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Multi-parameter, impulsive effects and positive periodic solutions of first-order functional differential equations

Xuemei Zhang^{1*} and Meiqiang Feng²

*Correspondence: zxm74@sina.com

¹Department of Mathematics and Physics, North China Electric Power University, Beijing, 102206, People's Republic of China

Full list of author information is available at the end of the article

Abstract

The authors consider the existence of positive periodic solution for the impulsive functional differential equation

$$u'(t) = -a(t)g(u(h_1(t)))u(t) + \lambda b(t)f(t, u(h_2(t)), \int_{-\zeta}^0 e(s)u(t+s) ds), t \in \mathbf{R}, t \neq t_k,$$

$$u(t_k^+) - u(t_k) = \mu J_k(t_k, u(t_k)), k \in \mathbf{Z},$$

where $\lambda > 0$ and $\mu > 0$ are two parameters. Several new and more general existence and multiplicity results are derived in terms of different values of $\lambda > 0$ and $\mu > 0$. Here we not only consider the case that g is bounded, but the case that g is not necessarily bounded is also considered. Our results improve those in (Ma *et al.* in *J. Math. Anal. Appl.* 384:527-535, 2011; Li *et al.* in *Comput. Math. Appl.* 56:2556-2560, 2008). Moreover, the parameter dependence of the periodic solution is also studied.

Keywords: multi-parameter; impulsive effect; positive periodic solution; functional differential equations; parameter dependence of periodic solution

1 Introduction

Impulsive functional differential equations are emerging as an important area of investigation, since various real-world simulation processes which depend on their prehistory and are subject to short time disturbances, such as population dynamics, biology, biotechnology, industrial robotic, pharmacokinetics, optimal control, *etc.*, can be expressed by functional differential equations with impulses; see [1–5]. Moreover, Cushing [6] pointed out that any biological or environmental parameters are naturally subject to fluctuations in time, due to seasonal effects of weather, food supply, mating habits, hunting or harvesting seasons, *etc.* Therefore, it is necessary and important to consider models with periodic ecological parameters or perturbations, which may be more natural (see [7–27]). For example, Liu and Li [9] considered the following first-order functional differential equation with a parameter:

$$u'(t) = -a(t)u(t) + \lambda b(t)f(u(t - \tau_0(t))). \quad (\text{A})$$

By using the eigenvalue theory of operators, the authors established several sufficient conditions for the existence of positive T -periodic solutions.

In [10], Wang investigated the following first-order functional differential equation with a parameter:

$$u'(t) = a(t)g(u(t))u(t) - \lambda b(t)f(u(t - \tau_0(t))), \tag{B}$$

and by virtue of the fixed point theorems in a cone, the author obtained the existence, non-existence, and multiplicity of positive ω -periodic solutions under suitable conditions imposed on the nonlinear term f .

Yan [28] pointed out that (A) and (B) include various models. For example, Cooke and Kaplan [29], and Cooke and Yorke [30] studied, respectively, the equation

$$u'(t) = -g(y(t)) + f(y(t - \tau))$$

for epidemic and population growth and gonorrhoea epidemics.

Mackey and Glass [31] considered the following equation:

$$u'(t) = -\gamma u(t) + \frac{\beta_0 \theta^n}{\theta^n + u^n(t - \tau)}$$

for model of hematopoiesis (blood cell production).

For some other recent excellent results and applications on (A) and (B), we refer the reader to papers by Jin and Wang [32], Ye *et al.* [33], Li *et al.* [34], Graef and Kong [35, 36], Franco *et al.* [37], Wu and Wang [38], Padhi and Srivastava [39], Olach [40], Amster and Idels [41], Chen *et al.* [42], Ma *et al.* [43] and Weng and Sun [44].

As regards the ideal of an impulsive perturbation, in [28], a pioneer paper concerning the solvability of periodic problem, Yan considered the following impulsive functional differential equation:

$$\begin{cases} u'(t) = g(t, u(t)) - \lambda f(t, u(t - \tau_0(t))), & t \in \mathbf{R}, t \neq t_k, \\ u(t_k^+) - u(t_k) = \mu I_k(t_k, u(t_k - \tau(t_k))), & k \in \mathbf{Z}. \end{cases} \tag{C}$$

By utilizing a well-known fixed point index theorem in cone, the author got some sufficient conditions of the existence and non-existence of positive periodic solutions of (C).

In this paper, we investigate the existence of positive periodic solutions of the impulsive functional differential equation

$$\begin{cases} u'(t) = -a(t)g(u(h_1(t)))u(t) \\ \quad + \lambda b(t)f(t, u(h_2(t)), \int_{-\zeta}^0 e(s)u(t+s) ds), & t \in \mathbf{R}, t \neq t_k, \\ u(t_k^+) - u(t_k) = \mu I_k(t_k, u(t_k)), & k \in \mathbf{Z}, \end{cases} \tag{1.1}$$

where $\lambda > 0$ and $\mu > 0$ are parameters, $e \in C([-\zeta, 0], \mathbf{R})$, $\int_{-\zeta}^0 e(s) ds = 1$, and $\{t_k, k \in \mathbf{Z}\}$ is an increasing sequence of real number with $\lim_{k \rightarrow \pm\infty} t_k = \pm\infty$. In addition, let

$$\omega > 0, \quad \mathbf{R} = (-\infty, +\infty), \quad \mathbf{R}_+ = [0, +\infty).$$

We make the following hypotheses:

(H₁) $a, b \in C(\mathbf{R}, \mathbf{R}_+)$ are ω -periodic functions satisfying $\int_0^\omega a(t) dt > 0$, $\int_0^\omega b(t) dt > 0$;

- (H₂) $h_i \in C(\mathbf{R}, \mathbf{R}_+)$ ($i = 1, 2$) is ω -periodic function, $g \in C(\mathbf{R}_+, \mathbf{R}_+)$;
- (H₃) $f \in C(\mathbf{R} \times \mathbf{R}_+ \times \mathbf{R}_+, \mathbf{R}_+)$ is ω -periodic function in t ; moreover, $f(t, x, y) > 0$ for all $t, x > 0$ and $y > 0$;
- (H₄) $I_k \in C(\mathbf{R} \times \mathbf{R}_+, \mathbf{R}_+)$ is ω -periodic function in t and $I_k(t, x) > 0$ for all t and $x > 0, k \in \mathbf{Z}$; moreover, there exists a positive integer p such that

$$I_{k+p}(t_{k+p}, x) = I_k(t_k, x), \quad t_{k+p} = t_k + \omega, k \in \mathbf{Z}.$$

Without loss of generality, we assume that $[0, \omega] \cap \{t_k, k \in \mathbf{Z}\} = \{t_0, t_1, \dots, t_{p-1}\}$.

Some special cases of (1.1) have been investigated. For example, Li *et al.* [34] discussed problem (1.1) in the case that $\mu \equiv 1$ and $I_k(t, u) = I_k(u), k \in \mathbf{Z}^+$. By using the well-known Leggett-Williams multiple fixed point theorem, the authors obtained the existence of multiple positive periodic solutions of problem (1.1). However, in [34], Li *et al.* only dealt with the case that there exist two positive constants l and L such that

$$0 < l \leq g(u) \leq L < \infty \quad \text{for all } u > 0. \tag{1.2}$$

It is interesting to know whether there is a positive periodic solution of (1.1) when g does not satisfy (1.2). Very recently, Ma *et al.* [43] studied the existence of positive periodic solutions of (1.1) under the following two cases:

$$0 < l \leq g(u) < \infty \quad \text{for all } u > 0, \tag{1.3}$$

$$0 < g(u) \leq L \quad \text{for all } u > 0, \tag{1.4}$$

where l and L are two positive constants. However, Ma *et al.* obtained the existence of at least one positive periodic solution only for $I_k(t, u) \equiv 0, k \in \mathbf{Z}$.

At the same time, we notice that there is almost no paper except [28] studying impulsive functional differential equations with two parameters. By using a well-known fixed point index theorem due to Krasnoselskii, the author obtained the following results in the case g is bounded and $e(t) \equiv 0, t \in [-\zeta, 0]$.

