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A new iterative algorithm for the sum of two different types of finitely many accretive operators in Banach space and its connection with capillarity equation

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Abstract

In this paper, we present a new iterative algorithm with errors to solve the problems of finding zeros of the sum of finitely many *m*-accretive operators and finitely many α-inversely strongly accretive operators in a real smooth and uniformly convex Banach space. Strong convergence theorems are established, which extend the corresponding works given by some authors. Moreover, the relationship among the zero of the sum of *m*-accretive operator and α-inversely strongly accretive operator, the solution of one kind variational inequality, and the solution of the capillarity equation is investigated.

MSC: 47H05; 47H09; 47H10

Keywords: α-inversely strongly accretive operator; sum; zero; iterative algorithm; strong convergence; variational inequality; capillarity equation

1 Introduction and preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* denote the dual space of E. We use '$\rightarrow$' and '$\rightharpoonup$' to denote strong and weak convergence either in E or in E^*, respectively. We denote the value of $f \in E^*$ at $x \in E$ by $\langle x, f\rangle$.

A Banach space E is said to be uniformly convex if, for each $\varepsilon \in (0,2]$, there exists $\delta > 0$ such that

$$\|x\| = \|y\| = 1, \qquad \|x-y\| \geq \varepsilon \quad \Rightarrow \quad \left\|\frac{x+y}{2}\right\| \leq 1-\delta.$$

A Banach space E is said to be smooth if

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists for each $x, y \in \{z \in E : \|z\| = 1\}$.

In addition, we define a function $\rho_E : [0,+\infty) \to [0,+\infty)$ called the modulus of smoothness of E as follows:

$$\rho_E(t) = \sup\left\{\frac{1}{2}\big(\|x+y\| + \|x-y\|\big) - 1 : x, y \in E, \|x\| = 1, \|y\| \leq t\right\}.$$

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It is well known that E is uniformly smooth if and only if $\frac{\rho_E(t)}{t} \to 0$, as $t \to 0$. Let $q > 1$ be a real number. A Banach space E is said to be q-uniformly smooth if there exists a positive constant C such that $\rho_E(t) \leq Ct^q$. It is obvious that q-uniformly smooth Banach space must be uniformly smooth.

The normalized duality mapping $J : E \to 2^{E^*}$ is defined by

$$Jx := \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}, \quad x \in E.$$

It is well known that J is single-valued and norm-to-norm uniformly continuous on each bounded subsets of E if E is a real smooth and uniformly convex Banach space; see [1]. Moreover, $J(cx) = cJx$, for all $x \in E$ and $c \in R^1$. In what follows, we still denote by J the single-valued normalized duality mapping. If, E is reduced to the Hilbert space H, then $J \equiv I$ is the identity mapping. The normalized duality mapping J is said to be weakly sequentially continuous if $\{x_n\}$ is a sequence in E which converges weakly to x it follows that $\{Jx_n\}$ converges in weak* to Jx. J is said to be weakly sequentially continuous at zero if $\{x_n\}$ is a sequence in E which converges weakly to 0 it follows that $\{Jx_n\}$ converges in weak* to 0.

Let C be a nonempty, closed, and convex subset of E and let Q be a mapping of E onto C. Then Q is said to be sunny [2] if $Q(Q(x) + t(x - Q(x))) = Q(x)$, for all $x \in E$ and $t \geq 0$.

A mapping Q of E into E is said to be a retraction [2] if $Q^2 = Q$. If a mapping Q is a retraction, then $Q(z) = z$ for every $z \in R(Q)$, where $R(Q)$ is the range of Q.

For a mapping $U : C \to C$, we use $\mathrm{Fix}(U)$ to denote the fixed point set of it; that is, $\mathrm{Fix}(U) := \{x \in C : Ux = x\}$.

For an operator $A : D(A) \subset E \to 2^E$, we use $A^{-1}0$ to denote the set of zeros of it; that is, $A^{-1}0 := \{x \in D(A) : Ax = 0\}$.

Let $T : C \to E$ be a mapping. Then T is said to be

(1) nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \text{for } \forall x, y \in C;$$

(2) k-Lipschitz if there exists $k > 0$ such that

$$\|Tx - Ty\| \leq k\|x - y\|, \quad \text{for } \forall x, y \in C.$$

In particular, if $0 < k < 1$, then T is called a contraction and if $k = 1$, then T reduces to a nonexpansive mapping;

(3) accretive if for all $x, y \in C$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \geq 0,$$

where J is the normalized duality mapping;

(4) α-inversely strongly accretive if for all $x, y \in C$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \geq \alpha\|Tx - Ty\|^2,$$

for some $\alpha > 0$;

(5) m-accretive if T is accretive and $R(I + \lambda T) = E$, for $\forall\lambda > 0$;
(6) strongly positive (see [3]) if E is a real smooth Banach space and there exists $\overline{\gamma} > 0$ such that

$$\langle Tx, Jx\rangle \geq \overline{\gamma}\|x\|^2, \quad \text{for } \forall x \in C.$$

In this case,

$$\|aI - bT\| = \sup_{\|x\|\leq 1} \left|\bigl\langle (aI - bT)x, J(x)\bigr\rangle\right|,$$

where I is the identity mapping and $a \in [0,1]$, $b \in [-1,1]$.

We denote by J_r^A (for $r > 0$) the resolvent of the accretive operator A; that is, $J_r^A := (I + rA)^{-1}$. It is well known that J_r^A is nonexpansive and $\mathrm{Fix}(J_r^A) = A^{-1}0$.

A subset C of E is said to be a sunny nonexpansive retract of E if there exists a sunny nonexpansive retraction of E onto C.

Many practical problems can be reduced to finding zeros of the sum of two accretive operators; that is, $0 \in (A + B)x$. Forward-backward splitting algorithms, which have recently received much attention to many mathematicians, were proposed by Lions and Mercier [4], by Passty [5], and, in a dual form for convex programming, by Han and Lou [6].

The classical forward-backward splitting algorithm is given in the following way:

$$x_{n+1} = (I + r_nB)^{-1}(I - r_nA)x_n, \quad n \geq 0. \tag{1}$$

Based on iterative algorithm (1), much work has been done for finding $x \in H$ such that $x \in (A + B)^{-1}0$, where A and B are α-inversely strongly accretive operator and m-accretive operator defined in the Hilbert space H, respectively. However, most of the existing work are undertaken in the frame of Hilbert spaces; see [4–10], *etc.*

Recently, Qin *et al.*, presented the following iterative algorithm in the frame of q-uniformly smooth Banach spaces E in [11]:

$$x_0 \in C, \quad x_{n+1} = \alpha_n f(x_n) + \beta_n(I + r_nB)^{-1}\bigl[(I - r_nA)x_n + e_n\bigr] + \gamma_n f_n, \quad n \geq 0, \tag{2}$$

where $\{e_n\}$ is the error sequence, f is a contraction, A and B are α-inversely strongly accretive operator and m-accretive operator, respectively. If $(A+B)^{-1}0 \neq \emptyset$, they proved that $\{x_n\}$ converges strongly to $x = Q_{(A+B)^{-1}0}f(x)$, where $Q_{(A+B)^{-1}0}$ is the unique sunny nonexpansive retraction of E onto $(A + B)^{-1}0$, under some conditions.

On the other hand, there are some excellent work done on approximating fixed points of nonexpansive mappings. For example, in 2009, Yao *et al.* presented the following iterative algorithm in the frame of Hilbert space in [12]:

$$\begin{cases} x_0 \in C, \\ y_n = P_C[(1 - \alpha_n)x_n], \\ x_{n+1} = (1 - \beta_n)x_n + \beta_n Ty_n, \quad n \geq 0, \end{cases} \tag{3}$$

where P_C is the metric projection from H onto C and $T : C \to C$ is a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$. They proved that $\{x_n\}$ constructed by (3) converges strongly to a fixed point of T.

In 2006, Marino and Xu, presented the following iterative algorithm in the frame of Hilbert spaces in [13]:

$$x_0 \in C, \quad x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Tx_n, \quad n \geq 0, \tag{4}$$

where f is a contraction, A is a strongly positive linear bounded operator, and T is nonexpansive. If $\mathrm{Fix}(T) \neq \emptyset$, they proved that $\{x_n\}$ converges strongly to $p \in \mathrm{Fix}(T)$, which solves the variational inequality $\langle (\gamma f - A)p, z - p\rangle \leq 0$, for $\forall z \in \mathrm{Fix}(T)$, under some conditions.

