quad \text{in } X \text{ as } j \to \infty. \tag{2.12}$$

By (2.7), it follows from (2.10)-(2.11) that

$$\begin{aligned}
&\limsup_{j\to\infty}\max_{k\in I}\left(\frac{f(y_j)}{\|y_j\|}-f^{\infty}w_j\right)\le\epsilon,\\
&\liminf_{j\to\infty}\min_{k\in I}\left(\frac{f(y_j)}{\|y_j\|}-f_{\infty}w_j\right)\ge-\epsilon,\\
&\limsup_{j\to\infty}\max_{k\in I}\left(\frac{g(y_j)}{\|y_j\|}-g^{\infty}w_j\right)\le\epsilon,\\
&\liminf_{j\to\infty}\min_{k\in I}\left(\frac{g(y_j)}{\|y_j\|}-g_{\infty}w_j\right)\ge-\epsilon.
\end{aligned}\tag{2.13}$$

Since ϵ is arbitrary, it follows that

$$\begin{aligned}
&\limsup_{j\to\infty}\frac{f(y_j)}{\|y_j\|}\le f^{\infty}w_0\quad\text{in }X,\\
&\liminf_{j\to\infty}\frac{f(y_j)}{\|y_j\|}\ge f_{\infty}w_0\quad\text{in }X,\\
&\limsup_{j\to\infty}\frac{g(y_j)}{\|y_j\|}\le g^{\infty}w_0\quad\text{in }X,\\
&\liminf_{j\to\infty}\frac{g(y_j)}{\|y_j\|}\ge g_{\infty}w_0\quad\text{in }X.
\end{aligned}\tag{2.14}$$

Let $\mu_j\to\hat{\mu}$, $\frac{f(y_j)}{\|y_j\|}\to v_0$ and $\frac{g(y_j)}{\|y_j\|}\to z_0$ as $j\to\infty$. Then, in view of (2.9),

$$w_0=\hat{\mu}T^{\infty}[av_0]+\mathscr{R}^{\infty}\left[\tau\left(g^{\infty}w_0-z_0\right)\right].\tag{2.15}$$

We claim that

$$\hat{\mu}\in\left[\frac{\mu_1}{f^{\infty}},\frac{\lambda_1}{f_{\infty}}\right].$$

Since

$$f_{\infty}w_0\le v_0\le f^{\infty}w_0\quad\text{and}\quad g_{\infty}w_0\le z_0\le g^{\infty}w_0,$$

it follows from (2.15) that

$$\hat{\mu}T^{\infty}[af_{\infty}w_0]+\mathscr{R}^{\infty}\left[\tau\left(\left(g^{\infty}-g^{\infty}\right)w_0\right)\right]\le w_0\le\hat{\mu}T^{\infty}\left[af^{\infty}w_0\right]+\mathscr{R}^{\infty}\left[\tau\left(\left(g^{\infty}-g_{\infty}\right)w_0\right)\right].$$

Moreover, we have

$$w_0\ge\hat{\mu}T^{\infty}[af_{\infty}w_0].\tag{2.16}$$

Since $\|w_0\|=1$ and $w_0\ge0$, the strong positivity of T^{∞} ensures that $w_0>0$ on $\hat{I}$.

Obviously, w_0 satisfies the following boundary value problem:

$$\begin{aligned} &-\Delta\big[p(k-1)w_0(k-1)\big]+q(k)w_0(k)=\hat{\mu}a(k)v_0, \quad k\in I,\\ &-\Delta w_0(0)+\alpha z_0=0, \qquad \Delta w_0(N)+\beta z_0=0. \end{aligned} \tag{2.17}$$