Theorem A (Theorem 3.1 of [28]) *Let the following conditions hold.*

- (A₁) $\lambda > 0$ and $\mu \geq 0$ are parameters.
- (A₂) $t_k, k \in \mathbf{Z}$, is an increasing sequence of real number with $\lim_{k \rightarrow \pm\infty} t_k = \pm\infty$.
- (A₃) h and $f : \mathbf{R} \times \mathbf{R}^+ \rightarrow \mathbf{R}^+$ satisfy Carathéodory conditions, that is, $h(t, y)$ and $f(t, y)$ are locally Lebesgue measurable in t for each fixed y and are continuous in y for each fixed t , are ω -periodic functions in t . Moreover, $f(t, y) > 0$ for all t and $y > 0$. $\tau : \mathbf{R} \rightarrow \mathbf{R}$ is locally bounded Lebesgue measurable ω -periodic function.
- (A₄) There exist ω -periodic functions a_1 and $a_2 : \mathbf{R} \rightarrow \mathbf{R}^+$ which are locally bounded Lebesgue measurable so that $a_1(t)y \leq h(t, y) \leq a_2(t)y$ for all $y > 0$ and $\lim_{y \rightarrow 0^+} h(t, y)y$ exists, $\int_0^\omega a_1(t) dt > 0$.
- (A₅) $I_k : \mathbf{R} \times [0, \infty) \rightarrow \mathbf{R}, k \in \mathbf{Z}$, satisfy Carathéodory conditions and are ω -periodic functions in t and there exists an integer ρ such that $I_{k+\rho}(t_{k+\rho}, y) = I_k(t_k, y), t_{k+\rho} = t_k + \omega, k \in \mathbf{Z}$. Moreover, $I_k(t, 0) = 0$ for all $k \in \mathbf{Z}$.
- (A₆) p and $q : \mathbf{R} \rightarrow \mathbf{R}^+$ are positive bounded Lebesgue measurable ω -periodic functions and is bounded away from zero.

Suppose further that f^0, f_∞, I_∞ , and I^0 are positive constants, where

$$f^0 = \limsup_{y \rightarrow 0^+} \max_{t \in [0, \omega]} \frac{f(t, y)}{p(t)y}, \quad f_\infty = \liminf_{y \rightarrow \infty} \min_{t \in [0, \omega]} \frac{f(t, y)}{p(t)y},$$

$$I^0 = \limsup_{y \rightarrow 0^+} \max_{t \in [0, \omega], k \in [0, \rho]} \frac{I_k(t, y)}{q(t)y}, \quad I_\infty = \liminf_{y \rightarrow \infty} \min_{t \in [0, \omega], k \in [0, \rho]} \frac{I_k(t, y)}{q(t)y}.$$

If

$$\beta(\lambda f^0 P + \mu I^0 Q) < 1$$

and

$$\alpha \sigma (\lambda f_\infty P + \mu I_\infty Q) > 1,$$

then problem (1.1) has a positive ω -periodic solution.

It is not difficult to see that the conditions of Theorem A are not the optimal conditions which guarantee the existence of at least one positive periodic solution for problem (1.1). In fact, if

$$\beta(\lambda f^0 P + \mu I^0 Q) < 1$$

or

$$\alpha \sigma (\lambda f_\infty P + \mu I_\infty Q) > 1,$$

we can prove that problem (1.1) has at least one positive periodic solution, respectively.

Motivated by the papers mentioned above, we will extend the results of [9, 10, 28, 32, 34, 43] to problem (1.1) in the case that g satisfies (1.2), (1.3), and (1.4), respectively. Moreover, the parameter dependence of the periodic solution is also studied.

The rest of the paper is organized as follows. In Section 2, we study the existence and multiplicity results of positive periodic solutions for problem (1.1) when the function g satisfies (1.2). In Section 3, we establish the dependence results of the positive periodic solution on the parameter for problem (1.1). The final section of the paper studies the existence of positive periodic solutions for problem (1.1) when the function g satisfies (1.3) or (1.4). Some remarks are also given to explain why we do not consider the multiplicity of positive periodic solutions and the dependence results of the positive periodic solution on the parameter when the function g satisfies (1.3) or (1.4).

2 Existence results under (1.2)

2.1 Main results

In this section, we state the main results, including existence and multiplicity results of positive periodic solutions for problem (1.1).

For convenience, we introduce the following notations:

$$f^0 = \limsup_{x \rightarrow 0^+, y \rightarrow 0^+} \max_{t \in J} \frac{f(t, x, y)}{x}, \quad f^\infty = \limsup_{x \rightarrow \infty, y \rightarrow \infty} \max_{t \in J} \frac{f(t, x, y)}{x},$$

$$f_0 = \liminf_{x \rightarrow 0^+, y \rightarrow 0^+} \min_{t \in J} \frac{f(t, x, y)}{x}, \quad f_\infty = \liminf_{x \rightarrow \infty, y \rightarrow \infty} \min_{t \in J} \frac{f(t, x, y)}{x},$$

$$\begin{aligned}
 I^0(k) &= \limsup_{x \rightarrow 0^+} \frac{I_k(t_k, x)}{x}, & I^\infty(k) &= \limsup_{x \rightarrow \infty} \frac{I_k(t_k, x)}{x}, \\
 I_0(k) &= \liminf_{x \rightarrow 0^+} \frac{I_k(t_k, x)}{x}, & I_\infty(k) &= \liminf_{x \rightarrow \infty} \frac{I_k(t_k, x)}{x},
 \end{aligned}$$

where $J = [0, \omega]$, $k = 0, 1, \dots, p - 1$. Moreover, we choose four numbers r, r_1, r_2 , and R satisfying

$$0 < r < r_1 < \delta r_2 < r_2 < R < +\infty, \tag{2.1}$$

where δ is defined in (2.9).

Theorem 2.1 *Let (H₁)-(H₄) hold. Assume that $0 < f_\infty \leq f^\infty < +\infty$, $0 < I_\infty(k) \leq I^\infty(k) < +\infty$, $k = 0, 1, \dots, p - 1$, and the function g satisfies (1.2). Then*

- (i) *there exist $\lambda_0 > 0$ and $\mu_0 > 0$ such that, for any $\lambda > \lambda_0$ and $\mu > \mu_0$, problem (1.1) has a positive periodic solution $u(t)$ with*

$$\delta r \leq u(t) \leq \frac{1}{\delta} R, \quad t \in J; \tag{2.2}$$

- (ii) *there exist $\bar{\lambda}_0 > 0$ and $\bar{\mu}_0 > 0$ such that, for any $0 < \lambda < \bar{\lambda}_0$ and $0 < \mu < \bar{\mu}_0$, problem (1.1) has a positive periodic solution $u(t)$, $t \in J$ with property (2.2).*

Theorem 2.2 *Let (H₁)-(H₄) hold. Assume that $0 < f_0 \leq f^0 < +\infty$, $0 < I_0(k) \leq I^0(k) < +\infty$, $k = 0, 1, \dots, p - 1$ and the function g satisfies (1.2). Then*

- (i) *there exist $\lambda_0 > 0$ and $\mu_0 > 0$ such that, for any $\lambda > \lambda_0$ and $\mu > \mu_0$, problem (1.1) has a positive periodic solution $u(t)$ with*

$$\delta r \leq u(t) \leq R, \quad t \in J; \tag{2.3}$$

- (ii) *there exist $\bar{\lambda}_0 > 0$ and $\bar{\mu}_0 > 0$ such that, for any $0 < \lambda < \bar{\lambda}_0$ and $0 < \mu < \bar{\mu}_0$, problem (1.1) has a positive periodic solution $u(t)$, $t \in J$ with property (2.3).*

Remark 2.1 Some ideas of Theorems 2.1 and 2.2 are from Yan [28].

Theorem 2.3 *Assume that (H₁)-(H₄) hold and the function g satisfies (1.2).*

- (i) *If $f^\infty = 0$ and $I^\infty(k) = 0$, $k = 0, 1, \dots, p - 1$, then there exist $\lambda_0 > 0$ and $\mu_0 > 0$ such that, for any $\lambda > \lambda_0$ and $\mu > \mu_0$, problem (1.1) has a positive periodic solution $u(t)$, $t \in J$ with property (2.2).*
- (ii) *If $f^0 = 0$ and $I^0(k) = 0$, $k = 0, 1, \dots, p - 1$, then there exist $\lambda_0 > 0$ and $\mu_0 > 0$ such that, for any $\lambda > \lambda_0$ and $\mu > \mu_0$, problem (1.1) has a positive periodic solution $u(t)$ with property (2.3).*
- (iii) *If $f^0 = f^\infty = I^\infty(k) = I^0(k) = 0$, $k = 0, 1, \dots, p - 1$, then there exist $\lambda_0 > 0$ and $\mu_0 > 0$ such that, for any $\lambda > \lambda_0$ and $\mu > \mu_0$, problem (1.1) has at least two positive periodic solutions $u_1(t)$ and $u_2(t)$ with*

$$\delta r \leq u_1(t) \leq r_1 < \delta r_2 \leq u_2(t) \leq R, \quad t \in J. \tag{2.4}$$

Theorem 2.4 Assume that (H_1) - (H_4) hold and the function g satisfies (1.2).

