Jung and Choi *Boundary Value Problems* (2016) 2016:54
DOI 10.1186/s13661-016-0564-0

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Singular potential biharmonic problem with fixed energy

Tacksun Jung[1*] and Q-Heung Choi[2]

*Correspondence:
tsjung@kunsan.ac.kr
[1]Department of Mathematics, Kunsan National University, Kunsan, 573-701, Korea
Full list of author information is available at the end of the article

Abstract

We investigate multiple solutions for the perturbation of a singular potential biharmonic problem with fixed energy. We get a theorem that shows the existence of at least one nontrivial weak solution under some conditions and fixed energy on which the corresponding functional of the equation satisfies the Palais-Smale condition. We obtain this result by variational method and critical point theory.

MSC: 35J35; 35J40

Keywords: perturbation of a biharmonic problem; singular potential; fixed energy; variational method; critical point theory; $(P.S.)_c$ condition

1 Introduction and statement of main result

Let Ω be a simply connected bounded domain of R^n with smooth boundary $\partial\Omega$, $n \geq 3$. Let C be a closed interval containing 0 in R, and $D = R^+ \backslash C$ be the complement of C in R^+. Let χ be any curve in Ω, $x : S^1 \to \Omega \subset R^n$ be a C^4 curve such that $x(t) \in \chi \subset \Omega$, and $u \circ x : S^1 \to R$ be the composition function of u and x such that $(u \circ x)(t) = u(x(t)) \in D = R^+ \backslash C$ for all $t \in S^1$. Then $u \circ x$ is a C^4 function. Let $c \in R$, Δ be the elliptic operator, and Δ^2 be the biharmonic operator. Let us introduce the following subset of $L^q(S^1, R)$:

$$\begin{aligned} H = \{ & u \circ x \in L^q(S^1, R) \mid \big((\Delta^2 u(x(t)) + c\Delta u(x(t))) \cdot u(x(t)) \big)^{\frac{1}{2}} \in L^q(S^1, R), \\ & u(x(t)) \in D = R^+ \backslash C, \forall x(t) \in \chi \subset \Omega, \forall t \in S^1 \}. \end{aligned}$$

Then H is the loop space on D. Let us endow H with the norm

$$\|u\|_H^2 = \left(\int_0^{2\pi} \left| (\Delta^2 u(x(t)) + c\Delta u(x(t))) \cdot u(x(t)) \right|^q x'(t)\, dt \right)^{\frac{1}{q}} \quad \text{for all } q \geq 1.$$

Then H is a Hilbert space. In this paper, we investigate the existence and multiplicity of weak solutions $u \circ x \in H$ for the perturbation of the biharmonic equation with singular potential

$$\Delta^2 (u \circ x) + c\Delta (u \circ x) = \Lambda (u \circ x) + |u \circ x|^{q-1} + \frac{1}{|u \circ x|^{p+1}}, \tag{1.1}$$

where Λ, p, and q are real constants such that $2 < q < p$ and $q < \frac{2n}{n-2}$. Throughout this paper, we deal with (1.1) with fixed energy

$$\Lambda u + \frac{1}{q}|u|^q - \frac{1}{p}\frac{1}{|u|^p} = h,$$

where h is a positive constant.

We assume that:

(A1) (fixed energy) there exists a positive constant $h > 0$ such that

$$\begin{aligned} &\Lambda u\bigl(x(t)\bigr) + \frac{1}{q}\bigl|u\bigl(x(t)\bigr)\bigr|^q - \frac{1}{p}\frac{1}{|u(x(t))|^p} \\ &\quad = h, \quad u\bigl(x(t)\bigr) \in D = R^+ \setminus C, x(t) \in \chi \subset \Omega, \forall t \in S^1; \end{aligned}$$

(A2) there exists a neighborhood Z of C in R such that, for some constant $A > 0$,

$$-\frac{1}{2}\Lambda u\bigl(x(t)\bigr)^2 - \frac{1}{q}\bigl|u\bigl(x(t)\bigr)\bigr|^q + \frac{1}{p}\frac{1}{|u(x(t))|^p} \geq \frac{A}{d^2(u(x(t)), C)}$$

for $u(x(t)) \in Z$, $x(t) \in \chi \subset \Omega$, $\forall t \in S^1$.