This combined with φ_1, ψ_1 satisfying (1.5) and (1.6) can get that

$$\begin{aligned} &(\hat{\mu}f_\infty-\lambda_1)\sum_{k=1}^{N}a(k)\varphi_1(k)w_0(k)\\ &\quad\le\sum_{k=1}^{N}a(k)\varphi_1(k)\big(\hat{\mu}v_0(k)-\lambda_1 w_0(k)\big)\\ &\quad=\sum_{k=1}^{N}\big[-\Delta\big[p(k-1)\Delta w_0(k-1)\big]\varphi_1(k)+\Delta\big[p(k-1)\Delta\varphi_1(k-1)\big]w_0(k)\big]\\ &\quad=p(N)\big[\varphi_1(N+1)w_0(N)-w_0(N+1)\varphi_1(N)\big]+p(0)\big[\varphi_1(0)w_0(1)-\varphi_1(1)w_0(0)\big]\\ &\quad=p(N)\beta\varphi_1(N+1)\big[z_0-g^\infty w_0(N+1)\big]+p(0)\alpha\varphi_1(0)\big[z_0-g^\infty w_0(0)\big]\\ &\quad\le p(N)\beta\varphi_1(N+1)\big[g^\infty w_0(N+1)-g^\infty w_0(N+1)\big]\\ &\qquad+p(0)\alpha\varphi_1(0)\big[g^\infty w_0(0)-g^\infty w_0(0)\big]\\ &\quad=0 \end{aligned} \tag{2.18}$$

and

$$\begin{aligned} &\big(\hat{\mu}f^\infty-\mu_1\big)\sum_{k=1}^{N}a(k)\psi_1(k)w_0(k)\\ &\quad\ge\sum_{k=1}^{N}a(k)\psi_1(k)\big(\hat{\mu}v_0(k)-\mu_1 w_0(k)\big)\\ &\quad=\sum_{k=1}^{N}\big[-\Delta\big[p(k-1)\Delta w_0(k-1)\big]\psi_1(k)+\Delta\big[p(k-1)\Delta\psi_1(k-1)\big]w_0(k)\big]\\ &\quad=p(N)\big[\psi_1(N+1)w_0(N)-w_0(N+1)\psi_1(N)\big]+p(0)\big[\psi_1(0)w_0(1)-\psi_1(1)w_0(0)\big]\\ &\quad=p(N)\beta\psi_1(N+1)\big[z_0-g_\infty w_0(N+1)\big]+p(0)\alpha\psi_1(0)\big[z_0-g_\infty w_0(0)\big]\\ &\quad\ge p(N)\beta\psi_1(N+1)\big[g_\infty w_0(N+1)-g^\infty w_0(N+1)\big]\\ &\qquad+p(0)\alpha\psi_1(0)\big[g_\infty w_0(0)-g_\infty w_0(0)\big]\\ &\quad=0. \end{aligned} \tag{2.19}$$

Thus

$$\frac{\mu_1}{f^\infty}\le\hat{\mu}\le\frac{\lambda_1}{f_\infty}.$$

Since $w_0>0$ on $\hat{I}$, (2.12) implies that $w_j>0$ on $\hat{I}$ for j large enough, and so is y_j from (2.8). This leads to the latter part of assertions of this proposition. □

3 Existence of a bifurcation interval from infinity

This section is devoted to studying the existence of a bifurcation interval from infinity for (1.1). To do this, we associate with (1.1) a nonlinear mapping $\Phi(\lambda, y) : (0, \infty) \times X \to X$ as follows:

$$\Phi(\lambda, y) := y - \lambda T^{\infty}\big[af\big(|y|\big)\big] - \mathscr{R}^{\infty}\big[\tau\big(g^{\infty}y - g\big(|y|\big)\big)\big]. \tag{3.1}$$

We note that a nonnegative $y \in X$ attains (1.1) if and only if $\Phi(\lambda, y) = 0$.

In this section, we shall apply Lemma 2.1 to show that for any $\sigma \in (0, \frac{\mu_1}{f^\infty})$, the interval $[\frac{\mu_1}{f^\infty} - \sigma, \frac{\lambda_1}{f_\infty} + \sigma]$ is a bifurcation interval from infinity for (3.1) and, consequently, $[\frac{\mu_1}{f^\infty} - \sigma, \frac{\lambda_1}{f_\infty} + \sigma]$ is a bifurcation interval from infinity for the nonnegative solutions of (1.1).