- (i) If $f_\infty = +\infty$ and $I_\infty(k) = +\infty, k = 0, 1, \dots, p - 1$, then there exist $\bar{\lambda}_0 > 0$ and $\bar{\mu}_0 > 0$ such that, for any $0 < \lambda < \bar{\lambda}_0$ and $0 < \mu < \bar{\mu}_0$, problem (1.1) has a positive periodic solution $u(t), t \in J$ with property (2.2).
- (ii) If $f_0 = +\infty$ and $I_0(k) = +\infty, k = 0, 1, \dots, p - 1$, then there exist $\bar{\lambda}_0 > 0$ and $\bar{\mu}_0 > 0$ such that, for any $0 < \lambda < \bar{\lambda}_0$ and $0 < \mu < \bar{\mu}_0$, problem (1.1) has a positive periodic solution $u(t), t \in J$ with property (2.3).
- (iii) If $f_0 = f_\infty = I^\infty(k) = I_0(k) = +\infty, k = 0, 1, \dots, p - 1$, then there exist $\bar{\lambda}_0 > 0$ and $\bar{\mu}_0 > 0$ such that, for any $0 < \lambda < \bar{\lambda}_0$ and $0 < \mu < \bar{\mu}_0$, problem (1.1) has at least two positive periodic solutions $u_1(t)$ and $u_2(t)$ with

$$\delta r \leq u_1(t) \leq r_1 < \delta r_2 \leq u_2(t) \leq \frac{1}{\delta} R, \quad t \in J. \tag{2.5}$$

Remark 2.2 Some ideas of the proof of Theorem 2.3 and Theorem 2.4 are from Wang [10].

2.2 Preliminaries

In our main results, we will make use of the following definitions and lemmas.

Definition 2.1 A function $u : \mathbf{R} \rightarrow (0, \infty)$ is called a positive solution of (1.1) if it satisfies:

- (i) $u(t)$ is absolutely continuous on each (t_k, t_{k+1}) ;
- (ii) for each $k \in \mathbf{Z}, u(t_k^+)$ and $u(t_k^-)$ exist and $u(t_k^-) = u(t_k)$;
- (iii) $u(t)$ satisfies (1.1).

Definition 2.2 A solution $u(t)$ of (1.1) on \mathbf{R} is called ω -periodic if $u(t + \omega) = u(t)$.

Let E be the Banach space

$$\{u | u : \mathbf{R} \rightarrow \mathbf{R} \text{ is continuous at } t \neq t_k, u(t_k^-) = u(t_k) \text{ and } u(t_k^+) \text{ exists, } k \in \mathbf{Z} \text{ and } u(t + \omega) = u(t)\}$$

with $\|u\| = \sup_{0 \leq t \leq \omega} |u(t)|$. We denote

$$\Omega_r := \{u \in E : \|u\| < r\}, \quad \partial\Omega = \{u \in E : \|u\| = r\}$$

for all $r > 0$ in the sequel.

Lemma 2.1 (See [34]) Assume that (H_1) - (H_4) hold and the function g satisfies (1.2). Then $u \in E$ is a solution of problem (1.1) if and only if u is a solution of the integral equation

$$u(t) = \lambda \int_t^{t+\omega} G(t, s) b(s) f\left(s, u(h_2(s)), \int_{-\zeta}^0 e(s) u(s + v) dv\right) ds + \mu \sum_{t \leq t_k < t + \omega} G(t, t_k) I_k(t_k, u(t_k)), \tag{2.6}$$

where

$$G(t, s) = \frac{e^{\int_t^s a(v)g(u(h_1(v))) dv}}{e^{\int_0^\omega a(v)g(u(h_1(v))) dv} - 1}, \quad s \in [t, t + \omega]. \tag{2.7}$$

Further, it follows from (2.7) and (1.2) that

$$\frac{1}{\sigma^L - 1} = \alpha \leq G(t, s) \leq \beta = \frac{\sigma^L}{\sigma^l - 1}, \quad s \in [t, t + \omega], \tag{2.8}$$

where $\sigma = e^{\int_0^\omega a(t) dt}$.

Define a cone K in E by

$$K = \{u \in E : u(t) \geq \delta \|u\|, t \in J\},$$

where

$$0 < \delta = \frac{\sigma^l - 1}{\sigma^L(\sigma^L - 1)} < 1. \tag{2.9}$$

It is easy to see K is a closed non-empty subset of E .

Define $T_\lambda^\mu : K \rightarrow E$ by

$$\begin{aligned} (T_\lambda^\mu u)(t) = & \lambda \int_t^{t+\omega} G(t, s) b(s) f\left(s, u(h_2(s)), \int_{-\zeta}^0 e(s) u(s + \nu) dv\right) ds \\ & + \mu \sum_{t \leq t_k < t + \omega} G(t, t_k) I_k(t_k, u(t_k)). \end{aligned} \tag{2.10}$$

From (2.10) and Lemma 2.1, it is easy to obtain the following result.

Lemma 2.2 (See [16]) *Assume that (H₁)-(H₄) hold. Problem (1.1) is equivalent to the fixed point problem of T_λ^μ in K .*

Lemma 2.3 (See Lemmas 2.1 and 2.2 in [28]) *Assume that (H₁)-(H₄) hold and the function g satisfies (1.2). Then $T_\lambda^\mu : K \rightarrow K$ is completely continuous.*

The following well-known result of the fixed point is crucial in our arguments.

Lemma 2.4 (See [45]) *Let P be a cone in a real Banach space E . Assume Ω_1, Ω_2 are bounded open sets in E with $0 \in \Omega_1, \bar{\Omega}_1 \subset \Omega_2$. If*

$$A : P \cap (\bar{\Omega}_2 \setminus \Omega_1) \rightarrow P$$

is completely continuous such that either

- (a) $\|Ax\| \leq \|x\|, \forall x \in P \cap \partial\Omega_1$ and $\|Ax\| \geq \|x\|, \forall x \in P \cap \partial\Omega_2$, or
- (b) $\|Ax\| \geq \|x\|, \forall x \in P \cap \partial\Omega_1$ and $\|Ax\| \leq \|x\|, \forall x \in P \cap \partial\Omega_2$,

then A has at least one fixed point in $P \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

2.3 Proofs of the main results

For convenience we write

$$\gamma = \int_0^\omega b(s) ds.$$

Proof of Theorem 2.1 Part (i). Noticing that $f(t, x, y) > 0, I_k(t, x) > 0$ ($k = 0, 1, \dots, p - 1$) for all $t, x > 0$, and $y > 0$, we can define, for $r > 0$,

$$m_r = \min_{t \in J, \delta r \leq x, y \leq r} \{f(t, x, y)\} > 0, \quad m^* = \min\{m_k, k = 0, 1, \dots, p - 1\} > 0,$$

where

$$m_k = \min_{t \in J, \delta r \leq x \leq r} \{I_k(t, x)\} > 0, \quad k = 0, 1, \dots, p - 1.$$

Since $\int_{-\zeta}^0 e(s) ds = 1$, it follows from $\delta r \leq x \leq r$ that

$$\delta r \leq \int_{-\zeta}^0 e(s)x(s+v) dv \leq r.$$

Let

$$\lambda_0 \geq (2\alpha m_r \gamma)^{-1} r, \quad \mu_0 \geq (2\alpha m^* p)^{-1} r.$$

Then for any $u \in K \cap \partial\Omega_r$, and $\lambda > \lambda_0, \mu > \mu_0$, we have

$$\begin{aligned} (T_\lambda^\mu u)(t) &= \lambda \int_t^{t+\omega} G(t, s)b(s)f\left(s, u(h_2(s)), \int_{-\zeta}^0 e(s)u(s+v) dv\right) ds \\ &\quad + \mu \sum_{t \leq t_k < t+\omega} G(t, t_k)I_k(t_k, u(t_k)) \\ &\geq \alpha \left[\lambda \int_0^\omega b(s)f\left(s, u(h_2(s)), \int_{-\zeta}^0 e(s)u(s+v) dv\right) ds + \mu \sum_{0 \leq t_k < \omega} I_k(t_k, u(t_k)) \right] \\ &\geq \alpha \left[\lambda \int_0^\omega b(s)f\left(s, u(h_2(s)), \int_{-\zeta}^0 e(s)u(s+v) dv\right) ds + \mu \sum_{k=0}^{p-1} I_k(t_k, u(t_k)) \right] \\ &\geq \alpha \left[\lambda m_r \int_0^\omega b(s) ds + \mu m^* p \right] \\ &> \alpha [\lambda_0 m_r \gamma + \mu_0 m^* p] \\ &\geq \frac{1}{2} r + \frac{1}{2} r = r, \end{aligned}$$

which implies that

$$\|T_\lambda^\mu u\| > \|u\|, \quad \forall u \in K \cap \partial\Omega_r, \lambda > \lambda_0 \text{ and } \mu > \mu_0. \tag{2.11}$$