Our problems are characterized as singular biharmonic problems with singularity at $\{u = 0\}$. We recommend the book [1] for the singular elliptic problems. Many authors considered the biharmonic boundary value problem or the fourth-order elliptic boundary value problems. In particular, Choi and Jung [2] showed that the problem

$$\begin{aligned} &\Delta^2 u + c\Delta u = bu^+ + s \quad \text{in } \Omega, \\ &u = 0, \qquad \Delta u = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.2}$$

has at least two nontrivial solutions when $c < \lambda_1$, $\lambda_1(\lambda_1 - c) < b < \lambda_2(\lambda_2 - c)$, and $s < 0$ or when $\lambda_1 < c < \lambda_2$, $b < \lambda_1(\lambda_1 - c)$, and $s > 0$. We obtained these results by using the variational reduction method. In [3], by using degree theory we also proved that when $c < \lambda_1$, $\lambda_1(\lambda_1 - c) < b < \lambda_2(\lambda_2 - c)$, and $s < 0$, problem (1.2) has at least three nontrivial solutions. Tarantello [4] also studied the problem

$$\begin{aligned} &\Delta^2 u + c\Delta u = b\bigl((u+1)^+ - 1\bigr), \\ &u = 0, \qquad \Delta u = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.3}$$

She showed that if $c < \lambda_1$ and $b \geq \lambda_1(\lambda_1 - c)$, then problem (1.3) has a negative solution. She obtained this result by degree theory. Micheletti and Pistoia [5] also proved that if $c < \lambda_1$ and $b \geq \lambda_2(\lambda_2 - c)$, then problem (1.3) has at least three solutions by variational linking theorem and Leray-Schauder degree theory. In this paper, we essentially work with variational techniques: We first prove that the associated functional of (1.1) satisfies the Palais-Smale condition, and then we use critical point theory.

Let $\lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots$ be the eigenvalues of the eigenvalue problem $-\Delta u = \lambda u$ in Ω, $u = 0$ on $\partial\Omega$, and let ϕ_k be eigenfunctions corresponding to the eigenvalues λ_k, $k \geq 1$, suitably normalized with respect to the $L^2(\Omega)$ inner product, where each eigenvalue λ_k

is repeated with its multiplicity. We note that $\phi_1(x) > 0$ for $x \in \Omega$. Then the eigenvalue problem

$$\Delta^2 u + c\Delta u = \nu u \quad \text{in } \Omega,$$
$$u = 0, \qquad \Delta u = 0 \quad \text{on } \partial\Omega,$$

has infinitely many eigenvalues

$$\nu_1 = \lambda_1(\lambda_1 - c) < \nu_2 = \lambda_2(\lambda_2 - c) \leq \cdots \leq \nu_k = \lambda_k(\lambda_k - c) \leq \cdots$$

and eigenfunctions ϕ_k corresponding to the eigenvalues $\nu_k = \lambda_k(\lambda_k - c)$, $k \geq 1$, suitably normalized with respect to the $L^2(\Omega)$ inner product. We note that there exists a constant $D > 0$ such that $\|u\|_{L^q(S^1,R)} \leq D\|u\|_H$ for $q \geq 1$ because $\lambda_i(\lambda_i - c) \to \infty$ as $i \to \infty$. In this paper we are trying to find weak solutions of equation (1.1) in H. The weak solutions of (1.1) in H satisfy

$$\int_0^{2\pi} \Big[\big(\Delta^2 u(x(t)) + c\Delta u(x(t))\big) \cdot v(x(t)) - \Lambda u(x(t)) v(x(t)) \\ - \big|u(x(t))\big|^{q-1} v(x(t)) - \frac{1}{|u(x(t))|^{p+1}} v(x(t)) \Big]\, dx = 0 \tag{1.4}$$

for all $v \circ x \in H$. We shall show in Section 2 that there exists a one-to-one correspondence between weak solutions of (1.1) and critical points of the continuous and Frechét-differentiable functional

$$J(u \circ x) \in C^1(H, R),$$
$$J(u \circ x) = A\big(u(x(t))\big) - \int_0^{2\pi} \left[\frac{1}{2}\Lambda\big(u(x(t))\big)^2 + \frac{1}{q}\big|u(x(t))\big|^q - \frac{1}{p}\frac{1}{|u(x(t))|^p} \right] x'(t)\, dt, \tag{1.5}$$

where

$$A\big(u(x(t))\big) = \frac{1}{2}\int_0^{2\pi} \big[\big|\Delta u(x(t))\big|^2 - c\big|\nabla u(x(t))\big|^2\big] x'(t)\, dt.$$

The Euler equation for J is (1.1).