In fact, if $[\frac{\mu_1}{f^\infty} - \sigma, \frac{\lambda_1}{f_\infty} + \sigma]$ is a bifurcation interval from infinity for (3.1), then, according to Definition 1.1, we have

(i) the solutions of (3.1) are *a priori* bounded in X for $\lambda = \frac{\mu_1}{f^\infty} - \sigma$ and $\lambda = \frac{\lambda_1}{f_\infty} + \sigma$.
(ii) there exists a sequence $\{(\mu_n, y_n)\} \subset \mathcal{S}$ such that $\{\mu_n\} \subset [\frac{\mu_1}{f^\infty} - \sigma, \frac{\lambda_1}{f_\infty} + \sigma]$ and $\|y_n\| \to \infty$.

Let $\{\mu_{n_j}\}$ be any convergent subsequence of $\{(\mu_n, y_n)\}$, and let

$$\lim_{j\to\infty} \mu_{n_j} = \mu^\sharp \quad \text{and} \quad \lim_{j\to\infty} \|y_{nj}\| = \infty.$$

We claim that

$$\mu^\sharp \in \left[\frac{\mu_1}{f^\infty}, \frac{\lambda_1}{f_\infty}\right] \quad \text{and} \quad y_{n_j} > 0 \quad \text{on } \hat{I} \text{ if } j \text{ is large enough.} \tag{3.2}$$

Indeed, as in the proof of Lemma 2.3, we have the same conclusion that there exist some $v_0, z_0 \in X$ and $\mu^\sharp$ such that

$$w_0 = \mu^\sharp T^{\infty}\big(a|v_0|\big) + \mathscr{R}^{\infty}\big[\tau\big(g^{\infty}w_0 - |z_0|\big)\big]. \tag{3.3}$$

Since

$$f_\infty w_0 \le |v_0| \le f^\infty w_0 \quad \text{and} \quad g_\infty w_0 \le |z_0| \le g^\infty w_0,$$

it follows from the strong positivity of T^∞ and the positivity of $\mathscr{R}^\infty$ that

$$\mu^\sharp T^{\infty}[af_\infty w_0] \le w_0 \le \mu^\sharp T^{\infty}\big[af^\infty w_0\big] + \mathscr{R}^{\infty}\big[\tau\big(g^\infty - g_\infty\big)w_0\big]. \tag{3.4}$$

This together with the strong positivity of T^∞ implies that

$$w_0 > 0 \quad \text{on } \hat{I}. \tag{3.5}$$

By using (2.18) and (2.19) with obvious changes, it follows that

$$\frac{\mu_1}{f^\infty} \le \mu^\sharp \le \frac{\lambda_1}{f_\infty}. \tag{3.6}$$

From (3.5), it follows that $w_j > 0$ on $\hat{I}$ for j large enough and so is y_j from (2.8). Therefore $[\frac{\mu_1}{f^\infty} - \sigma, \frac{\lambda_1}{f_\infty} + \sigma]$ is actually an interval of bifurcation from infinity for (1.1).

In what follows, we shall apply Lemma 2.1 to show that $[\frac{\mu_1}{f^\infty} - \sigma, \frac{\lambda_1}{f_\infty} + \sigma]$ is a bifurcation interval from infinity for (3.1), two lemmas on the nonexistence of solutions will be first shown. Let $\Phi_\chi : (0,\infty) \times X \to X$ be defined as

$$\Phi_\chi(\lambda, u) := u - \lambda T^\infty\bigl[af\bigl(|u|\bigr)\bigr] - \chi(\lambda)\mathscr{R}^\infty\bigl[\tau\bigl(g^\infty u - g\bigl(|u|\bigr)\bigr)\bigr]. \tag{3.7}$$

Here $\chi : [0, \frac{\mu_1}{f^\infty}] \to [0,1]$ is a smooth cut-off function such that

$$\chi(\lambda) = \begin{cases} 0 & \text{near } \lambda = 0, \\ 1 & \text{near } \lambda = \frac{\mu_1}{f^\infty}. \end{cases} \tag{3.8}$$

Lemma 3.1 *Let* (H1)-(H3) *hold and* $\Lambda \subset \mathbb{R}^+$ *be a compact interval with* $\Lambda \cap [\frac{\mu_1}{f^\infty}, \frac{\lambda_1}{f_\infty}] = \emptyset$. *Then there exists a constant* $r > 0$ *such that*

$$\Phi_\chi(\lambda, y) \neq 0, \quad \lambda \in \Lambda, y \in X : \|y\| \geq r. \tag{3.9}$$