Our paper is organized in the following way: in Section 2, inspired by the work in [11–13], we shall present the following iterative algorithm with errors in a real smooth and uniformly convex Banach space:

$$\begin{cases} x_0 \in C, \\ y_n = Q_C[(1-\alpha_n)(x_n + e_n)], \\ z_n = (1-\beta_n)x_n + \beta_n[a_0 y_n + \sum_{i=1}^N a_i J_{r_{n,i}}^{A_i}(y_n - r_{n,i}B_i y_n)], \\ x_{n+1} = \gamma_n \eta f(x_n) + (I - \gamma_n T)z_n, \quad n \geq 0, \end{cases} \tag{A}$$

where C is a nonempty, closed, and convex sunny nonexpansive retract of E, Q_C is the sunny nonexpansive retraction of E onto C, $\{e_n\} \subset E$ is the error sequence, $\{A_i\}_{i=1}^N$ is a finite family of m-accretive operators and $\{B_i\}_{i=1}^N$ is a finite family of α-inversely strongly accretive operators. $T: E \to E$ is a strongly positive linear bounded operator with coefficient $\overline{\gamma}$ and $f: E \to E$ is a contraction with coefficient $k \in (0,1)$. $J_{r_{n,i}}^{A_i} = (I + r_{n,i}A_i)^{-1}$, for $i = 1, 2, \ldots, N$, $\sum_{m=0}^N a_m = 1$, $0 < a_m < 1$, for $m = 0, 1, 2, \ldots, N$. More detail of iterative algorithm (A) will be presented in Section 2. Then $\{x_n\}$ is proved to converge strongly to $p_0 \in \bigcap_{i=1}^N (A_i + B_i)^{-1}0$, which is also a solution of one kind variational inequality.

Our main contributions in Section 2 are:

(i) the discussion is undertaken in the frame of real smooth and uniformly convex Banach space, which is more general than that in Hilbert space or in q-uniformly smooth Banach space;

(ii) the assumption that 'the normalized duality mapping J is weakly sequentially continuous' in most of the existing related work is weaken to 'J is weakly sequentially continuous at zero';

(iii) a new path convergence theorem (Lemma 8) is obtained which is a direct extension of the corresponding result in [13] from Hilbert space to real smooth and uniformly convex Banach space;

(iv) the connection between zeros of the sum of m-accretive operators and α-inversely strongly accretive operators and the solution of one kind variational inequalities is being set up.

In Section 3, one kind capillarity equation is discussed, from which we can see the connection among the unique solution of this equation, the unique solution of one kind variational inequality and the iterative algorithm presented in Section 2.

Next, we list some results we need in sequel:

Lemma 1 (see [1]) *Let E be a Banach space and $f: E \to E$ be a contraction. Then f has a unique fixed point $u \in E$.*

Lemma 2 (see [14]) *Let E be a real uniformly convex Banach space, C be a nonempty, closed, and convex subset of E and $T : C \to E$ be a nonexpansive mapping such that $\mathrm{Fix}(T) \neq \emptyset$, then $I - T$ is demiclosed at zero.*

Lemma 3 (see [15]) *In a real Banach space E, the following inequality holds*:

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle, \quad \forall x, y \in E,$$

where $j(x + y) \in J(x + y)$.

Lemma 4 (see [16]) *Let $\{a_n\}$ and $\{c_n\}$ be two sequences of nonnegative real numbers satisfying*

$$a_{n+1} \leq (1 - t_n)a_n + b_n + c_n, \quad \forall n \geq 0,$$

where $\{t_n\} \subset (0,1)$ and $\{b_n\}$ is a number sequence. Assume that $\sum_{n=0}^{\infty} t_n = +\infty$, $\limsup_{n\to\infty} \frac{b_n}{t_n} \leq 0$, and $\sum_{n=0}^{\infty} c_n < +\infty$. Then $\lim_{n\to\infty} a_n = 0$.

Lemma 5 (see [17]) *Let E be a Banach space and let A be an m-accretive operator. For $\lambda > 0$, $\mu > 0$, and $x \in E$, one has*

$$J_\lambda x = J_\mu\left(\frac{\mu}{\lambda}x + \left(1 - \frac{\mu}{\lambda}\right)J_\lambda x\right),$$

where $J_\lambda = (I + \lambda A)^{-1}$ and $J_\mu = (I + \mu A)^{-1}$.

Lemma 6 (see [18]) *Let E be a real Banach space and let C be a nonempty, closed, and convex subset of E. Suppose $A : C \to E$ is a single-valued operator and $B : E \to 2^E$ is m-accretive. Then*

$$\mathrm{Fix}\big((I + rB)^{-1}(I - rA)\big) = (A + B)^{-1}0, \quad \textit{for } \forall r > 0.$$

Lemma 7 (see [19]) *Assume T is a strongly positive bounded operator with coefficient $\overline{\gamma} > 0$ on a real smooth Banach space E and $0 < \rho \leq \|T\|^{-1}$. Then $\|I - \rho T\| \leq 1 - \rho\overline{\gamma}$.*

2 Strong convergence theorems

Lemma 8 *Let E be a real smooth and uniformly convex Banach space and C be a nonempty, closed, and convex sunny nonexpansive retract of E, and let Q_C be the sunny nonexpansive retraction of E onto C. Let $f : E \to E$ be a fixed contractive mapping with coefficient $k \in (0,1)$, $T : E \to E$ be a strongly positive linear bounded operator with coefficient $\overline{\gamma}$ and $U : C \to C$ be a nonexpansive mapping. Suppose that the duality mapping $J : E \to E^*$ is weakly sequentially continuous at zero, $0 < \eta < \frac{\overline{\gamma}}{2k}$ and $\mathrm{Fix}(U) \neq \emptyset$. If for each $t \in (0,1)$, define $T_t : E \to E$ by*

$$T_t x := t\eta f(x) + (I - tT)UQ_C x, \tag{5}$$

then T_t has a fixed point x_t, for each $0 < t \leq \|T\|^{-1}$, which is convergent strongly to the fixed point of U, as $t \to 0$. That is, $\lim_{t\to0} x_t = p_0 \in \mathrm{Fix}(U)$. Moreover, p_0 satisfies the following *variational inequality*: *for* $\forall z \in \mathrm{Fix}(U)$,

$$\langle (T - \eta f)p_0, J(p_0 - z)\rangle \leq 0. \tag{6}$$

Proof Step 1. T_t is a contraction, for $0 < t < \|T\|^{-1}$.

In fact, noticing Lemma 7, we have

$$\begin{aligned}\|T_t x - T_t y\| &\leq t\eta\|f(x) - f(y)\| + \|(I - tT)(UQ_C x - UQ_C y)\| \\ &\leq kt\eta\|x - y\| + (1 - t\overline{\gamma})\|x - y\| \\ &= [1 - t(\overline{\gamma} - k\eta)]\|x - y\|,\end{aligned}$$

which implies that T_t is a contraction since $0 < \eta < \frac{\overline{\gamma}}{2k}$.

Then Lemma 1 implies that T_t has a unique fixed point, denoted by x_t, which uniquely solves the fixed point equation $x_t = t\eta f(x_t) + (I - tT)UQ_C x_t$.

Step 2. $\{x_t\}$ is bounded, for $t \in (0, \|T\|^{-1})$.

For $p \in \mathrm{Fix}(U) \subset C$, we have $p = UQ_C p$, then

$$\begin{aligned}\|x_t - p\| &= \|(I - tT)(UQ_C x_t - p) + t(\eta f(x_t) - Tp)\| \\ &\leq (1 - t\overline{\gamma})\|x_t - p\| + t\|\eta f(x_t) - Tp\| \\ &= (1 - t\overline{\gamma})\|x_t - p\| + t\|\eta(f(x_t) - f(p)) + (\eta f(p) - Tp)\| \\ &\leq (1 - t\overline{\gamma})\|x_t - p\| + t(k\eta\|x_t - p\| + \|\eta f(p) - Tp\|) \\ &= [1 - t(\overline{\gamma} - k\eta)]\|x_t - p\| + t\|\eta f(p) - Tp\|.\end{aligned}$$

This ensures that

$$\|x_t - p\| \leq \frac{\|\eta f(p) - Tp\|}{\overline{\gamma} - k\eta}.$$

Thus $\{x_t\}$ is bounded, which implies that both $\{f(x_t)\}$ and $\{TUQ_C x_t\}$ are bounded.

Step 3. $x_t - UQ_C x_t \to 0$, as $t \to 0$.

Noticing the result of Step 2, we have $\|x_t - UQ_C x_t\| = t\|\eta f(x_t) - TUQ_C x_t\| \to 0$, as $t \to 0$.

Step 4. $\langle (T - \eta f)x - (T - \eta f)y, J(x - y)\rangle \geq (\overline{\gamma} - k\eta)\|x - y\|^2$, for $\forall x, y \in E$.

In fact,

$$\begin{aligned}&\langle (T - \eta f)x - (T - \eta f)y, J(x - y)\rangle \\ &\quad = \langle Tx - Ty, J(x - y)\rangle - \eta\langle f(x) - f(y), J(x - y)\rangle \\ &\quad \geq \overline{\gamma}\|x - y\|^2 - k\eta\|x - y\|^2 = (\overline{\gamma} - k\eta)\|x - y\|^2.\end{aligned}$$

Step 5. If the variational inequality (6) has a solution, then the solution must be unique.