Our main result is as follows.

Theorem 1.1 (Fixed energy problem) *Assume that* $\lambda_k < c < \lambda_{k+1}$, $2 < q < p$, $q < \frac{2n}{n-2}$, $\lambda_{k+m}(\lambda_{k+m} - c) < \Lambda < \lambda_{k+m+1}(\lambda_{k+m+1} - c)$, $k \geq 1$, $m \geq 1$, *and conditions* (A1) *and* (A2) *hold. Then* (1.1) *has at least one nontrivial weak solution* $u(x)$ *such that*

$$u(x) \neq 0.$$

For the proof of Theorem 1.1, we apply the variational technique. Under the assumptions of Theorem 1.1, we show that the functional $J(u \circ x)$ satisfies the Palais-Smale condition, so that we can use the variational linking method in critical point theory. The outline of the proof of Theorem 1.1 is as follows. In Section 2, we introduce the eigenvalues and eigenfunctions of the eigenvalue problem $\Delta^2 u + c\Delta u - \Lambda u = \Lambda_k u$ in Ω, $u = 0$, $\Delta u = 0$ on $\partial\Omega$, introduce the eigenspaces spanned by the eigenfunctions of $\Lambda_k = \lambda_k(\lambda_k - c) - \Lambda$, investigate the properties of eigenspaces and prove that the functional $J(u \circ x)$ satisfies the Palais-Smale condition. In Section 3, we divide the whole space H into two subspaces $H^+(S,R)$ and $H^-(S,R)$, find some inequalities of $J(u \circ x)$ on two linked sublevel sets, and prove Theorem 1.1.

2 Eigenspace and Palais-Smale condition

Let us consider the eigenvalue problem

$$\begin{aligned} &\Delta^2 u + c\Delta u - \Lambda u = \Lambda_i u \quad \text{in } \Omega, \\ &u = 0, \qquad \Delta u = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{2.1}$$

Let Λ_i, $i \geq 1$, be eigenvalues of the eigenvalue problem (2.1), that is,

$$\Lambda_i = \lambda_i(\lambda_i - c) - \Lambda.$$

If $\lambda_{k+m}(\lambda_{k+m} - c) < \Lambda < \lambda_{k+m+1}(\lambda_{k+m+1} - c)$, then

$$\Lambda_1 < \Lambda_2 < \cdots < \Lambda_{k+m} < 0 < \Lambda_{k+m+1} < \Lambda_{k+m+2} < \cdots, \quad \forall k \geq 1, m \geq 1$$

and

$$\lim_{i\to\infty} \frac{\Lambda_i}{\lambda_i(\lambda_i - c)} = 1.$$

Let $c_{\lambda_i(\lambda_i-c)}$ be eigenvectors of $\lambda(\lambda - c) - \Lambda$ corresponding to the eigenvalues Λ_i. Let us set

$$\begin{aligned} W_{\lambda_i(\lambda_i-c)} &= \operatorname{span}\bigl\{\phi_i \mid \bigl(\Delta^2 + c\Delta\bigr)\phi_i = \lambda_i(\lambda_i - c)\phi_i\bigr\}, \\ H_{\lambda_i(\lambda_i-c)}\bigl(S^1,R\bigr) &= \bigl\{c_{\lambda_i(\lambda_i-c)}(\phi \circ x) \in H\bigl(S^1,R\bigr) \mid c \in R, \phi \in W_{\lambda_i(\lambda_i-c)}, \\ &\qquad c_{\lambda_i(\lambda_i-c)}\phi\bigl(x(t)\bigr) \in D = R^+\setminus C, x(t) \in \chi \subset \Omega, \forall t \in S^1\bigr\}. \end{aligned}$$