Proof Assume on the contrary that there exist $\lambda_j \geq 0$, $y_j \in X$ and $\lambda_0 \in \Lambda$ such that

$$\Phi_\chi(\lambda_j, y_j) = 0, \quad \lambda_j \to \lambda_0, \|y_j\| \to \infty \text{ as } j \to \infty.$$

The same argument as in the proof of Lemma 2.3 gives a contradiction that $\frac{\mu_1}{f^\infty} \leq \lambda_0 \leq \frac{\lambda_1}{f_\infty}$. This is a contradiction. The proof of Lemma 3.1 is complete. □

Lemma 3.2 *Let* (H1)-(H3) *hold. Then for any* $\lambda > \frac{\lambda_1}{f_\infty}$ *fixed, there exists a constant* $r > 0$ *such that*

$$\Phi(\lambda, y) \neq t\varphi_1, \quad t \in [0,1], y \in X : \|y\| \geq r. \tag{3.10}$$

Proof Assume on the contrary that there exist $\mu_0 \in (\frac{\lambda_1}{f_\infty}, \infty)$, $t_0, t_j \in [0,1]$, and $y_j \in X$ can be taken such that

$$\Phi(\mu_0, y_j) = t_j\varphi_1, \quad t_j \to t_0, \|y_j\| \to \infty \text{ as } j \to \infty.$$

Using the same argument as in the proof of Lemma 2.3, we can obtain a subsequence of $\{y_j\}$, still denoted by $\{y_j\}$, which may satisfy that $y_j > 0$ on $\hat{I}$ for all $j > 1$. It follows that

$$\begin{aligned} &y_j = \mu_0 T^\infty\bigl[af(y_j)\bigr] + \mathscr{R}^\infty\bigl[\tau\bigl(g^\infty y_j - g(y_j)\bigr)\bigr] + t_j\varphi_1, \\ &\quad t_j \to t_0 \in [0,1], \|y_j\| \to \infty \text{ as } j \to \infty. \end{aligned} \tag{3.11}$$

Thus

$$\begin{aligned} &-\Delta\bigl[p(k-1)\Delta y_j(k-1)\bigr] + q(k)y_j(k) = \mu_0 a(k)f(y_j) + t_j\lambda_1\varphi_1, \quad k \in I, \\ &-\Delta y_j(0) + \alpha g\bigl(y_j(0)\bigr) = 0, \qquad \Delta y_j(N) + \beta g\bigl(y_j(N+1)\bigr) = 0. \end{aligned} \tag{3.12}$$

Moreover, it follows from φ_1 satisfies (1.5) and (3.12) that

$$\begin{aligned}
&\sum_{k=1}^{N}\left[\mu_0 a(k)f(y_j)\varphi_1(k)+t_j\lambda_1\varphi_1^2(k)-\lambda_1 a(k)y_j(k)\varphi_1(k)\right]\\
&\quad=\sum_{k=1}^{N}\left[-\Delta\left[p(k-1)\Delta y_j(k-1)\right]\varphi_1(k)+\Delta\left[p(k-1)\Delta\varphi_1(k-1)\right]y_j(k)\right]\\
&\quad=p(N)\beta\varphi_1(N+1)\left[g\big(y_i(N+1)\big)-g^\infty y_j(N+1)\right]\\
&\qquad+p(0)\alpha\varphi_1(0)\left[g\big(y_j(0)\big)-g^\infty y_j(0)\right].
\end{aligned}\tag{3.13}$$

This implies that

$$\begin{aligned}
&p(N)\beta\varphi_1(N+1)\left[g\big(y_i(N+1)\big)-g^\infty y_j(N+1)\right]+p(0)\alpha\varphi_1(0)\left[g\big(y_j(0)\big)-g^\infty y_j(0)\right]\\
&\quad\ge(\mu_0 f_\infty-\lambda_1)\sum_{k=1}^{N}a(k)y_j(k)\varphi_1(k)+\mu_0\sum_{k=1}^{N}a(k)h_1\big(y_j(k)\big)\varphi_1(k).
\end{aligned}$$