Suppose both $u_0 \in \mathrm{Fix}(U)$ and $v_0 \in \mathrm{Fix}(U)$ are the solutions of the variational inequality (6). Then we have

$$\langle (T - \eta f)v_0, J(v_0 - u_0)\rangle \leq 0 \tag{7}$$

and

$$\langle (T-\eta f)u_0, J(u_0-v_0)\rangle \le 0. \tag{8}$$

Adding up (7) and (8), we obtain

$$\langle (T-\eta f)u_0-(T-\eta f)v_0, J(u_0-v_0)\rangle \le 0.$$

In view of the result of Step 4, we have $u_0 = v_0$.

Step 6. $x_t \to p_0 \in \mathrm{Fix}(U)$, as $t \to 0$, which satisfies the variational inequality (6).

For $\forall z \in \mathrm{Fix}(U)$, $x_t - z = t(\eta f(x_t) - Tz) + (I - tT)(UQ_C x_t - z)$. Thus Lemma 3 implies that

$$\begin{aligned}\|x_t-z\|^2 &\le \|I-tT\|^2\|UQ_Cx_t-UQ_Cz\|^2+2t\langle \eta f(x_t)-Tz, J(x_t-z)\rangle\\ &\le (1-t\overline{\gamma})\|x_t-z\|^2+2t\langle \eta f(x_t)-Tz, J(x_t-z)\rangle.\end{aligned}$$

Then

$$\begin{aligned}\|x_t-z\|^2 &\le \frac{2}{\overline{\gamma}}\langle \eta f(x_t)-Tz, J(x_t-z)\rangle\\ &= \frac{2}{\overline{\gamma}}\bigl[\eta\langle f(x_t)-f(z), J(x_t-z)\rangle+\langle \eta f(z)-T(z), J(x_t-z)\rangle\bigr]\\ &\le \frac{2}{\overline{\gamma}}\bigl[\eta k\|x_t-z\|^2+\langle \eta f(z)-Tz, J(x_t-z)\rangle\bigr].\end{aligned}$$

Therefore, for $\forall z \in \mathrm{Fix}(U)$, we have

$$\|x_t-z\|^2 \le \frac{2}{\overline{\gamma}-2k\eta}\langle \eta f(z)-Tz, J(x_t-z)\rangle. \tag{9}$$

Since $\{x_t\}$ is bounded as $t \to 0^+$, we can choose $\{t_n\} \subset (0,1)$ such that $t_n \to 0^+$ and $x_{t_n} \rightharpoonup p_0$. From Lemma 2 and the result of Step 3, we see that $p_0 = UQ_Cp_0 = Up_0$. Thus $p_0 \in \mathrm{Fix}(U)$. Substituting z by p_0 in (9), then we can deduce that $x_{t_n} \to p_0$ since J is weakly sequentially continuous at zero. Next, we shall prove that p_0 solves the variational inequality (6).

Since $x_t = t\eta f(x_t) + (I - tT)UQ_C x_t$,

$$(T-\eta f)x_t = -\frac{1}{t}(I-tT)(I-UQ_C)x_t.$$

For $\forall z \in \mathrm{Fix}(U)$, since U is nonexpansive,

$$\begin{aligned}&\langle (T-\eta f)x_t, J(x_t-z)\rangle\\ &\quad= -\frac{1}{t}\langle (I-tT)(I-UQ_C)x_t, J(x_t-z)\rangle\\ &\quad= -\frac{1}{t}\langle (I-UQ_C)x_t-(I-UQ_C)z, J(x_t-z)\rangle+\langle T(I-UQ_C)x_t, J(x_t-z)\rangle\\ &\quad= -\frac{1}{t}\bigl[\|x_t-z\|^2-\langle UQ_Cx_t-UQ_Cz, J(x_t-z)\rangle\bigr]+\langle T(I-UQ_C)x_t, J(x_t-z)\rangle\\ &\quad\le \langle T(I-UQ_C)x_t, J(x_t-z)\rangle.\end{aligned} \tag{10}$$

Since $x_{t_n} \to p_0$, we have $(I - UQ_C)x_{t_n} \to (I - UQ_C)p_0 = 0$, as $n \to \infty$. Since $\{x_{t_n}\}$ is bounded, $(T - \eta f)x_{t_n} \to (T - \eta f)p_0$ and J is uniformly continuous on each bounded subset of E, taking the limits on both sides of (10) we have $\langle (T - \eta f)p_0, J(p_0 - z)\rangle \leq 0$, for $z \in \mathrm{Fix}(U)$. Thus p_0 satisfies (6).

In a summary, we infer that each cluster point of $\{x_t\}$ is equal to p_0, which is the unique solution of the variational inequality (6).

This completes the proof. □

Remark 9 Lemma 8 is a direct extension of Theorem 3.2 in [13] from Hilbert space to real smooth and uniformly convex Banach space.

Theorem 10 *Let E be a real smooth and uniformly convex Banach space and C be a nonempty, closed, and convex sunny nonexpansive retract of E, and let Q_C be the sunny nonexpansive retraction of E onto C. Let $f : E \to E$ be a fixed contractive mapping with coefficient $k \in (0,1)$, $T : E \to E$ be a strongly positive linear bounded operator with coefficient $\overline{\gamma}$. Suppose that the duality mapping $J : E \to E^*$ is weakly sequentially continuous at zero, and $0 < \eta < \frac{\overline{\gamma}}{2k}$. Let $A_i : C \to 2^E$ be m-accretive operator and $B_i : C \to E$ be α-inversely strongly accretive operator, where $i = 1,2,\dots,N$. Suppose that, for $\forall r > 0$ and $i = 1,2,\dots,N$,*

$$\big\langle B_i x - B_i y, J\big[(I - rB_i)x - (I - rB_i)y\big]\big\rangle \geq 0.$$

Let $\{x_n\}$ be generated by the iterative algorithm (A), $0 < a_m < 1$, *for $m = 0,1,2,\dots,N$, $\sum_{m=0}^{N} a_m = 1$. Suppose $\{e_n\}_{n=0}^{\infty} \subset E$, $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are three sequences in $(0,1)$ and $\{r_{n,i}\} \subset (0,+\infty)$ satisfying the following conditions*:

(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$, $\gamma_n \to 0$, $\frac{\gamma_{n-1}}{\gamma_n} \to 1$, $\beta_n \to 1$, $\alpha_n \to 0$, *as* $n \to \infty$;

(ii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < +\infty$, $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < +\infty$, $\sum_{n=0}^{\infty} (1 - \gamma_n \overline{\gamma})\alpha_n\beta_n < +\infty$;

(iii) $\sum_{n=0}^{\infty} |r_{n+1,i} - r_{n,i}| < +\infty$ *and* $r_{n,i} \geq \varepsilon > 0$, *for* $n \geq 0$ *and* $i = 1,2,\dots,N$;

(iv) $\sum_{n=0}^{\infty} \|e_n\| < +\infty$.

If $\bigcap_{i=1}^{N} (A_i + B_i)^{-1}0 \neq \emptyset$, then $\{x_n\}$ converges strongly to a point $p_0 \in \bigcap_{i=1}^{N} (A_i + B_i)^{-1}0$, which is the unique solution of the following variational inequality: for $\forall z \in \bigcap_{i=1}^{N} (A_i + B_i)^{-1}0$,

$$\big\langle (T - \eta f)p_0, J(p_0 - z)\big\rangle \leq 0. \qquad (*)$$

Proof Let $u_{n,i} = (I - r_{n,i}B_i)y_n$, $v_n = a_0 y_n + \sum_{i=1}^{N} a_i J_{r_{n,i}}^{A_i} u_{n,i}$, for $n \geq 0$, where $i = 1,2,\dots,N$.

We shall split the proof into five steps:

Step 1. $\{x_n\}$, $\{u_{n,i}\}$ $(i = 1,2,\dots,N)$, $\{y_n\}$, $\{v_n\}$, and $\{z_n\}$ are all bounded.

From the assumptions on B_i, in view of Lemma 3, we have, for $\forall x, y \in C$,

$$\begin{aligned}\big\|(I - rB_i)x - (I - rB_i)y\big\|^2 &\leq \|x - y\|^2 - 2r\big\langle B_i x - B_i y, J\big[(I - rB_i)x - (I - rB_i)y\big]\big\rangle \\ &\leq \|x - y\|^2,\end{aligned}$$

which implies that $(I - rB_i)$ is nonexpansive, for $r > 0$.