If $0 < f^\infty < +\infty, 0 < I^\infty(k) < +\infty, k = 0, 1, \dots, p - 1$, then there exist $l_1 > 0, l_2 > 0$, and $R > r > 0$ such that

$$f(t, x, y) < l_1 x, \quad I_k(t, x) < l_2 x \quad (\forall t \in J, x \geq R, y \geq R, k = 0, 1, \dots, p - 1),$$

where l_1 satisfies

$$2l_1 \beta \lambda \gamma \leq 1, \tag{2.12}$$

l_2 satisfies

$$2l_2\beta\mu p \leq 1. \tag{2.13}$$

Let $\eta = \frac{R}{\delta}$. Thus, when $u \in K \cap \partial\Omega_\eta$ we have

$$u(t) \geq \delta\|u\| = \delta\eta = R, \quad t \in J.$$

Since $\int_{-\zeta}^0 e(s) ds = 1$, it follows from $x \geq R$ that

$$\int_{-\zeta}^0 e(s)x(s+v) dv \geq R.$$

Therefore, it follows from (2.10) that

$$\begin{aligned} (T_\lambda^\mu u)(t) &= \lambda \int_t^{t+\omega} G(t,s)b(s)f\left(s,u(h_2(s)), \int_{-\zeta}^0 e(s)u(s+v) dv\right) ds \\ &\quad + \mu \sum_{t \leq t_k < t+\omega} G(t,t_k)I_k(t_k, u(t_k)) \\ &\leq \beta \left[\lambda \int_0^\omega b(s)f\left(s,u(h_2(s)), \int_{-\zeta}^0 e(s)u(s+v) dv\right) ds + \mu \sum_{0 \leq t_k < \omega} I_k(t_k, u(t_k)) \right] \\ &\leq \beta \left[\lambda \int_0^\omega b(s)f\left(s,u(h_2(s)), \int_{-\zeta}^0 e(s)u(s+v) dv\right) ds + \mu \sum_{k=0}^{p-1} I_k(t_k, u(t_k)) \right] \\ &\leq \beta \left[\lambda \int_0^\omega b(s)l_1\|u\| ds + \mu \sum_{k=0}^{p-1} l_2\|u\| \right] \\ &\leq \beta \left[\lambda l_1 \int_0^\omega b(s) ds \|u\| + \mu l_2 p \|u\| \right] \\ &\leq \beta [\lambda l_1 \gamma \|u\| + \mu l_2 p \|u\|] \\ &\leq \frac{1}{2} \|u\| + \frac{1}{2} \|u\| \\ &= \|u\|. \end{aligned}$$

This yields

$$\|T_\lambda^\mu u\| \leq \|u\|, \quad \forall u \in K \cap \partial\Omega_\eta. \tag{2.14}$$

Applying (b) of Lemma 2.4 to (2.11) and (2.14) shows that T_λ^μ has a fixed point $u \in K \cap (\bar{\Omega}_\eta \setminus \Omega_r)$ with $r \leq \|u\| \leq \eta = \frac{1}{\delta}R$. Hence, since for $u \in K$ we have $u(t) \geq \delta\|u\|$, $t \in J$, it follows that (2.2) holds. This gives the proof of part (i).

Part (ii). Noticing that $f(t,x,y) > 0$, $I_k(t,x) > 0$ ($k = 0, 1, \dots, p-1$) for all $t, x > 0$, and $y > 0$, we can define, for $r > 0$,

$$M_r = \max_{t \in J, 0 \leq x, y \leq r} \{f(t,x,y)\} > 0, \quad M^* = \max\{M_k, k = 0, 1, \dots, p-1\} > 0,$$

where

$$M_k = \max_{t \in J, 0 \leq x \leq r} \{I_k(t, x)\}, \quad k = 0, 1, \dots, p-1.$$

Since $\int_{-\zeta}^0 e(s) ds = 1$, it follows from $0 \leq x \leq r$ that

$$0 \leq \int_{-\zeta}^0 e(s)x(s+v) dv \leq r.$$

Let

$$\bar{\lambda}_0 \leq \frac{1}{2\beta M_r \gamma} r, \quad \bar{\mu}_0 \leq \frac{1}{2\beta p M^*} r.$$

Then, for $u \in K \cap \partial\Omega_r$ and $\lambda < \bar{\lambda}_0, \mu < \bar{\mu}_0$, we have

$$\begin{aligned} (T_\lambda^\mu u)(t) &= \lambda \int_t^{t+\omega} G(t,s)b(s)f\left(s, u(h_2(s)), \int_{-\zeta}^0 e(s)u(s+v) dv\right) ds \\ &\quad + \mu \sum_{t \leq t_k < t+\omega} G(t,t_k)I_k(t_k, u(t_k)) \\ &\leq \beta \left[\lambda \int_0^\omega b(s)f\left(s, u(h_2(s)), \int_{-\zeta}^0 e(s)u(s+v) dv\right) ds + \mu \sum_{0 \leq t_k < \omega} I_k(t_k, u(t_k)) \right] \\ &\leq \beta \left[\lambda \int_0^\omega b(s)f\left(s, u(h_2(s)), \int_{-\zeta}^0 e(s)u(s+v) dv\right) ds + \mu \sum_{k=0}^{p-1} I_k(t_k, u(t_k)) \right] \\ &\leq \beta \left[\lambda \int_0^\omega b(s)M_r ds + \mu \sum_{k=0}^{p-1} M^* \right] \\ &\leq \beta \left[\lambda M_r \int_0^\omega b(s) ds + \mu p M^* \right] \\ &\leq \beta [\lambda M_r \gamma + \mu M^* p] \\ &< \beta [\bar{\lambda}_0 M_r \gamma + \bar{\mu}_0 M^* p] \\ &\leq \frac{1}{2} r + \frac{1}{2} r \\ &= r. \end{aligned}$$

This implies

$$\|T_\lambda^\mu u\| < \|u\|, \quad \forall u \in K \cap \partial\Omega_r. \tag{2.15}$$

If $0 < f_\infty < +\infty, 0 < I_\infty(k) < +\infty, k = 0, 1, \dots, p-1$, then there exist $l_3 > 0, l_4 > 0$, and $R > r > 0$ such that

$$f(t, x, y) > l_3 x, \quad I_k(t, x) > l_4 x \quad (\forall t \in J, x \geq R, y \geq R, k = 0, 1, \dots, p-1),$$

where l_3 satisfies

$$2\alpha \delta \lambda l_3 \gamma \geq 1, \tag{2.16}$$

l_4 satisfies

$$2\alpha\delta\mu l_4 p \geq 1. \tag{2.17}$$

Let $\bar{\eta} = \frac{R}{\delta}$. Thus, when $u \in K \cap \partial\Omega_{\bar{\eta}}$ we have

$$u(t) \geq \delta\|u\| = \delta\bar{\eta} = R, \quad t \in J.$$

Since $\int_{-\zeta}^0 e(s) ds = 1$, it follows from $x \geq R$ that

$$\int_{-\zeta}^0 e(s)x(s+v) dv \geq R.$$

Therefore, for any $u \in K \cap \partial\Omega_{\bar{\eta}}$, we have

$$\begin{aligned} (T_\lambda^\mu u)(t) &= \lambda \int_t^{t+\omega} G(t,s)b(s)f\left(s,u(h_2(s)), \int_{-\zeta}^0 e(s)u(s+v) dv\right) ds \\ &\quad + \mu \sum_{t \leq t_k < t+\omega} G(t,t_k)I_k(t_k, u(t_k)) \\ &\geq \alpha \left[\lambda \int_0^\omega b(s)f\left(s,u(h_2(s)), \int_{-\zeta}^0 e(s)u(s+v) dv\right) ds + \mu \sum_{0 \leq t_k < \omega} I_k(t_k, u(t_k)) \right] \\ &\geq \alpha \left[\lambda \int_0^\omega b(s)f\left(s,u(h_2(s)), \int_{-\zeta}^0 e(s)u(s+v) dv\right) ds + \mu \sum_{k=0}^{p-1} I_k(t_k, u(t_k)) \right] \\ &\geq \alpha \left[\lambda \int_0^\omega b(s)l_3\delta\|u\| ds + \mu l_4\delta\|u\|^p \right] \\ &\geq \alpha\delta(\lambda l_3\gamma\|u\| + \mu l_4 p\|u\|) \\ &\geq \frac{1}{2}\|u\| + \frac{1}{2}\|u\| \\ &= \|u\|, \end{aligned}$$

which implies that

$$\|T_\lambda^\mu u\| \geq \|u\|, \quad \forall u \in K \cap \partial\Omega_{\bar{\eta}}. \tag{2.18}$$