Then

$$H = \bigoplus_{\Lambda_i \geq \Lambda_1} H_{\lambda_i(\lambda_i-c)}\bigl(S^1,R\bigr). \tag{2.2}$$

Lemma 2.1 *Assume that* $\lambda_k < c < \lambda_{k+1}$, $2 < q < p$, $q < \frac{2n}{n-2}$, $\lambda_{k+m}(\lambda_{k+m} - c) < \Lambda < \lambda_{k+m+1}(\lambda_{k+m+1} - c)$, $k \geq 1$, $m \geq 1$, *and that conditions* (A1) *and* (A2) *hold. Let* $u \circ x \in L^q(S^1,R)$ *and* $\Lambda(u \circ x) + |u \circ x|^{q-1} + \frac{1}{|u\circ x|^{p+1}} \in L^q(S^1,R)$. *Then all the solutions of*

$$\Delta^2(u \circ x) + c\Delta(u \circ x) = \Lambda(u \circ x) + |u \circ x|^{q-1} + \frac{1}{|u \circ x|^{p+1}}$$

belong to H.

Proof Equation (1.1) can be rewritten as

$$u \circ x = \left(\Delta^2 + c\Delta\right)^{-1}\left(\Lambda(u \circ x) + |u \circ x|^{q-1} + \frac{1}{|u \circ x|^{p+1}}\right).$$

Then there exists a constant $D_1 > 0$ such that

$$\begin{aligned}
\|u \circ x\|_H^2 &= \left\|\left(\Delta^2 + c\Delta\right)^{-1}\left(\Lambda(u \circ x) + |u \circ x|^{q-1} + \frac{1}{|u \circ x|^{p+1}}\right)\right\|_H^2 \\
&= \left\|\left(\Delta^2 + c\Delta\right)\left(\Delta^2 + c\Delta\right)^{-1}\left(\Lambda(u \circ x) + |u \circ x|^{q-1} + \frac{1}{|u \circ x|^{p+1}}\right)\right. \\
&\quad \left.\cdot \left(\Delta^2 + c\Delta\right)^{-1}\left(\Lambda(u \circ x) + |u \circ x|^{q-1} + \frac{1}{|u \circ x|^{p+1}}\right)\right\|_{L^q(S^1,R)} \\
&= \left\|(\Delta - c\nabla)\left(\Delta^2 + c\Delta\right)^{-1}\left(\Lambda(u \circ x) + |u \circ x|^{q-1} + \frac{1}{|u \circ x|^{p+1}}\right)\right\|_{L^q(S^1,R)}^2 \\
&\leq D_1\left\|\Lambda(u \circ x) + |u \circ x|^{q-1} + \frac{1}{|u \circ x|^{p+1}}\right\|_{L^q(S^1,R)}^2.
\end{aligned}$$

Thus,

$$\|u \circ x\|_H < \infty,$$

and the lemma is proved. □

Lemma 2.2 *Assume that* $\lambda_k < c < \lambda_{k+1}$, $2 < q < p$, $q < \frac{2n}{n-2}$, $\lambda_{k+m}(\lambda_{k+m} - c) < \Lambda < \lambda_{k+m+1}(\lambda_{k+m+1} - c)$, $k \geq 1$, $m \geq 1$, *and that conditions* (A1) *and* (A2) *hold. Then the functional* $J(u \circ x)$ *is continuous and Fréchet differentiable with Fréchet derivative in* H,

$$\begin{aligned}
DJ(u \circ x) \cdot (v \circ x) = \int_0^{2\pi} \Big[&\left(\Delta^2 u(x(t)) + c\Delta u(x(t))\right) \cdot v(x(t)) - \Lambda u(x(t)) \cdot v(x(t)) \\
&- \left|u(x(t))\right|^{q-1} \cdot v(x(t)) - \frac{v(x(t))}{|u(x(t))|^{p+1}}\Big] x'(t)\, dt, \quad \forall v \circ x \in H.
\end{aligned}$$

Moreover $DJ \in C$, *that is*, $J \in C^1$.