Then noticing the facts that both $(I - r_{n,i}B_i)$ and $J_{r_{n,i}}^{A_i}$ are nonexpansive, for $i = 1,2,\dots,N$, we have, for $\forall p \in \bigcap_{i=1}^{N} (A_i + B_i)^{-1}0 \subset C$,

$$\|y_n - p\| \leq (1 - \alpha_n)\|x_n - p\| + \|e_n\| + \alpha_n\|e_n + p\|. \qquad (11)$$

Using Lemma 6, we have, for $p \in \bigcap_{i=1}^{N}(A_i + B_i)^{-1}0$,

$$\begin{aligned}
\|z_n - p\| &\le (1-\beta_n)\|x_n - p\| + \beta_n\left(a_0\|y_n - p\| + \sum_{i=1}^{N} a_i \left\|J_{r_{n,i}}^{A_i}(I - r_{n,i}B_i)(y_n - p)\right\|\right) \\
&\le (1-\beta_n)\|x_n - p\| + \beta_n a_0\|y_n - p\| + \beta_n \sum_{i=1}^{N} a_i\|y_n - p\| \\
&= (1-\beta_n)\|x_n - p\| + \beta_n\|y_n - p\|. \qquad (12)
\end{aligned}$$

Then Lemma 7 implies that for $n \ge 0$,

$$\begin{aligned}
\|x_{n+1} - p\| &\le \gamma_n\left\|\eta f(x_n) - Tp\right\| + \left\|(I - \gamma_n T)(z_n - p)\right\| \\
&\le \gamma_n \eta k\|x_n - p\| + \gamma_n\left\|\eta f(p) - Tp\right\| + (1 - \gamma_n\overline{\gamma})\|z_n - p\|. \qquad (13)
\end{aligned}$$

Noticing (11)-(13), we have, for $n \ge 0$,

$$\begin{aligned}
&\|x_{n+1} - p\| \\
&\quad\le \gamma_n\eta k\|x_n - p\| + \gamma_n\left\|\eta f(p) - Tp\right\| \\
&\qquad + (1 - \gamma_n\overline{\gamma})\big[(1-\beta_n)\|x_n - p\| + \beta_n\|y_n - p\|\big] \\
&\quad\le \big[\gamma_n\eta k + (1 - \gamma_n\overline{\gamma})(1-\beta_n) + (1 - \gamma_n\overline{\gamma})(1-\alpha_n)\beta_n\big]\|x_n - p\| + \gamma_n\left\|\eta f(p) - Tp\right\| \\
&\qquad + (1 - \gamma_n\overline{\gamma})\beta_n\|e_n\| + (1 - \gamma_n\overline{\gamma})\beta_n\alpha_n\|e_n + p\| \\
&\quad= \big[1 - \alpha_n\beta_n(1 - \gamma_n\overline{\gamma}) - \gamma_n(\overline{\gamma} - k\eta)\big]\|x_n - p\| + \gamma_n\left\|\eta f(p) - Tp\right\| \\
&\qquad + (1 - \gamma_n\overline{\gamma})\beta_n\|e_n\| + (1 - \gamma_n\overline{\gamma})\alpha_n\beta_n\|e_n + p\| \\
&\quad\le \big[1 - \gamma_n(\overline{\gamma} - k\eta)\big]\|x_n - p\| + \gamma_n\left\|\eta f(p) - Tp\right\| + 2\|e_n\| + (1 - \gamma_n\overline{\gamma})\alpha_n\beta_n\|p\| \\
&\quad\le \max\left\{\|x_n - p\|, \frac{\|\eta f(p) - Tp\|}{\overline{\gamma} - k\eta}\right\} + 2\|e_n\| + (1 - \gamma_n\overline{\gamma})\alpha_n\beta_n\|p\|. \qquad (14)
\end{aligned}$$

By using the inductive method, we can easily get the following result from (14):

$$\|x_{n+1} - p\| \le \max\left\{\|x_0 - p\|, \frac{\|\eta f(p) - Tp\|}{\overline{\gamma} - k\eta}\right\} + 2\sum_{k=0}^{n}\|e_k\| + \|p\|\sum_{k=0}^{n}(1 - \gamma_k\overline{\gamma})\alpha_k\beta_k.$$

Therefore, from assumptions (ii) and (iv), we know that $\{x_n\}$ is bounded.

For $\forall p \in \bigcap_{i=1}^{N}(A_i + B_i)^{-1}0$, since $\|y_n - p\| \le \|(1-\alpha_n)(x_n + e_n) - p\| \le \|x_n\| + \|e_n\| + \|p\|$, $\{y_n\}$ is bounded, which implies that $\{u_{n,i}\}$ is bounded in view of the fact that $I - r_{n,i}B_i$ is nonexpansive, for each $i = 1, 2, \dots, N$.

Moreover, $\{J_{r_{n,i}}^{A_i}(I - r_{n,i}B_i)y_n\}$ is bounded since $J_{r_{n,i}}^{A_i}$ is nonexpansive, for $i = 1, 2, \dots, N$. Thus $\{v_n\}$ is bounded, which ensures that $\{z_n\}$ is bounded. Since $r_{n,i} \ge \varepsilon > 0$, $B_i y_n = \frac{y_n - u_{n,i}}{r_{n,i}}$ is bounded, for $n \ge 0$ and $i = 1, 2, \dots, N$.

Set $M_1 = \sup\{\|u_{n,i}\|, \|J_{r_{n,i}}^{A_i}u_{n,i}\|, \|B_i y_n\|, \|x_n\|, \|v_n\|, \|z_n\|, \|Tz_n\|, \eta\|f(x_n)\| : n \ge 0, i = 1, 2, \dots, N\}$.

Step 2. $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.

First, we shall discuss $\|J^{A_i}_{r_{n,i}} u_{n,i} - J^{A_i}_{r_{n-1,i}} u_{n-1,i}\|$, for $n \geq 1$.
If $r_{n-1,i} \leq r_{n,i}$, then by using Lemma 5, we have

$$\begin{aligned}
&\left\|J^{A_i}_{r_{n,i}} u_{n,i} - J^{A_i}_{r_{n-1,i}} u_{n-1,i}\right\| \\
&\quad = \left\|J^{A_i}_{r_{n-1,i}} \left(\frac{r_{n-1,i}}{r_{n,i}} u_{n,i} + \left(1 - \frac{r_{n-1,i}}{r_{n,i}}\right) J^{A_i}_{r_{n,i}} u_{n,i}\right) - J^{A_i}_{r_{n-1,i}} u_{n-1,i}\right\| \\
&\quad \leq \left\|\frac{r_{n-1,i}}{r_{n,i}} u_{n,i} + \left(1 - \frac{r_{n-1,i}}{r_{n,i}}\right) J^{A_i}_{r_{n,i}} u_{n,i} - u_{n-1,i}\right\| \\
&\quad \leq \frac{r_{n-1,i}}{r_{n,i}} \|u_{n,i} - u_{n-1,i}\| + \left(1 - \frac{r_{n-1,i}}{r_{n,i}}\right) \left\|J^{A_i}_{r_{n,i}} u_{n,i} - u_{n-1,i}\right\| \\
&\quad \leq \|u_{n,i} - u_{n-1,i}\| + \frac{r_{n,i} - r_{n-1,i}}{\varepsilon} \left\|J^{A_i}_{r_{n,i}} u_{n,i} - u_{n-1,i}\right\|.
\end{aligned} \tag{15}$$

If $r_{n,i} \leq r_{n-1,i}$, then imitating the proof of (15), we have

$$\left\|J^{A_i}_{r_{n,i}} u_{n,i} - J^{A_i}_{r_{n-1,i}} u_{n-1,i}\right\| \leq \|u_{n,i} - u_{n-1,i}\| + \frac{r_{n-1,i} - r_{n,i}}{\varepsilon} \left\|J^{A_i}_{r_{n,i}} u_{n,i} - u_{n-1,i}\right\|. \tag{16}$$

Combining (15) and (16), we have, for $n \geq 1$,

$$\begin{aligned}
&\left\|J^{A_i}_{r_{n,i}} u_{n,i} - J^{A_i}_{r_{n-1,i}} u_{n-1,i}\right\| \\
&\quad \leq \|u_{n,i} - u_{n-1,i}\| + \frac{|r_{n-1,i} - r_{n,i}|}{\varepsilon} \left\|J^{A_i}_{r_{n,i}} u_{n,i} - u_{n-1,i}\right\| \\
&\quad \leq \|u_{n,i} - u_{n-1,i}\| + \frac{2|r_{n-1,i} - r_{n,i}|}{\varepsilon} M_1 \\
&\quad \leq \left\|(I - r_{n,i} B_i)(y_n - y_{n-1})\right\| + |r_{n,i} - r_{n-1,i}| \|B_i y_{n-1}\| + \frac{2|r_{n-1,i} - r_{n,i}|}{\varepsilon} M_1 \\
&\quad \leq \|y_n - y_{n-1}\| + |r_{n,i} - r_{n-1,i}| \|B_i y_{n-1}\| + \frac{2|r_{n-1,i} - r_{n,i}|}{\varepsilon} M_1.
\end{aligned} \tag{17}$$