Applying (a) of Lemma 2.4 to (2.15) and (2.18) shows that T_λ^μ has a fixed point $u \in K \cap (\bar{\Omega}_{\bar{\eta}} \setminus \Omega_{\bar{r}})$ with $r \leq \|u\| \leq \bar{\eta} = \frac{1}{\delta}R$. Hence, since for $u \in K$ we have $u(t) \geq \delta\|u\|$, $t \in J$, it follows that (2.3) holds. This finishes the proof of part (ii). \square

Proof of Theorem 2.2 Part (i). Noticing that $f(t,x,y) > 0$, $I_k(t,x) > 0$ ($k = 0, 1, \dots, p-1$) for all $t, x > 0$, and $y > 0$, we can define, for $R > 0$,

$$m_R = \min_{t \in J, \delta R \leq x, y \leq R} \{f(t,x,y)\} > 0, \quad m^* = \min\{m_k, k = 0, 1, \dots, p-1\} > 0,$$

where

$$m_k = \min_{t \in J, \delta R \leq x \leq R} \{I_k(t,x)\}, \quad k = 0, 1, \dots, p-1.$$

Let

$$\lambda_0 \geq (2\alpha m_R \gamma)^{-1} R, \quad \mu_0 \geq (2\alpha m^* p)^{-1} R.$$

Then, for any $u \in K \cap \partial\Omega_R$ and $\lambda > \lambda_0, \mu > \mu_0$, we have

$$\begin{aligned} (T_\lambda^\mu u)(t) &= \lambda \int_t^{t+\omega} G(t,s)b(s)f\left(s,u(h_2(s)), \int_{-\zeta}^0 e(s)u(s+v) dv\right) ds \\ &\quad + \mu \sum_{t \leq t_k < t+\omega} G(t,t_k)I_k(t_k,u(t_k)) \\ &\geq \alpha \left[\lambda \int_0^\omega b(s)f\left(s,u(h_2(s)), \int_{-\zeta}^0 e(s)u(s+v) dv\right) ds + \mu \sum_{0 \leq t_k < \omega} I_k(t_k,u(t_k)) \right] \\ &\geq \alpha \left[\lambda \int_0^\omega b(s)f\left(s,u(h_2(s)), \int_{-\zeta}^0 e(s)u(s+v) dv\right) ds + \mu \sum_{k=0}^{p-1} I_k(t_k,u(t_k)) \right] \\ &\geq \alpha \left[\lambda m_R \int_0^\omega b(s) ds + \mu m^* p \right] \\ &> \alpha [\lambda_0 m_R \gamma + \mu_0 m^* p] \\ &\geq \frac{1}{2} R + \frac{1}{2} R = R, \end{aligned}$$

which implies that

$$\|T_\lambda^\mu u\| > \|u\|, \quad \forall u \in K \cap \partial\Omega_R, \lambda > \lambda_0 \text{ and } \mu > \mu_0. \tag{2.19}$$

If $0 < f^0 < +\infty, 0 < I^0(k) < +\infty, k = 0, 1, \dots, p-1$, then there exist $l_1 > 0, l_2 > 0$, and $0 < r < R$ such that

$$f(t,x,y) < l_1 x, \quad I_k(t,x) < l_2 x \quad (\forall t \in J, 0 \leq x,y \leq r, k = 0, 1, \dots, p-1),$$

where l_1 and l_2 satisfy (2.12) and (2.13), respectively.

Therefore, for $u \in K \cap \partial\Omega_r$, we have

$$\begin{aligned} (T_\lambda^\mu u)(t) &= \lambda \int_t^{t+\omega} G(t,s)b(s)f\left(s,u(h_2(s)), \int_{-\zeta}^0 e(s)u(s+v) dv\right) ds \\ &\quad + \mu \sum_{t \leq t_k < t+\omega} G(t,t_k)I_k(t_k,u(t_k)) \\ &\leq \beta \left[\lambda \int_0^\omega b(s)f\left(s,u(h_2(s)), \int_{-\zeta}^0 e(s)u(s+v) dv\right) ds + \mu \sum_{0 \leq t_k < \omega} I_k(t_k,u(t_k)) \right] \\ &\leq \beta \left[\lambda \int_0^\omega b(s)f\left(s,u(h_2(s)), \int_{-\zeta}^0 e(s)u(s+v) dv\right) ds + \mu \sum_{k=0}^{p-1} I_k(t_k,u(t_k)) \right] \\ &\leq \beta \left[\lambda \int_0^\omega b(s)l_1 \|u\| ds + \mu \sum_{k=0}^{p-1} l_2 \|u\| \right] \\ &\leq \beta \left[\lambda l_1 \int_0^\omega b(s) ds \|u\| + \mu l_2 p \|u\| \right] \end{aligned}$$

$$\begin{aligned} &\leq \beta[\lambda I_1 \gamma \|u\| + \mu I_2 p \|v\|] \\ &\leq \frac{1}{2} \|u\| + \frac{1}{2} \|u\| \\ &= \|u\|. \end{aligned}$$

This yields

$$\|T_\lambda^\mu u\| \leq \|u\|, \quad \forall u \in K \cap \partial\Omega_r. \tag{2.20}$$

Applying (a) of Lemma 2.4 to (2.19) and (2.20) shows that T_λ^μ has a fixed point $u \in K \cap (\bar{\Omega}_R \setminus \Omega_r)$ with $r \leq \|u\| \leq R$. Hence, since for $u \in K$ we have $u(t) \geq \delta \|u\|$, $t \in J$, it follows that (2.3) holds. This gives the proof of part (i).

Part (ii). Noticing that $f(t, x, y) > 0$, $I_k(t, x) > 0$ ($k = 0, 1, \dots, p-1$) for all $t, x > 0$, and $y > 0$, we can define, for $R > 0$,

$$M_R = \max_{t \in J, 0 \leq x, y \leq R} \{f(t, x, y)\} > 0, \quad M^* = \max\{M_k, k = 0, 1, \dots, p-1\} > 0,$$

where

$$M_k = \max_{t \in J, 0 \leq x \leq R} \{I_k(t, x)\}, \quad k = 0, 1, \dots, p-1.$$

Let

$$\bar{\lambda}_0 \leq \frac{1}{2\beta M_R \gamma} R, \quad \bar{\mu}_0 \leq \frac{1}{2\beta p M^*} R.$$

Then, for $u \in K \cap \partial\Omega_R$ and $\lambda < \bar{\lambda}_0$, $\mu < \bar{\mu}_0$, we have

$$\begin{aligned} (T_\lambda^\mu u)(t) &= \lambda \int_t^{t+\omega} G(t, s) b(s) f\left(s, u(h_2(s)), \int_{-\zeta}^0 e(s) u(s+v) dv\right) ds \\ &\quad + \mu \sum_{t \leq t_k < t+\omega} G(t, t_k) I_k(t_k, u(t_k)) \\ &\leq \beta \left[\lambda \int_0^\omega b(s) f\left(s, u(h_2(s)), \int_{-\zeta}^0 e(s) u(s+v) dv\right) ds + \mu \sum_{0 \leq t_k < \omega} I_k(t_k, u(t_k)) \right] \\ &\leq \beta \left[\lambda \int_0^\omega b(s) f\left(s, u(h_2(s)), \int_{-\zeta}^0 e(s) u(s+v) dv\right) ds + \mu \sum_{k=0}^{p-1} I_k(t_k, u(t_k)) \right] \\ &\leq \beta \left[\lambda \int_0^\omega b(s) M_R ds + \mu \sum_{k=0}^{p-1} M^* \right] \\ &\leq \beta \left[\lambda M_R \int_0^\omega b(s) ds + \mu p M^* \right] \\ &\leq \beta [\lambda M_R \gamma + \mu M^* p] \\ &< \beta [\bar{\lambda}_0 M_R \gamma + \bar{\mu}_0 M^* p] \\ &\leq \frac{1}{2} R + \frac{1}{2} R \\ &= R. \end{aligned}$$

This implies

$$\|T_\lambda^\mu u\| < \|u\|, \quad \forall u \in K \cap \partial\Omega_R. \tag{2.21}$$

If $0 < f_0 < +\infty$, $0 < I_0(k) < +\infty$, $k = 0, 1, \dots, p-1$, then there exist $l_3 > 0$, $l_4 > 0$, and $0 < r < R$ such that

$$f(t, x, y) > l_3x, \quad I_k(t, x) > l_4x \quad (\forall t \in J, 0 \leq x, y \leq r, k = 0, 1, \dots, p-1),$$

where l_3 and l_4 satisfy (2.16) and (2.17), respectively.