Hence assertion (2.10) gives

$$\begin{aligned}
&p(N)\beta\varphi_1(N+1)\frac{[g(y_i(N+1))-g^\infty y_j(N+1)]}{\|y_j\|}+p(0)\alpha\varphi_1(0)\frac{[g(y_j(0))-g^\infty y_j(0)]}{\|y_j\|}\\
&\quad\ge f_\infty\left(\mu_0-\frac{\lambda_1+\mu_0\varepsilon}{f_\infty}\right)\sum_{k=1}^{N}a(k)\frac{y_j(k)}{\|y_j\|}\varphi_1(k)-\frac{\mu_0\|a\|d_\varepsilon}{\|y_j\|}\sum_{k=1}^{N}\varphi_1(k).
\end{aligned}\tag{3.14}$$

Now use again for (3.12) the same procedure as in the proof of Lemma 2.3, then we see that some subsequence of $\{y_j/\|y_j\|\}$, still denoted by $\{y_j/\|y_j\|\}$, tends to a positive function w_0 in X. Take $\epsilon>0$ so small that $\mu_0-\frac{\lambda_1+\mu_0\epsilon}{f_\infty}>0$. Then combining (3.14) with (2.15) leads to a contradiction that

$$\begin{aligned}
0&=p(N)\beta\varphi_1(N+1)\left[g^\infty w_0(N+1)-g^\infty w_0(N+1)\right]+p(0)\alpha\varphi_1(0)\left[g^\infty w_0(0)-g^\infty w_0(0)\right]\\
&\ge\lim_{j\to\infty}\left\{p(N)\beta\varphi_1(N+1)\frac{[g(y_j(N+1))-g^\infty y_j(N+1)]}{\|y_j\|}\right.\\
&\qquad\left.+p(0)\alpha\varphi_1(0)\frac{[g(y_j(0))-g^\infty y_j(0)]}{\|y_j\|}\right\}\\
&\ge\lim_{j\to\infty}\left\{f_\infty\left(\mu_0-\frac{\lambda_1+\mu_0\varepsilon}{f_\infty}\right)\sum_{k=1}^{N}a(k)\frac{y_j(k)}{\|y_j\|}\varphi_1(k)-\frac{\mu_0\|a\|d_\varepsilon}{\|y_j\|}\sum_{k=1}^{N}\varphi_1(k)\right\}\\
&=f_\infty\left(\mu_0-\frac{\lambda_1+\mu_0\epsilon}{f_\infty}\right)\sum_{k=1}^{N}a(k)w_0(k)\varphi_1(k)>0.
\end{aligned}$$

The proof of Lemma 3.2 is complete. □

Lemma 3.3 *Let* $\lambda_n^-=\frac{\mu_1}{f^\infty}-\frac{1}{n}$ *and* $\lambda_n^+=\frac{\lambda_1}{f_\infty}+\frac{1}{n}$, *where* $n>\frac{f^\infty}{\mu_1}$ *is an integer. Assume that* (H1)-(H3) *hold. Then there exists a constant* $r_n>0$ *satisfying* $r_n\to\infty$ *as* $n\to\infty$ *such that* *for any n large enough,*

$$\deg\bigl(\Phi\bigl(\lambda_n^-,\cdot\bigr),B_{r_n},0\bigr)=1, \tag{3.15}$$

$$\deg\bigl(\Phi\bigl(\lambda_n^+,\cdot\bigr),B_{r_n},0\bigr)=0. \tag{3.16}$$

Proof First we show assertion (3.15). From Lemma 3.1, there exists $r_n > 0$ such that $r_n \to \infty$ as $n \to \infty$ satisfying

$$\Phi_\chi(\lambda,y)\neq 0,\quad \forall\lambda\in\bigl[0,\lambda_n^-\bigr],\forall y\in X:\|y\|=r_n.$$