Let $M_2 = (\frac{2}{\varepsilon} + 1) M_1$, and using (17), we have, for $n \geq 1$,

$$\begin{aligned}
\|v_n - v_{n-1}\| &\leq a_0 \|y_n - y_{n-1}\| + \sum_{i=1}^{N} a_i \left\|J^{A_i}_{r_{n,i}} u_{n,i} - J^{A_i}_{r_{n-1,i}} u_{n-1,i}\right\| \\
&\leq \|y_n - y_{n-1}\| + M_2 \sum_{i=1}^{N} a_i |r_{n,i} - r_{n-1,i}|.
\end{aligned} \tag{18}$$

Using (18), we have, for $n \geq 1$,

$$\begin{aligned}
\|z_n - z_{n-1}\| &\leq (1 - \beta_n) \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| \|x_{n-1}\| \\
&\quad + \beta_n \|v_n - v_{n-1}\| + |\beta_n - \beta_{n-1}| \|v_{n-1}\| \\
&\leq (1 - \beta_n) \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| \|x_{n-1}\| + \beta_n \|y_n - y_{n-1}\| \\
&\quad + \beta_n M_2 \sum_{i=1}^{N} a_i |r_{n,i} - r_{n-1,i}| + |\beta_n - \beta_{n-1}| \|v_{n-1}\|.
\end{aligned} \tag{19}$$

Noticing that for $n \geq 1$,

$$\|y_n - y_{n-1}\| \leq (1-\alpha_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|x_{n-1}\| + (1-\alpha_n)\|e_n - e_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|e_{n-1}\|. \tag{20}$$

Using (19) and (20), we have, for $n \geq 1$,

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
&\quad \leq \gamma_n \eta \big\|f(x_n) - f(x_{n-1})\big\| + \eta|\gamma_n - \gamma_{n-1}|\big\|f(x_{n-1})\big\| + \|I - \gamma_n T\|\|z_n - z_{n-1}\| \\
&\qquad + |\gamma_n - \gamma_{n-1}|\|Tz_{n-1}\| \\
&\quad \leq \gamma_n \eta k\|x_n - x_{n-1}\| + \eta|\gamma_n - \gamma_{n-1}|\big\|f(x_{n-1})\big\| + (1-\gamma_n\overline{\gamma})\|z_n - z_{n-1}\| \\
&\qquad + |\gamma_n - \gamma_{n-1}|\|Tz_{n-1}\| \\
&\quad \leq \big[(1-\gamma_n\overline{\gamma})(1-\alpha_n\beta_n) + \gamma_n\eta k\big]\|x_n - x_{n-1}\| + 2M_1|\gamma_n - \gamma_{n-1}| \\
&\qquad + 2M_1(1-\gamma_n\overline{\gamma})|\beta_n - \beta_{n-1}| + (1-\gamma_n\overline{\gamma})M_2\beta_n\sum_{i=1}^N a_i|r_{n,i} - r_{n-1,i}| \\
&\qquad + (1-\gamma_n\overline{\gamma})\beta_n|\alpha_n - \alpha_{n-1}|\big(M_1 + \|e_{n-1}\|\big) + (1-\gamma_n\overline{\gamma})\beta_n(1-\alpha_n)\|e_n - e_{n-1}\| \\
&\quad \leq \big[1 - \gamma_n(\overline{\gamma} - \eta k)\big]\|x_n - x_{n-1}\| + 2M_1|\gamma_n - \gamma_{n-1}| \\
&\qquad + 2M_1(1-\gamma_n\overline{\gamma})|\beta_n - \beta_{n-1}| + (1-\gamma_n\overline{\gamma})M_2\beta_n\sum_{i=1}^N a_i|r_{n,i} - r_{n-1,i}| \\
&\qquad + (1-\gamma_n\overline{\gamma})\beta_n|\alpha_n - \alpha_{n-1}|\big(M_1 + \|e_{n-1}\|\big) + (1-\gamma_n\overline{\gamma})\beta_n(1-\alpha_n)\|e_n - e_{n-1}\|.
\end{aligned} \tag{21}$$

From the assumptions on $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{r_{n,i}\}$, and $\{e_n\}$, in view of (21), and Lemma 4, we have $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.

Step 3. Set $W_n = [a_0 I + \sum_{i=1}^N a_i J_{r_{n,i}}^{A_i}(I - r_{n,i}B_i)]$, then $W_n : C \to C$ is nonexpansive and $\mathrm{Fix}(W_n) = \bigcap_{i=1}^N (A_i + B_i)^{-1}0$.

It is obvious that W_n is nonexpansive and $\bigcap_{i=1}^N (A_i + B_i)^{-1}0 \subset \mathrm{Fix}(W_n)$. So we are left to show that $\mathrm{Fix}(W_n) \subset \bigcap_{i=1}^N (A_i + B_i)^{-1}0$.

In fact, if $p \in \mathrm{Fix}(W_n)$, then $W_n p = p$. For $\forall q \in \bigcap_{i=1}^N (A_i + B_i)^{-1}0 \subset \mathrm{Fix}(W_n)$, we have

$$\begin{aligned}
\|p - q\| &\leq a_0\|p - q\| + a_1\big\|J_{r_{n,1}}^{A_1}(I - r_{n,1}B_1)p - q\big\| + \cdots + a_N\big\|J_{r_{n,N}}^{A_N}(I - r_{n,N}B_N)p - q\big\| \\
&\leq (a_0 + a_1 + \cdots + a_{N-1})\|p - q\| + a_N\big\|J_{r_{n,N}}^{A_N}(I - r_{n,N}B_N)p - q\big\| \\
&= (1 - a_N)\|p - q\| + a_N\big\|J_{r_{n,N}}^{A_N}(I - r_{n,N}B_N)p - q\big\| \\
&\leq \|p - q\|.
\end{aligned}$$

Therefore, $\|p - q\| = (1 - a_N)\|p - q\| + a_N\|J_{r_{n,N}}^{A_N}(I - r_{n,N}B_N)p - q\|$, which implies that $\|p - q\| = \|J_{r_{n,N}}^{A_N}(I - r_{n,N}B_N)p - q\|$. Similarly, $\|p - q\| = \|J_{r_{n,1}}^{A_1}(I - r_{n,1}B_1)p - q\| = \cdots = \|J_{r_{n,N-1}}^{A_{N-1}}(I - r_{n,N-1}B_{N-1})p - q\|$.

Then $\|p - q\| = \|\frac{a_1}{\sum_{i=1}^N a_i}(J_{r_{n,1}}^{A_1}(I - r_{n,1}B_1)p - q) + \frac{a_2}{\sum_{i=1}^N a_i}(J_{r_{n,2}}^{A_2}(I - r_{n,2}B_2)p - q) + \cdots + \frac{a_N}{\sum_{i=1}^N a_i}(J_{r_{n,N}}^{A_N}(I - r_{n,N}B_N)p - q)\|$, which implies from the strictly convexity of E that $p - q = J_{r_{n,1}}^{A_1}(I - r_{n,1}B_1)p - q = J_{r_{n,2}}^{A_2}(I - r_{n,2}B_2)p - q = \cdots = J_{r_{n,N}}^{A_N}(I - r_{n,N}B_N)p - q$.

Therefore, $J_{r_{n,i}}^{A_i}(I - r_{n,i}B_i)p = p$, for $i = 1, 2, \ldots, N$. Then $p \in \bigcap_{i=1}^N (A_i + B_i)^{-1}0$. Thus $\mathrm{Fix}(W_n) \subset \bigcap_{i=1}^N (A_i + B_i)^{-1}0$.

Step 4. $W_n y_n - y_n \to 0$, as $n \to \infty$, where W_n is the same as that in Step 3.

In fact, since both $\{x_n\}$ and $\{W_n y_n\}$ are bounded and $\beta_n \to 1$, as $n \to +\infty$,

$$z_n - W_n y_n = (1 - \beta_n)(x_n - W_n y_n) \to 0, \quad \text{as } n \to +\infty.$$

Since both $\{f(x_n)\}$ and $\{Tz_n\}$ are bounded and $\gamma_n \to 0$, as $n \to +\infty$,

$$x_{n+1} - z_n = \gamma_n\bigl[\eta f(x_n) - Tz_n\bigr] \to 0, \quad \text{as } n \to +\infty.$$

Therefore

$$x_n - W_n y_n = (x_n - x_{n+1}) + (x_{n+1} - z_n) + (z_n - W_n y_n) \to 0,$$

as $n \to +\infty$, in view of the fact of Step 2.

Since $\sum_{n=0}^\infty e_n < +\infty$ and $\alpha_n \to 0$, as $n \to \infty$,

$$\begin{aligned}\|W_n y_n - y_n\| &= \bigl\|Q_C W_n y_n - Q_C\bigl[(1 - \alpha_n)(x_n + e_n)\bigr]\bigr\| \\ &\le \|W_n y_n - x_n\| + \alpha_n\|x_n\| + (1 - \alpha_n)\|e_n\| \to 0, \quad \text{as } n \to \infty.\end{aligned}$$

Moreover, $x_{n+1} - y_n \to 0$, as $n \to \infty$.