Therefore, for any $u \in K \cap \partial\Omega_r$, we have

$$\begin{aligned} (T_\lambda^\mu u)(t) &= \lambda \int_t^{t+\omega} G(t, s)b(s)f\left(s, u(h_2(s)), \int_{-\zeta}^0 e(s)u(s+v) dv\right) ds \\ &\quad + \mu \sum_{t \leq t_k < t+\omega} G(t, t_k)I_k(t_k, u(t_k)) \\ &\geq \alpha \left[\lambda \int_0^\omega b(s)f\left(s, u(h_2(s)), \int_{-\zeta}^0 e(s)u(s+v) dv\right) ds + \mu \sum_{0 \leq t_k < \omega} I_k(t_k, u(t_k)) \right] \\ &\geq \alpha \left[\lambda \int_0^\omega b(s)f\left(s, u(h_2(s)), \int_{-\zeta}^0 e(s)u(s+v) dv\right) ds + \mu \sum_{k=0}^{p-1} I_k(t_k, u(t_k)) \right] \\ &\geq \alpha \left[\lambda \int_0^\omega b(s)l_3\|u\| ds + \mu l_4\|u\|^p \right] \\ &\geq \alpha [\lambda l_3\|u\|\gamma + \mu l_4\|u\|^p] \\ &\geq \frac{1}{2}\|u\| + \frac{1}{2}\|u\| \\ &= \|u\|, \end{aligned}$$

which implies that

$$\|T_\lambda^\mu u\| \geq \|u\|, \quad \forall u \in K \cap \partial\Omega_r. \tag{2.22}$$

Applying (a) of Lemma 2.4 to (2.21) and (2.22) shows that T_λ^μ has a fixed point $u \in K \cap (\bar{\Omega}_R \setminus \Omega_r)$ with $r \leq \|u\| \leq R$. Hence, since for $u \in K$ we have $u(t) \geq \delta\|u\|$, $t \in J$, it follows that (2.3) holds. This finishes the proof of part (ii). \square

Proof of Theorem 2.3 Similar to the proof of Theorem 2.1(i) and Theorem 2.2(i), respectively, one can show that Theorem 2.3(i) and (ii) hold.

Consider part (iii). Choose two numbers r_1 and r_2 satisfying (2.1). By Theorem 2.1(i) and Theorem 2.2(i), there exist $\lambda_0 > 0$ and $\mu_0 > 0$ such that

$$\|T_\lambda^\mu u\| > \|u\|, \quad \forall u \in K \cap \partial\Omega_{r_i}, i = 1, 2. \tag{2.23}$$

Since $f^0 = f^\infty = I^\infty(k) = I^0(k) = 0$, $k = 0, 1, \dots, p-1$, from the proof of Theorem 2.1(i), Theorem 2.2(i), and (2.1), it follows that

$$\|T_\lambda^\mu u\| \leq \|u\|, \quad \forall u \in K \cap \partial\Omega_r \tag{2.24}$$

and

$$\|T_\lambda^\mu u\| \leq \|u\|, \quad \forall u \in K \cap \partial\Omega_R. \tag{2.25}$$

Applying Lemma 2.4 to (2.23)-(2.25) shows that T_λ^μ has two fixed points u_1 and u_2 such that $u_1 \in K \cap (\bar{\Omega}_{r_1} \setminus \Omega_r)$ and $u_2 \in K \cap (\bar{\Omega}_R \setminus \Omega_{r_2})$. These are the desired distinct positive periodic solutions of problem (1.1) for $\lambda_0 > 0$ and $\mu_0 > 0$ satisfying (2.4). Then the result of part (iii) follows. \square

Proof of Theorem 2.4 Similar to the proof of those Theorem 2.1(ii) and Theorem 2.2(ii), respectively, one can show that Theorem 2.4(i) and (ii) hold.

Now consider part (iii). Choose two numbers r_1 and r_2 satisfying (2.1). By Theorem 2.1(ii) and Theorem 2.2(ii), there exist $\bar{\lambda}_0 > 0$ and $\bar{\mu}_0 > 0$ such that

$$\|T_\lambda^\mu u\| < \|u\|, \quad \forall 0 < \lambda < \bar{\lambda}_0, 0 < \mu < \bar{\mu}_0, u \in K \cap \partial\Omega_{r_i}, i = 1, 2. \tag{2.26}$$

Since $f_0 = f_\infty = I_\infty(k) = I_0(k) = +\infty, k = 0, 1, \dots, p - 1$, from the proof of Theorem 2.1(ii) and Theorem 2.2(ii) and (2.1), it follows that

$$\|T_\lambda^\mu u\| \geq \|u\|, \quad \forall u \in K \cap \partial\Omega_r \tag{2.27}$$

and

$$\|T_\lambda^\mu u\| \geq \|u\|, \quad \forall u \in K \cap \partial\Omega_R. \tag{2.28}$$

Applying Lemma 2.4 to (2.26)-(2.28) shows that T_λ^μ has two fixed points u_1 and u_2 such that $u_1 \in K \cap (\bar{\Omega}_{r_1} \setminus \Omega_r)$ and $u_2 \in K \cap (\bar{\Omega}_R \setminus \Omega_{r_2})$. These are the desired distinct positive periodic solutions of problem (1.1) for $0 < \lambda < \bar{\lambda}_0$ and $0 < \mu < \bar{\mu}_0$ satisfying (2.5). Then proof of part (iii) is complete. \square

3 The parameter dependence of positive periodic solution under (1.2)

In this section, we consider the parameter dependence of positive periodic solution in the case $\lambda = \mu$. For convenience, let

$$\mathbf{M} = \max_{t \in J, 0 \leq x, y \leq d} f(t, x, y), \quad \mathbf{M}^* = \max\{\mathbf{M}_k, k = 0, 1, \dots, p - 1\},$$

where

$$\mathbf{M}_k = \max_{t \in J, 0 \leq x \leq d} I_k(t, x), \quad k = 0, 1, \dots, p - 1,$$

and d is a positive number.

(H₃)' Let $f \in C(\mathbf{R} \times \mathbf{R}_+ \times \mathbf{R}_+, \mathbf{R}_+)$ be ω -periodic functions in t .

Theorem 3.1 Assume that (H₁), (H₂), (H₃)', and (H₄) hold and the function g satisfies (1.2). Then the following two conclusions hold.

- (H₅) If $f^0 = 0, I^0(k) = 0,$ and $f_\infty = \infty, I_\infty(k) = \infty, k = 0, 1, \dots, p - 1,$ then for every $\lambda > 0$ problem (1.1) has a positive periodic solution $u_\lambda(t)$ satisfying $\lim_{\lambda \rightarrow 0^+} \|u_\lambda\| = \infty.$
- (H₆) If $f_0 = \infty, I_0(k) = \infty,$ and $f^\infty = 0, I^\infty(k) = 0, k = 0, 1, \dots, p - 1,$ then for every $\lambda > 0$ problem (1.1) has a positive periodic solution $u_\lambda(t)$ satisfying $\lim_{\lambda \rightarrow 0^+} \|u_\lambda\| = 0.$

Remark 3.1 Assume that $f_\infty = \infty$ or $I_\infty(k) = \infty, k = 0, 1, \dots, p - 1$ in (H₅) or $f_0 = \infty$ or $I_0(k) = \infty, k = 0, 1, \dots, p - 1$ in (H₆). Then the conclusions of Theorem 3.1 also hold.

Remark 3.2 Theorem 3.1 improves and develops Theorem 2.3.7 in [45] essentially.

Remark 3.3 Theorem 3.1 only considers the special case $\mu = \lambda.$ However, we do not give any information on the general case $\mu \neq \lambda.$

Proof of Theorem 3.1 We need only to prove this theorem under condition (H₅) since the proof is similar when (H₆) holds.