Since $\chi(0)=0$ and $\chi(\lambda_n^-)=1$ for n large enough from (3.8), by the homotopy invariance and normalization of the topology degree, it follows that for any n large enough,

$$\begin{aligned}\deg\bigl(\Phi\bigl(\lambda_n^-,\cdot\bigr),B_{r_n},0\bigr)&=\deg\bigl(\Phi_\chi\bigl(\lambda_n^-,\cdot\bigr),B_{r_n},0\bigr)=\deg\bigl(\Phi_\chi(0,\cdot),B_{r_n},0\bigr)\\&=\deg(I_X,B_{r_n},0)=1.\end{aligned}$$

Next, we show assertion (3.16). We may derive from Lemma 3.2 that

$$\Phi\bigl(\lambda_n^+,y\bigr)\neq t\varphi_1,\quad \forall t\in[0,1],\forall y\in X:\|y\|\geq r_n.$$

So for any n large enough, by the homotopy invariance, it follows that

$$\deg\bigl(\Phi\bigl(\lambda_n^+,\cdot\bigr),B_{r_n},0\bigr)=\deg\bigl(\Phi\bigl(\lambda_n^+,\cdot\bigr)-\varphi_1,B_{r_n},0\bigr)=0.$$ □

Proof of Theorem 1.1 For any fixed $n\in\mathbb{N}$ with $\frac{\mu_1}{f^\infty}-\frac{1}{n}>0$, set $\alpha_n=\frac{\mu_1}{f^\infty}-\frac{1}{n}$, $\beta_n=\frac{\lambda_1}{f_\infty}+\frac{1}{n}$. It is easy to verify that for any fixed n large enough, there exists r_n such that $r_n\to\infty$ as $n\to\infty$ satisfying that for any $r\geq r_n$, it follows from Lemmas 3.1-3.3 that all conditions of Lemma 2.1 are satisfied. So there exists a closed connected component $\mathcal{C}_n$ of solutions (3.1) such that $\mathcal{C}_n$ is unbounded in $[\alpha_n,\beta_n]\times X$ and either

(i) $\mathcal{C}_n$ is unbounded in λ direction, or
(ii) there exists an interval $[c,d]$ such that $(\alpha_n,\beta_n)\cap(c,d)=\emptyset$ and $\mathcal{C}_n$ bifurcates from infinity in $[c,d]\times X$.

By Lemma 3.1, the case (ii) cannot occur. Thus $\mathcal{C}_n$ is unbounded bifurcated from $[\alpha_n,\beta_n]\times\{\infty\}$ in $\mathbb{R}\times X$. Furthermore, set $\sigma=\frac{1}{n}$ for n large enough, we have from Lemma 3.1 that for any closed interval $I\subset[\frac{\mu_1}{f^\infty}-\sigma,\frac{\lambda_1}{f_\infty}+\sigma]\backslash[\frac{\mu_1}{f^\infty},\frac{\lambda_1}{f_\infty}]$, if $y\in\{y\in X\mid(\lambda,y)\text{ is a solution of (3.1)},\lambda\in I\}$, then $\|y\|\to\infty$ in X is impossible. So $\mathcal{C}_{1/\sigma}$ must be bifurcated from $[\frac{\mu_1}{f^\infty}-\sigma,\frac{\lambda_1}{f_\infty}+\sigma]\times\{\infty\}$. □

Next, we are devoted to the proof of Theorem 1.2, which characterized the bifurcation components of (1.1).

Proof of Theorem 1.2 Under condition (1.7), assume to the contrary that there exists a positive solution y_j of (1.1) with $\lambda=\lambda_j\geq\frac{\lambda_1}{f_\infty}$, and

$$\|y_j\|\to\infty,\quad j\to\infty.$$

If $w_j = \frac{y_j}{\|y_j\|}$, then the same argument as in the proof of Lemma 2.3 shows the existence of a positive function $w_0 \in X$ such that a subsequence of $\{w_j\}$, still denoted by w_j, tends to w_0 in X. It follows that for any j large enough, we have

$$w_j(k) > \frac{\min_{k\in\hat{I}} w_0(k)}{2} \quad \text{on } I, \tag{3.17}$$

which implies that

$$\min_{\hat{I}} u_j \to \infty \quad \text{as } j \to \infty. \tag{3.18}$$

Set

$$h_{1*} = \liminf_{u\to\infty} h_1(u) \in (-\infty, \infty],$$
$$\gamma_2^* = \limsup_{u\to\infty} \gamma_2(u) \in [-\infty, \infty).$$