Step 5. $\limsup_{n\to+\infty}\langle \eta f(p_0) - Tp_0, J(x_{n+1} - p_0)\rangle \le 0$, where $p_0 \in \bigcap_{i=1}^N (A_i + B_i)^{-1}0$, which is the unique solution of the variational inequality $(*)$.

Noticing the result of Step 3 and using Lemma 8, we know that there exists z_t such that $z_t = t\eta f(z_t) + (I - tT)W_n Q_C z_t$ for $t \in (0,1)$. Moreover, $z_t \to p_0 \in \mathrm{Fix}(W_n) = \bigcap_{i=1}^N (A_i + B_i)^{-1}0$, as $t \to 0$. And, p_0 is the unique solution of the variational inequality $(*)$.

Since $\|z_t\| \le \|z_t - p_0\| + \|p_0\|$, then $\{z_t\}$ is bounded, as $t \to 0$. Using Lemma 3, we have

$$\begin{aligned}\|z_t - y_n\|^2 &= \|z_t - W_n y_n + W_n y_n - y_n\|^2 \\ &\le \|z_t - W_n y_n\|^2 + 2\bigl\langle W_n y_n - y_n, J(z_t - y_n)\bigr\rangle \\ &= \bigl\|t\eta f(z_t) + (I - tT)W_n Q_C z_t - W_n y_n\bigr\|^2 + 2\bigl\langle W_n y_n - y_n, J(z_t - y_n)\bigr\rangle \\ &\le \|W_n Q_C z_t - W_n y_n\|^2 + 2t\bigl\langle \eta f(z_t) - TW_n Q_C z_t, J(z_t - W_n y_n)\bigr\rangle \\ &\quad + 2\bigl\langle W_n y_n - y_n, J(z_t - y_n)\bigr\rangle \\ &\le \|z_t - y_n\|^2 + 2t\bigl\langle \eta f(z_t) - TW_n Q_C z_t, J(z_t - W_n y_n)\bigr\rangle \\ &\quad + 2\|W_n y_n - y_n\|\|z_t - y_n\|,\end{aligned}$$

which implies that

$$t\bigl\langle TW_n Q_C z_t - \eta f(z_t), J(z_t - W_n y_n)\bigr\rangle \le \|W_n y_n - y_n\|\|z_t - y_n\|.$$

So, $\lim_{t\to 0}\limsup_{n\to+\infty}\langle TW_n Q_C z_t - \eta f(z_t), J(z_t - W_n y_n)\rangle \le 0$ in view of Step 4.

Since $z_t \to p_0$, $W_n Q_C z_t \to W_n Q_C p_0 = p_0$, as $t \to 0$ in view of Step 3. Noticing the fact that

$$\begin{aligned}
&\langle Tp_0 - \eta f(p_0), J(p_0 - W_n y_n)\rangle \\
&\quad = \langle Tp_0 - \eta f(p_0), J(p_0 - W_n y_n) - J(z_t - W_n y_n)\rangle + \langle Tp_0 - \eta f(p_0), J(z_t - W_n y_n)\rangle \\
&\quad = \langle Tp_0 - \eta f(p_0), J(p_0 - W_n y_n) - J(z_t - W_n y_n)\rangle \\
&\qquad + \langle Tp_0 - \eta f(p_0) - TW_n Q_C z_t + \eta f(z_t), J(z_t - W_n y_n)\rangle \\
&\qquad + \langle TW_n Q_C z_t - \eta f(z_t), J(z_t - W_n y_n)\rangle,
\end{aligned}$$

we have $\limsup_{n\to+\infty}\langle Tp_0 - \eta f(p_0), J(p_0 - W_n y_n)\rangle \le 0$.

Since $\langle Tp_0 - \eta f(p_0), J(p_0 - x_{n+1})\rangle = \langle Tp_0 - \eta f(p_0), J(p_0 - x_{n+1}) - J(p_0 - W_n y_n)\rangle + \langle Tp_0 - \eta f(p_0), J(p_0 - W_n y_n)\rangle$ and $x_{n+1} - W_n y_n \to 0$, then $\limsup_{n\to\infty}\langle \eta f(p_0) - Tp_0, J(x_{n+1} - p_0)\rangle \le 0$.

Step 6. $x_n \to p_0$, as $n \to +\infty$, where $p_0 \in \bigcap_{i=1}^N (A_i + B_i)^{-1}0 \subset C$ is the same as that in Step 5.

Let $M_3 = \sup\{\|(1-\alpha_n)(x_n + e_n) - p_0\| : n \ge 0\}$. By using Lemma 3 again, we have

$$\|y_n - p_0\|^2 \le (1-\alpha_n)^2\|x_n - p_0\|^2 + 2\langle (1-\alpha_n)e_n - \alpha_n p_0, J[(1-\alpha_n)(x_n + e_n) - p_0]\rangle. \tag{22}$$

Using (22) and the result of Step 3, we have

$$\begin{aligned}
\|z_n - p_0\|^2 &\le (1-\beta_n)\|x_n - p_0\|^2 + \beta_n\|W_n y_n - W_n p_0\|^2 \\
&\le (1-\beta_n)\|x_n - p_0\|^2 + \beta_n\|y_n - p_0\|^2 \\
&\le (1-\alpha_n\beta_n)\|x_n - p_0\|^2 \\
&\quad + 2\beta_n\langle (1-\alpha_n)e_n - \alpha_n p_0, J[(1-\alpha_n)(x_n + e_n) - p_0]\rangle.
\end{aligned}\tag{23}$$

Using (23) and Lemma 3, we have, for $n \ge 0$,

$$\begin{aligned}
&\|x_{n+1} - p_0\|^2 \\
&\quad = \|\gamma_n(\eta f(x_n) - Tp_0) + (I - \gamma_n T)(z_n - p_0)\|^2 \\
&\quad \le (1-\gamma_n\overline{\gamma})^2\|z_n - p_0\|^2 + 2\gamma_n\langle \eta f(x_n) - Tp_0, J(x_{n+1} - p_0)\rangle \\
&\quad \le (1-\gamma_n\overline{\gamma})^2(1-\alpha_n\beta_n)\|x_n - p_0\|^2 + 2\gamma_n\eta\langle f(x_n) - f(p_0), J(x_{n+1} - p_0) - J(x_n - p_0)\rangle \\
&\qquad + 2\gamma_n\eta\langle f(x_n) - f(p_0), J(x_n - p_0)\rangle + 2\gamma_n\langle \eta f(p_0) - Tp_0, J(x_{n+1} - p_0)\rangle \\
&\qquad + 2(1-\gamma_n\overline{\gamma})^2\beta_n(1-\alpha_n)\langle e_n, J[(1-\alpha_n)(x_n + e_n) - p_0]\rangle \\
&\qquad - 2\alpha_n\beta_n(1-\gamma_n\overline{\gamma})^2\langle p_0, J[(1-\alpha_n)(x_n + e_n) - p_0]\rangle \\
&\quad \le [1 - \gamma_n(\overline{\gamma} - 2\eta k)]\|x_n - p_0\|^2 + 2M_3[\|e_n\| + \alpha_n\beta_n(1-\gamma_n\overline{\gamma})\|p_0\|] \\
&\qquad + \gamma_n[2\langle \eta f(p_0) - Tp_0, J(x_{n+1} - p_0)\rangle + 2\eta\|x_n - p_0\|\|x_{n+1} - x_n\|].
\end{aligned}\tag{24}$$

Let $\delta_n^{(1)} = \gamma_n(\overline{\gamma} - 2\eta k)$, $\delta_n^{(2)} = \gamma_n[2\langle \eta f(p_0) - Tp_0, J(x_{n+1} - p_0)\rangle + 2\eta\|x_n - p_0\|\|x_{n+1} - x_n\|]$, $\delta_n^{(3)} = 2M_3[\|e_n\| + \alpha_n\beta_n(1-\gamma_n\overline{\gamma})\|p_0\|]$. Then (24) can be simplified as $\|x_{n+1} - p_0\|^2 \le (1 - \delta_n^{(1)})\|x_n - p_0\|^2 + \delta_n^{(2)} + \delta_n^{(3)}$.

Using the assumptions (ii) and (iv), the results of Steps 1, 2, and 5 and by using Lemma 4, we know that $x_n \to p_0$, as $n \to +\infty$.

This completes the proof. □

Remark 11 The assumption that 'the α-inversely strongly accretive operator $B_i : C \to E$ satisfies for $\forall r > 0$ and $i = 1, 2, \dots, N$, $\langle B_i x - B_i y, J[(I - rB_i)x - (I - rB_i)y]\rangle \geq 0$' is valid, and we can find an example in Section 3 (Remark 26).