For given $\lambda > 0,$ similar to the proof of (2.18) and (2.20), it follows from (H₅) that there exists $0 < r < R$ such that

$$\|T_\lambda^\mu u\| \leq \|u\|, \quad u \in K \cap \partial\Omega_r$$

and

$$\|T_\lambda^\mu Tu\| \geq \|u\|, \quad u \in K \cap \partial\Omega_R.$$

Hence, condition (a) of Lemma 2.4 is satisfied for the operator $T_\lambda^\mu,$ which implies that T_λ^μ has a fixed point u_λ in $\bar{\Omega}_R \setminus \Omega_r.$

It remains to prove $\|u_\lambda\| = +\infty$ as $\lambda \rightarrow 0^+.$ In fact, if not, there exist a number $d > 0$ and a sequence $\lambda_n \rightarrow 0^+$ such that

$$\|u_{\lambda_n}\| \leq d \quad (n = 1, 2, 3, \dots).$$

Furthermore, the sequence $\{\|u_{\lambda_n}\|\}$ contains a subsequence that converges to a number η ($0 \leq \eta \leq d$). For simplicity, suppose that $\{\|u_{\lambda_n}\|\}$ itself converges to $\eta.$

If $\eta > 0,$ then $\|u_{\lambda_n}\| > \frac{\eta}{2}$ for sufficiently large n ($n > N$), and therefore

$$\begin{aligned} \frac{1}{\lambda_n} &= \frac{\| \int_t^{t+\omega} G(t,s)b(s)f(s, u(h_2(s)), \int_{-\zeta}^0 e(s)u(s+v) dv) ds + \sum_{t \leq t_k \leq t+\omega} G(t, t_k)I_k(t_k, u_{\lambda_n}(t_k)) \|}{\|u_{\lambda_n}\|} \\ &\leq \frac{\beta[\int_0^\omega b(s)f(s, u(h_2(s)), \int_{-\zeta}^0 e(s)u(s+v) dv) ds + \sum_{0 \leq t_k \leq \omega} I_k(t_k, u_{\lambda_n}(t_k))]}{\|u_{\lambda_n}\|} \\ &\leq \frac{\beta[\gamma M + pM^*]}{\|u_{\lambda_n}\|} \\ &\leq \frac{2\beta[\gamma M + pM^*]}{\eta} \quad (n > N), \end{aligned}$$

which contradicts $\lambda_n \rightarrow 0^+.$

If $\eta = 0,$ then $\|u_{\lambda_n}\| \rightarrow 0$ for sufficiently large n ($n > N$), and therefore it follows from (H₅) that for any $\varepsilon > 0$ there exists $r_3 > 0$ such that

$$f(t, x, y) \leq \varepsilon x, \quad I_k(t, x) \leq \varepsilon x, \quad \forall 0 \leq x, y \leq r_3,$$

and hence it follows from (2.10) that

$$\begin{aligned} \frac{1}{\lambda_n} &= \frac{\| \int_t^{t+\omega} G(t,s)b(s)f(s,u(h_2(s)), \int_{-\zeta}^0 e(s)u(s+v) dv) ds + \sum_{t \leq t_k \leq t+\omega} G(t,t_k)I_k(t_k,u_{\lambda_n}(t_k)) \|}{\|u_{\lambda_n}\|} \\ &\leq \frac{\beta[\int_0^\omega b(s)f(s,u(h_2(s)), \int_{-\zeta}^0 e(s)u(s+v) dv) ds + \sum_{0 \leq t_k \leq \omega} I_k(t_k,u_{\lambda_n}(t_k))]}{\|u_{\lambda_n}\|} \\ &\leq \frac{\beta[\gamma \varepsilon \|u_{\lambda_n}\| + p \varepsilon \|u_{\lambda_n}\|]}{\|u_{\lambda_n}\|} \\ &= \beta(\gamma + p)\varepsilon. \end{aligned}$$

Since ε is arbitrary, we have $\lambda_n \rightarrow +\infty$ ($n \rightarrow +\infty$) in contradiction with $\lambda_n \rightarrow 0^+$. Therefore, $\|u_\lambda\| \rightarrow +\infty$ as $\lambda \rightarrow 0^+$ and the proof is complete. \square

4 Existence under (1.3) and (1.4)

Firstly, let us denote that for each fixed constant $\rho > 0$,

$$h^*(\rho) := \max\{g(s) | 0 \leq s \leq \rho\}, \tag{4.1}$$

$$h_*(\rho) := \min\{g(s) | 0 \leq s \leq \rho\}. \tag{4.2}$$

Theorem 4.1 *Assume that (H₁)-(H₄) hold and g satisfies (1.3).*

(H₇) *If $f^0 = 0$ and $I^0(k) = 0, k = 0, 1, \dots, p - 1$, then there exist $\lambda_\rho > 0$ and $\mu_\rho > 0$ such that problem (1.1) has a positive periodic solution u with*

$$\sup_{t \in J} u(t) \leq \rho \quad \text{for } \lambda > \lambda_\rho \text{ and } \mu > \mu_\rho. \tag{4.3}$$

(H₈) *If $f_0 = +\infty$ and $I_0(k) = +\infty, k = 0, 1, \dots, p - 1$, such that problem (1.1) has a positive periodic solution u with*

$$\sup_{t \in J} u(t) \leq \rho \quad \text{for } 0 < \lambda < \lambda_\rho \text{ and } 0 < \mu < \mu_\rho. \tag{4.4}$$

Corollary 4.1 *Assume that (H₁)-(H₄) hold and the function g satisfies (1.3).*

- (i) *If $f^0 = 0$ and $I^0(k) = 0, k = 0, 1, \dots, p - 1$, then there exists $\lambda_\rho > 0$ or $\mu_\rho > 0$ such that problem (1.1) has a positive periodic solution $u(t)$ with property (4.3).*
- (ii) *If $f_0 = +\infty$ or $I_0(k) = +\infty, k = 0, 1, \dots, p - 1$, then there exist $\bar{\lambda}_\rho > 0$ and $\bar{\mu}_\rho > 0$ such that problem (1.1) has a positive periodic solution $u(t), t \in J$ with property (4.4).*

Let ρ be the constant in Theorem 4.1. Define a cone $K[\rho]$ in E by

$$K[\rho] = \left\{ u \in E : u(t) \geq \frac{h^*(\rho)(1 - h_*(\rho))}{1 - h^*(\rho)} \|u\| \right\}. \tag{4.5}$$

For every $\iota > 0$, let

$$\Omega_\iota = \{ u \in K[R] : \|u\| < \iota \}, \quad \partial\Omega_\iota = \{ u \in K[R] : \|u\| = \iota \}$$

and

$$\bar{\Omega}_t = \{u \in K[R] : \|u\| \leq t\}.$$

Define an operator $T_\lambda : E \rightarrow E$ by

$$\begin{aligned} (Tu)(t) = & \lambda \int_t^{t+\omega} G_u(t,s)b(s)f\left(s, u(h_2(s)), \int_{-\zeta}^0 e(s)u(s+v) dv\right) ds \\ & + \mu \sum_{t \leq t_k < t+\omega} G_u(t, t_k)I_k(t_k, u(t_k)), \end{aligned} \tag{4.6}$$

where

$$G_u(t, s) = \frac{e^{\int_t^s a(v)g(u(h_1(v))) dv}}{e^{\int_0^\omega a(v)g(u(h_1(v))) dv} - 1}, \quad s \in [t, t + \omega]. \tag{4.7}$$

It follows from (4.1) and (4.2) that

$$h_*(\rho) \leq g(u) \leq h^*(\rho), \quad u \in \bar{\Omega}_\rho,$$

and hence, for $u \in \bar{\Omega}_\rho$, we have

$$\frac{1}{\sigma^{h^*(\rho)} - 1} = \alpha(\rho) \leq G_u(t, s) \leq \beta(\rho) = \frac{\sigma^{h_*(\rho)}}{\sigma^{h_*(\rho)} - 1}, \quad s \in [t, t + \omega], \tag{4.8}$$

where σ is defined in (2.8).

Similarly, for $0 < r < \rho$, we can define

$$h^*(r) := \max\{g(s) | 0 \leq s \leq r\}, \quad h_*(r) := \min\{g(s) | 0 \leq s \leq r\}. \tag{4.9}$$

Obviously,

$$h_*(r) \leq g(u) \leq h^*(r)$$

and, for $u \in \bar{\Omega}_r$, we have

$$\begin{aligned} \frac{\sigma^{h^*(\rho)}}{1 - \sigma^{h^*(\rho)}} & \leq \frac{\sigma^{h^*(r)}}{1 - \sigma^{h^*(r)}} \leq G_u(t, s) \\ & \leq \frac{1}{1 - \sigma^{h_*(r)}} \leq \frac{1}{1 - \sigma^{h_*(\rho)}}, \quad s \in [t, t + \omega], u \in \bar{\Omega}_r. \end{aligned} \tag{4.10}$$

Lemma 4.1 (See [43]) *Assume that (H₁)-(H₄) hold. Let $\rho > 0$ be fixed. Then for each r : $0 < r \leq \rho$, we have $T(\bar{\Omega}_r) \subset K[\rho]$ and $T : \bar{\Omega}_r \rightarrow K[\rho]$ is compact and continuous.*

Remark 4.1 (See [43]) Notice that for the cone $K[\rho]$ defined by (4.3) we cannot obtain

$$T(K[\rho]) \subset K[\rho].$$

However, Lemma 4.1 shows that, for every $r \in (0, \rho]$,

$$T(\bar{\Omega}_r) \subset K[\rho],$$

which is enough for us to apply the fixed point theorem in cones.