Note that we consider only the cases $h_{1*} \in (-\infty, \infty)$ and $\gamma_2^* \in (-\infty, \infty)$. Either the case $h_{1*} = \infty$ or the case $\gamma_2^* = -\infty$ can be dealt with in a similar way with a minor modification. It follows from (3.18) that, for any $\epsilon > 0$, there exists $j_1 \geq 1$ such that for any $j \geq j_1$,

$$h_{1*} - \epsilon < h_1\big(u_j(k)\big) \quad \text{on } \hat{I},$$
$$\gamma_2\big(u_j(k)\big) < \gamma_2^* + \epsilon \quad \text{on } \hat{I}.$$

Thus, for any $j \geq j_1$,

$$\begin{aligned}
&(\lambda_1 - \lambda_j f_\infty) \sum_{k=1}^{N} a(k) y_j(k) \varphi_1(k) \\
&\quad \geq \lambda_j \sum_{k=1}^{N} a(k) h_1\big(u_j(k)\big)\varphi_1(k) - \alpha p(0)\varphi_1(0)\big[g\big(u_j(0)\big) - g^\infty u_j(0)\big] \\
&\qquad - \beta p(N)\varphi_1(N+1)\big[g\big(u_j(N+1)\big) - g^\infty u_j(N+1)\big] \\
&\quad \geq \lambda_j \sum_{k=1}^{N} a(k) h_1\big(u_j(k)\big)\varphi_1(k) - \alpha p(0)\varphi_1(0)\gamma_2\big(u_j(0)\big) \\
&\qquad - \beta p(N)\varphi_1(N+1)\gamma_2\big(u_j(N+1)\big) \\
&\quad \geq \frac{\lambda_1(h_{1*} - \epsilon)}{f_\infty} \sum_{k=1}^{N} a(k)\varphi_1(k) - \big(\gamma_2^* + \epsilon\big)\big[\alpha p(0)\varphi_1(0) + \beta p(N)\varphi_1(N+1)\big].
\end{aligned}$$

On the other hand, we have

$$\sum_{k=1}^{N} a(k)\varphi_1(k) = \frac{1}{\lambda_1}\sum_{k=1}^{N} q(k)\varphi_1(k) + \frac{g^\infty[\alpha p(0)\varphi_1(0) + \beta p(N)\varphi_1(N+1)]}{\lambda_1}.$$

These two assertions combined, we obtain that for any $j \geq j_1$,

$$\begin{aligned}
&(\lambda_1 - \lambda_j f_\infty) \sum_{k=1}^{N} a(k) y_j(k) \varphi_1(k) \\
&\quad > \frac{h_{1*} - \epsilon}{f_\infty} \sum_{k=1}^{N} q(k) \varphi_1(k) \\
&\qquad + \left(\frac{\lambda_1 (h_{1*} - \epsilon) g^\infty}{f_\infty \lambda_1} - \left(\gamma_2^* + \epsilon\right) \right) \left[\alpha p(0) \varphi_1(0) + \beta p(N) \varphi_1(N+1) \right] \\
&\quad > 0.
\end{aligned}$$

On the right-hand side, we see from (1.7) that

$$\lim_{\epsilon \to 0^+} \left(\frac{\lambda_1 (h_{1*} - \epsilon) g^\infty}{f_\infty \lambda_1} - \left(\gamma_2^* + \epsilon\right) \right) = \frac{h_{1*} g^\infty}{f_\infty} - \gamma_2^* > 0.$$

This means that for any j large enough,

$$(\lambda_1 - \lambda_j f_\infty) \sum_{k=1}^{N} a(k) y_j(k) \varphi_1(k) > 0,$$

which contradicts the assumption $\lambda_j \geq \frac{\lambda_1}{f_\infty}$. Case (1.7) has been proved. Case (1.8) can be also verified by the same arguments, and the proof of Theorem 1.2 is complete. □

Competing interests

The authors declare that they have no competing interests regarding the publication of this paper.

Authors' contributions

RM completed the main study, carried out the results of this article and YL drafted the manuscript, checked the proofs and verified the calculation. All the authors read and approved the final manuscript.

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