Lemma 12 (see [11]) *Let E be a real q-uniformly smooth Banach space with constant K_q and C be a nonempty, closed, and convex subset of E. Let $A : C \to E$ be an α-inversely strongly accretive operator. Then for $\forall r \leq (\frac{q\alpha}{K_q})^{\frac{1}{q-1}}$, $(I - rA)$ is nonexpansive.*

Corollary 13 *Let E be a real q-uniformly smooth Banach space with constant K_q and also be a uniformly convex Banach space. Let C, Q_C, f, k, η, T, J, A_i, a_m $(m = 0, 1, 2, \dots, N)$, $\overline{\gamma}$, $\{e_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{r_{n,i}\}$ satisfy the same conditions as those in Theorem 10. Let $B_i : C \to E$ be α-inversely strongly accretive operator, where $i = 1, 2, \dots, N$. Let $\{x_n\}$ be generated by the iterative algorithm (A). Suppose further that*

(v) $r_{n,i} \leq (\frac{q\alpha}{K_q})^{\frac{1}{q-1}}$, *for $n \geq 0$ and $i = 1, 2, \dots, N$.*

If $\bigcap_{i=1}^N (A_i + B_i)^{-1}0 \neq \emptyset$, then $\{x_n\}$ converges strongly to a point $p_0 \in \bigcap_{i=1}^N (A_i + B_i)^{-1}0$, which is the unique solution of the variational inequality $(*)$.

Proof Lemma 12 ensures that $(I - r_{n,i}B_i)$ is nonexpansive, for $n \geq 0$ and $i = 1, 2, \dots, N$. Then copy the proof of Theorem 10, the result follows.

This completes the proof. □

Corollary 14 *If $i \equiv 1$, then iterative algorithm* (A) *becomes the following one*:

$$\begin{cases} x_0 \in E, \\ y_n = Q_C[(1 - \alpha_n)(x_n + e_n)], \quad n \geq 0, \\ z_n = (1 - \beta_n)x_n + \beta_n[a_0 y_n + (1 - a_0)J_{r_n}^A(y_n - r_n B y_n)], \quad n \geq 0, \\ x_{n+1} = \gamma_n \eta f(x_n) + (I - \gamma_n T)z_n, \quad n \geq 0. \end{cases} \tag{B}$$

Let E, C, Q_C, f, k, η, T, J, $\overline{\gamma}$, $\{e_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ satisfy the same conditions as those in Theorem 10. Let $A : C \to E$ be m-accretive operator and $B : C \to E$ be α-inversely strongly accretive operator satisfying that

$$\langle Bx - By, J\bigl[(I - rB)x - (I - rB)y\bigr]\rangle \geq 0, \quad \text{for } \forall r > 0, \forall x, y \in C.$$

Suppose that $0 < a_0 < 1$, $\{r_n\} \subset (0, +\infty)$ such that $\sum_{n=0}^{\infty} |r_{n+1} - r_n| < +\infty$ and $r_n \geq \varepsilon > 0$ for $n \geq 0$.

If $(A + B)^{-1}0 \neq \emptyset$, then $\{x_n\}$ generated by the iterative algorithm (B) *converges strongly to $p_0 \in (A + B)^{-1}0$, which is the unique solution of the following variational inequality*: *for $\forall z \in (A + B)^{-1}0$,*

$$\langle (T - \eta f)p_0, J(p_0 - z)\rangle \leq 0. \tag{$**$}$$

Corollary 15 *If $B_i \equiv 0$, then iterative algorithm* (A) *becomes the following one for approximating common zeros of finitely many m-accretive operators*:

$$\begin{cases} x_0 \in E, \\ y_n = Q_C[(1-\alpha_n)(x_n + e_n)], \quad n \geq 0, \\ z_n = (1-\beta_n)x_n + \beta_n(a_0 y_n + \sum_{i=1}^N a_i J_{r_{n,i}}^{A_i} y_n), \quad n \geq 0, \\ x_{n+1} = \gamma_n \eta f(x_n) + (I - \gamma_n T)z_n, \quad n \geq 0. \end{cases} \tag{C}$$

Let E, C, Q_C, f, k, η, T, J, $\overline{\gamma}$, a_m $(m = 1,2,\ldots,N)$, $\{e_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$, $\{r_{n,i}\}$ satisfy the same conditions as those in Theorem 10. *Let $A_i : C \to E$ be m-accretive operator, $i = 1,2,\ldots,N$.*

If $\bigcap_{i=1}^N A_i^{-1}0 \neq \emptyset$, then $\{x_n\}$ generated by (C) *converges strongly to a point $p_0 \in \bigcap_{i=1}^N A_i^{-1}0$, which is the unique solution of the following variational inequality*: *for $\forall z \in \bigcap_{i=1}^N A_i^{-1}0$,*

$$\langle (T - \eta f)p_0, J(p_0 - z)\rangle \leq 0. \tag{***}$$

Corollary 16 *If $A_i \equiv 0$, then iterative algorithm* (A) *becomes to the following one for approximating common zeros of finitely many α-inversely strongly accretive operators*:

$$\begin{cases} x_0 \in E, \\ y_n = Q_C[(1-\alpha_n)(x_n + e_n)], \quad n \geq 0, \\ z_n = (1-\beta_n)x_n + \beta_n[a_0 y_n + \sum_{i=1}^N a_i(y_n - r_{n,i}B_i y_n)], \quad n \geq 0, \\ x_{n+1} = \gamma_n \eta f(x_n) + (I - \gamma_n T)z_n, \quad n \geq 0. \end{cases} \tag{D}$$

Let E, C, Q_C, f, k, η, T, J, $\overline{\gamma}$, a_m $(m = 1,2,\ldots,N)$, $\{e_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$, $\{r_{n,i}\}$ satisfy the same conditions as those in Theorem 10. *Let $B_i : C \to E$ be an α-inversely strongly accretive operator satisfying for $\forall r > 0$ and $i = 1,2,\ldots,N$,*

$$\langle B_i x - B_i y, J[(I - rB_i)x - (I - rB_i)y]\rangle \geq 0.$$

If $\bigcap_{i=1}^N B_i^{-1}0 \neq \emptyset$, then $\{x_n\}$ generated by (D) *converges strongly to a point $p_0 \in \bigcap_{i=1}^N B_i^{-1}0$, which is the unique solution of the following variational inequality*: *for $\forall z \in \bigcap_{i=1}^N B_i^{-1}0$,*

$$\langle (T - \eta f)p_0, J(p_0 - z)\rangle \leq 0. \tag{****}$$

3 Connection with nonlinear capillarity equation

Remark 17 In the next of this paper, we have four purposes: (1) give a new example to show that the assumption that 'the set of zeros of the sum of an m-accretive operator and an α-inversely strongly monotone operator is nonempty' is valid; that is, $(A+B)^{-1}0 \neq \emptyset$ is meaningful; (2) set up the relation between the solution of the capillarity equation and the zero of the sum of an m-accretive operator and an α-inversely strongly accretive operator; (3) apply the iterative algorithm studied in Section 2 to approximate the solution of the capillarity equation; (4) set up the relationship between the solution of the capillarity equation and the solution of one kind variational inequality.

Remark 18 In the following, assume $\frac{2N}{N+1} < p < +\infty$, $1 \leq q, r < +\infty$ if $p \geq N$, and $1 \leq q, r \leq \frac{Np}{N-p}$ if $p < N$, where $N \geq 1$. $\|\cdot\|_p$ denotes the norm in $L^p(\Omega)$. Let $\frac{1}{p} + \frac{1}{p'} = 1$.

We shall examine the following capillarity equation, which is a special case in [20]:

$$\begin{cases} -\operatorname{div}[(1 + \frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}})|\nabla u|^{p-2}\nabla u] + \lambda(|u|^{q-2}u + |u|^{r-2}u) + u(x) = 0, & \text{a.e. in } \Omega, \\ -\langle \vartheta, (1 + \frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}})|\nabla u|^{p-2}\nabla u\rangle = 0, & \text{a.e. on } \Gamma, \end{cases} \tag{E}$$

where Ω is a bounded conical domain of a Euclidean space R^N with its boundary $\Gamma \in C^1$ (*cf.* [21]). $|\cdot|$ denotes the Euclidean norm in R^N, $\langle \cdot, \cdot \rangle$ the Euclidean inner-product and ϑ the exterior normal derivative of Γ. λ is a nonnegative constant.

Theorem 19 (see [20]) *The capillarity equation* (E) *has a unique solution* $u(x) \in L^p(\Omega)$.

Lemma 20 (see [20]) *Define the mapping* $B_{p,q,r} : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ *by*

$$\begin{aligned} \langle v, B_{p,q,r}u \rangle = {} & \int_\Omega \left\langle \left(1 + \frac{|\nabla u|^p}{\sqrt{1 + |\nabla u|^{2p}}}\right)|\nabla u|^{p-2}\nabla u, \nabla v \right\rangle dx \\ & + \lambda \int_\Omega \left|u(x)\right|^{q-2} u(x)v(x)\,dx \\ & + \lambda \int_\Omega \left|u(x)\right|^{r-2} u(x)v(x)\,dx, \end{aligned}$$

for any $u, v \in W^{1,p}(\Omega)$. *Then* $B_{p,q,r}$ *is everywhere defined, strictly monotone, hemi-continuous and coercive.*

Lemma 21 (see [20]) *Define a mapping* $A : L^p(\Omega) \to 2^{L^p(\Omega)}$ *as follows*:

$$D(A) = \left\{u \in L^p(\Omega) | \textit{there exists an } f \in L^p(\Omega), \textit{such that } f \in B_{p,q,r}u\right\}.$$

For $u \in D(A)$, *let* $Au = \{f \in L^p(\Omega) | f \in B_{p,q,r}u\}$.