Lemma 4.2 (See [43]) *Assume that (H₁)-(H₄) hold. If u is a fixed point of T in $K[\rho]$, then u is a periodic solution of problem (1.1).*

Proof of Theorem 4.1 Let $R = \rho$ and $\alpha(\rho)$ replace α , $\beta(\rho)$ replace β in the proof of Theorem 2.2(i) and (ii) and repeat the argument of Theorem 2.2(i) and (ii). Then we can prove that Theorem 4.1 holds. □

Remark 4.2 Let the condition (1.4) replace (1.3) in Theorem 4.1, the results of Theorem 4.1 also hold.

Remark 4.3 Since the function $G_u(t, s)$ is related to u and the operator T does not satisfy

$$T(K[\rho]) \subset K[\rho],$$

we have the following facts.

- (i) $u \rightarrow 0$ means there exists a sufficiently small positive number r such that $u \in [0, r]$.
So one can easily choose a positive number ρ to control r .
- (ii) $u \rightarrow \infty$ means there exists a sufficiently large positive number R such that $u \in [R, \infty]$, which implies that it is very difficult to find a positive number η to control R .

Therefore, we only prove the existence of positive periodic solutions for problem (1.1) under (1.3) and f and I_k ($k = 0, 1, \dots, p - 1$) satisfying one of the following conditions:

- (i) $f^0 = 0$ and $I^0(k) = 0, k = 0, 1, \dots, p - 1$;
- (ii) $f_0 = \infty$ and $I_0(k) = \infty, k = 0, 1, \dots, p - 1$.

However, we do not give any information on the existence of positive periodic solutions for problem (1.1) in the four following cases:

- (iii) (1.3) holds, $f^\infty = 0$ and $I^\infty(k) = 0, k = 0, 1, \dots, p - 1$;
- (iv) (1.3) holds, $f_\infty = \infty$ and $I_\infty(k) = \infty, k = 0, 1, \dots, p - 1$;
- (v) (1.4) holds, $f^\infty = 0$ and $I^\infty(k) = 0, k = 0, 1, \dots, p - 1$;
- (vi) (1.4) holds, $f_\infty = \infty$ and $I_\infty(k) = \infty, k = 0, 1, \dots, p - 1$.

This and the proof of Theorem 3.1 imply the following.

Remark 4.4 If (1.3) or (1.4) hold, then we cannot obtain the parameter dependence results of positive periodic solution for problem (1.1).

5 An example

To illustrate how our main results can be used in practice, we present an example. Let

$$p = 1, \quad e(t) \equiv \frac{1}{\xi}, \quad \omega = 2\pi, \quad r = 1. \tag{5.1}$$

Example 5.1 Consider the following equation:

$$\begin{cases} u'(t) = -a(t)g(u(h_1(t)))u(t) \\ \quad + \lambda b(t)f(t, u(h_2(t)), \int_{-\zeta}^0 e(s)u(t+s) ds), \quad t \in \mathbf{R}, t \neq t_k, \\ u(t_k^+) - u(t_k) = \mu I_k(t_k, u(t_k)), \quad k \in \mathbf{Z}, \end{cases} \tag{5.2}$$

where $\lambda > 0$ and $\mu > 0$ are parameters, and

$$\begin{aligned} a(t) &= \frac{1}{2\pi}, & b(t) &= 1 - \sin t, & g(u(h_1(t))) &= \frac{1}{3} + e^{-u(t)}, \\ I_k(t_k, u(t_k)) &= \left(\frac{3}{2} + \sin(t_k)\right)u(t_k), \quad k \in \mathbf{Z}, \\ f\left(t, u(h_2(t)), \int_{-\zeta}^0 e(s)u(t+s) ds\right) &= \left(\frac{1}{2} + \frac{1}{3} \sin t\right)u(t) + \int_{-\zeta}^0 \frac{1}{\zeta} u(t+s) ds. \end{aligned} \tag{5.3}$$

Conclusion 5.1 For any $\lambda > \frac{3e^{\frac{4}{3}}(e^{\frac{4}{3}}-1)^2}{14\pi(e^{\frac{1}{3}}-1)}$ and $\mu > \frac{e^{\frac{4}{3}}(e^{\frac{4}{3}}-1)^2}{3(e^{\frac{1}{3}}-1)}$, problem (5.1) has a positive 2π -periodic solution.

Proof It follows from (5.3) that (H₁)-(H₄) of Theorem 2.1 hold. Noticing that $p = 1$, we have

$$[0, \omega] \cap [t_k, k \in \mathbf{Z}] = \{t_0\}.$$

Letting $t_0 = \pi$, then

$$\frac{1}{3} \leq g(u) = \frac{1}{3} + e^{-u} \leq \frac{4}{3}, \quad f_\infty = \frac{7}{6} < \frac{11}{6} = f^\infty, \quad I_\infty = I^\infty = \frac{3}{2}.$$

So we can take

$$l = \frac{1}{3}, \quad L = \frac{4}{3}.$$

On the other hand, it follows from (5.1) that

$$\begin{aligned} \sigma &= e^{\int_0^\omega a(t)dt} = e, & \alpha &= \frac{1}{\sigma^L - 1} = \frac{1}{e^{\frac{4}{3}} - 1}, \\ \beta &= \frac{\sigma^L}{\sigma^l - 1} = \frac{e^{\frac{4}{3}}}{e^{\frac{1}{3}} - 1}, & \delta &= \frac{\sigma^l - 1}{\sigma^L(\sigma^L - 1)} = \frac{e^{\frac{1}{3}} - 1}{e^{\frac{4}{3}}(e^{\frac{4}{3}} - 1)}, \\ \gamma &= \int_0^{2\pi} (1 - \sin t) dt = 2\pi, & m_r &= \frac{7(e^{\frac{1}{3}} - 1)}{6e^{\frac{4}{3}}(e^{\frac{4}{3}} - 1)}, & m^* &= \frac{3(e^{\frac{1}{3}} - 1)}{2e^{\frac{4}{3}}(e^{\frac{4}{3}} - 1)} \end{aligned}$$

and

$$\begin{aligned} \lambda_0 &= (2\alpha m_r \gamma)^{-1} r = \frac{3e^{\frac{4}{3}}(e^{\frac{4}{3}} - 1)^2}{14\pi(e^{\frac{1}{3}} - 1)}, \\ \mu^0 &= (2\alpha m^* p)^{-1} r = \frac{e^{\frac{4}{3}}(e^{\frac{4}{3}} - 1)^2}{3(e^{\frac{1}{3}} - 1)}. \end{aligned}$$

Therefore, it follows from Theorem 2.1 that there exist $\lambda_0 = \frac{3e^{\frac{4}{3}}(e^{\frac{4}{3}}-1)^2}{14\pi(e^{\frac{1}{3}}-1)} > 0$ and $\mu_0 = \frac{e^{\frac{4}{3}}(e^{\frac{4}{3}}-1)^2}{3(e^{\frac{1}{3}}-1)} > 0$ such that, for any $\lambda > \lambda_0$ and $\mu > \mu_0$, problem (5.1) has a positive 2π -periodic solution. \square

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

XZ completed the main study and carried out the results of this article. MF checked the proofs and verified the calculation. All authors read and approved the final manuscript.

Author details

¹Department of Mathematics and Physics, North China Electric Power University, Beijing, 102206, People's Republic of China. ²School of Applied Science, Beijing Information Science & Technology University, Beijing, 100192, People's Republic of China.

Acknowledgements

This work is sponsored by the project NSFC (11301178), the Fundamental Research Funds for the Central Universities (2014ZZD10, 2014MS58) and the Scientific Research Common Program of Beijing Municipal Commission of Education (KM201511232018). The authors are grateful to the anonymous referees for their constructive comments and suggestions, which have greatly improved this paper.

Received: 26 January 2015 Accepted: 24 July 2015 Published online: 11 August 2015

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