Then $A : L^p(\Omega) \to 2^{L^p(\Omega)}$ *is m-accretive.*

Lemma 22 *Define a mapping* $C : L^p(\Omega) \to L^p(\Omega)$ *by* $Cu = u(x)$, *for* $\forall u(x) \in L^p(\Omega)$.

Then C *is* 1*-inversely strongly accretive.*

Proof Let $J_p : L^p(\Omega) \to L^{p'}(\Omega)$ denote the normalized duality mapping. Then it is easy to check that $J_p u = |u|^{p-1} \operatorname{sgn} u \|u\|_p^{2-p}$, $\forall u \in L^p(\Omega)$.

Thus for $\forall u(x), v(x) \in L^p(\Omega)$, $\langle Cu - Cv, J_p(u - v)\rangle = \int_\Omega |u - v|^p \|u - v\|_p^{2-p}\,dx = \|u - v\|_p^2$, which implies that C is 1-inversely strongly accretive.

This completes the proof. □

Theorem 23 $u(x) \in L^p(\Omega)$ *is the unique solution of* (E) *if and only if* $u(x) \in (A + C)^{-1}0$.

Proof If $u(x)$ is the solution of (E), then

$$-\operatorname{div}\left[\left(1 + \frac{|\nabla u|^p}{\sqrt{1 + |\nabla u|^{2p}}}\right)|\nabla u|^{p-2}\nabla u\right] + \lambda\left(|u|^{q-2}u + |u|^{r-2}u\right) + u(x) = 0, \quad \text{a.e. in } \Omega.$$

Thus for $\forall\varphi \in C_0^\infty(\Omega)$, by using the property of generalized functions, we have

$$\begin{aligned}
0 &= \left\langle \varphi, -\operatorname{div}\left[\left(1+\frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}}\right)|\nabla u|^{p-2}\nabla u\right] + \lambda\left(|u|^{q-2}u+|u|^{r-2}u\right)+u(x)\right\rangle \\
&= \int_\Omega -\operatorname{div}\left[\left(1+\frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}}\right)|\nabla u|^{p-2}\nabla u\right]\varphi\,dx \\
&\quad + \int_\Omega \left[\lambda\left(|u|^{q-2}u+|u|^{r-2}u\right)+u(x)\right]\varphi\,dx \\
&= \int_\Omega \left\langle \left(1+\frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}}\right)|\nabla u|^{p-2}\nabla u, \nabla\varphi\right\rangle dx \\
&\quad + \int_\Omega \left[\lambda\left(|u|^{q-2}u+|u|^{r-2}u\right)+u(x)\right]\varphi\,dx \\
&= \langle \varphi, B_{p,q,r}u + Cu\rangle = \langle \varphi, Au+Cu\rangle.
\end{aligned}$$

Then $u(x) \in (A+C)^{-1}0$.

On the other hand, if $u(x) \in (A+C)^{-1}0$, then for $\forall\varphi \in C_0^\infty(\Omega)$,

$$\begin{aligned}
0 &= \langle \varphi, Au+Cu\rangle = \langle \varphi, B_{p,q,r}u + Cu\rangle \\
&= \int_\Omega \left\langle \left(1+\frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}}\right)|\nabla u|^{p-2}\nabla u, \nabla\varphi\right\rangle dx \\
&\quad + \lambda\int_\Omega \left(|u|^{q-2}u+|u|^{r-2}u\right)\varphi\,dx + \int_\Omega u(x)\varphi\,dx \\
&= \left\langle \varphi, -\operatorname{div}\left[\left(1+\frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}}\right)|\nabla u|^{p-2}\nabla u\right]\right. \\
&\quad \left. + \lambda\left(|u|^{q-2}u+|u|^{r-2}u\right)+u(x)\right\rangle.
\end{aligned}$$

Then $-\operatorname{div}[(1+\frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}})|\nabla u|^{p-2}\nabla u]+\lambda(|u|^{q-2}u+|u|^{r-2}u)+u(x)=0$, a.e. $x\in\Omega$. By using the Green's formula, we know that for any $v \in W^{1,p}(\Omega)$,

$$\begin{aligned}
&\int_\Gamma \left\langle \vartheta, \left(1+\frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}}\right)|\nabla u|^{p-2}\nabla u\right\rangle v|_\Gamma\,d\Gamma(x) \\
&\quad = \int_\Omega \operatorname{div}\left[\left(1+\frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}}\right)|\nabla u|^{p-2}\nabla u\right] v\,dx \\
&\qquad + \int_\Omega \left\langle \left(1+\frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}}\right)|\nabla u|^{p-2}\nabla u, \nabla v\right\rangle dx \\
&\quad = \int_\Omega \left[\lambda\left(|u|^{q-2}u+|u|^{r-2}u\right)+u(x)\right]dx + \langle v, B_{p,q,r}u\rangle \\
&\qquad - \int_\Omega \lambda\left(|u|^{q-2}u+|u|^{r-2}u\right)dx \\
&\quad = \langle v, Au+Cu\rangle = 0.
\end{aligned}$$

Thus $-\langle \vartheta, (1+\frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}})|\nabla u|^{p-2}\nabla u\rangle = 0$, a.e. on Γ.

Then $u(x) \in (A + C)^{-1}0$ implies that $u(x)$ is the solution of (E).

This completes the proof. □

Theorem 24 *Suppose A and C are the same as those in Lemmas* 21 *and* 22, *respectively. Let $T : L^p(\Omega) \to L^p(\Omega)$ be any strongly positive linear bounded operator with coefficient $\overline{\gamma}$ and $f : L^p(\Omega) \to L^p(\Omega)$ be a contraction with coefficient k. Suppose the following conditions are satisfied*:

(i) $0 < \eta < \frac{\overline{\gamma}}{2k}$, *and* $0 < a < 1$;

(ii) $\{e_n\}_{n=0}^{\infty} \subset L^p(\Omega)$, $\sum_{n=0}^{\infty} \|e_n\| < +\infty$;

(iii) $\{\alpha_n\}$, $\{\beta_n\}$, *and* $\{\gamma_n\}$ *are three sequences in* $(0,1)$. $\gamma_n \to 0$, $\frac{\gamma_{n-1}}{\gamma_n} \to 1$, $\beta_n \to 1$, $\alpha_n \to 0$, *as* $n \to \infty$. $\sum_{n=0}^{\infty} \gamma_n = \infty$, $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < +\infty$, $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < +\infty$, $\sum_{n=0}^{\infty} (1 - \gamma_n\overline{\gamma})\alpha_n\beta_n < +\infty$;

(iv) $\{r_n\} \subset (0,1)$ *such that* $\sum_{n=0}^{\infty} |r_{n+1} - r_n| < +\infty$, *and* $1 \geq r_n \geq \varepsilon > 0$ *for* $n \geq 0$.

If we construct the following iterative algorithm:

$$\begin{cases} u_0(x) \in L^p(\Omega), \\ v_n(x) = (1 - \alpha_n)(u_n(x) + e_n(x)), \\ w_n(x) = (1 - \beta_n)u_n(x) + \beta_n[av_n(x) + (1 - a)J_{r_n}^A(v_n(x) - r_nCv_n(x))], \\ u_{n+1}(x) = \gamma_n\eta f(u_n) + (I - \gamma_nT)w_n(x), \quad n \geq 0, \end{cases} \tag{F}$$

then $u_n(x)$ converges strongly to $u(x) \in (A + C)^{-1}0$, which is the unique solution of the capillarity equation (E) *and satisfies the following variational inequality*: *for* $\forall z(x) \in (A + C)^{-1}0$,

$$\langle (T - \eta f)u(x), J_p(u(x) - z) \rangle \leq 0.$$

Remark 25 From Theorem 24 we can easily see the relationship among the solution of the capillarity equation, the solution of a variational inequality, and the zero of sum of an m-accretive operator and an α-inversely strongly accretive operator.

Remark 26 Let C be the 1-inversely strong accretive operator defined in Lemma 22, then it is obvious that C satisfies

$$\langle Cx - Cy, J_p[(I - rC)x - (I - rC)y] \rangle \geq 0, \quad \text{for } 1 \geq r > 0, x, y \in L^p(\Omega).$$

Thus the assumption imposed on B_i in Theorem 10 is valid.

Competing interests

The authors declare that there is no conflict of interests regarding the publication of this article.

Authors' contributions

All authors contributed equally to this manuscript. All authors read and approved the final manuscript.

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