Proof First, we shall prove that $J(u \circ x)$ is continuous. Let $u \circ x, v \circ x \in H$. Then since $u(x(t)), v(x(t)) \in D = R^+ \backslash C$, $\forall t \in S^1$, it follows that $u(x(t)) > 0$, $v(x(t)) > 0$, and $u(x(t)) + v(x(t)) > 0$. Thus, $\frac{1}{|u(x(t))|^p}$ and $\frac{1}{|u(x(t))+v(x(t))|^p}$ are well defined, continuous, and C^1. Thus, we have

$$\begin{aligned}
&\left|J(u \circ x + v \circ x) - J(u \circ x)\right| \\
&\quad = \left|\frac{1}{2}\int_0^{2\pi} \left(\Delta^2\left(u(x(t)) + v(x(t))\right) + c\Delta\left(u(x(t)) + v(x(t))\right)\right) \cdot \left(u(x(t)) + v(x(t))\right) x'(t)\, dt\right. \\
&\quad\quad - \int_0^{2\pi}\left[\frac{1}{2}\Lambda\left(u(x(t)) + v(x(t))\right)^2 + \frac{1}{q}\left|u(x(t)) + v(x(t))\right|^q\right. \\
&\quad\quad \left. - \frac{1}{p}\frac{1}{|u(x(t)) + v(x(t))|^p}\right] x'(t)\, dt
\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{2}\int_0^{2\pi} \big(\Delta^2 u(x(t)) + c\Delta u(x(t))\big) \cdot u(x(t))x'(t)\,dt \\
&+ \int_0^{2\pi} \left[\frac{1}{2}\Lambda u(x(t))^2 + \frac{1}{q}\big|u(x(t))\big|^q - \frac{1}{p}\frac{1}{|u(x(t))|^p}\right]x'(t)\,dt\bigg|.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
&\big|J(u\circ x + v\circ x) - J(u\circ x)\big| \\
&= \bigg|\frac{1}{2}\int_0^{2\pi}\big[\big(\Delta^2 u(x(t)) + c\Delta u(x(t))\big)\cdot v(x(t)) + \big(\Delta^2 v(x(t)) + c\Delta v(x(t))\big)\cdot u(x(t)) \\
&\quad + \big(\Delta^2 v(x(t)) + c\Delta v(x(t))\big)\cdot v(x(t))\big]x'(t)\,dt \\
&\quad - \int_0^{2\pi}\bigg[\bigg(\frac{1}{2}\Lambda\big(u(x(t)) + v(x(t))\big)^2 + \frac{1}{q}\big|u(x(t)) + v(x(t))\big|^q \\
&\quad - \frac{1}{p}\frac{1}{|u(x(t)) + v(x(t))|^p}\bigg) \\
&\quad - \bigg(\frac{1}{2}\Lambda u(x(t))^2 + \frac{1}{q}\big|u(x(t))\big|^q - \frac{1}{p}\frac{1}{|u(x(t))|^p}\bigg)\bigg]x'(t)\,dt\bigg| \\
&\le \bigg|\frac{1}{2}\int_0^{2\pi}\big[\big(\Delta^2 u(x(t)) + c\Delta u(x(t))\big)\cdot v(x(t))\big]x'(t)\,dt\bigg| \\
&\quad + \bigg|\frac{1}{2}\int_0^{2\pi}\big(\big(\Delta^2 v(x(t)) + c\Delta v(x(t))\big)\cdot u(x(t))\big)x'(t)\,dt\bigg| \\
&\quad + \bigg|\frac{1}{2}\int_0^{2\pi}\big(\big(\Delta^2 v(x(t)) + c\Delta v(x(t))\big)\cdot v(x(t))\big)x'(t)\,dt\bigg| \\
&\quad + \bigg|\int_0^{2\pi}\bigg[\bigg(\frac{1}{2}\Lambda\big(u(x(t)) + v(x(t))\big)^2 + \frac{1}{q}\big|u(x(t)) + v(x(t))\big|^q \\
&\quad - \frac{1}{p}\frac{1}{|u(x(t)) + v(x(t))|^p}\bigg) \\
&\quad - \bigg(\frac{1}{2}\Lambda u(x(t))^2 + \frac{1}{q}\big|u(x(t))\big|^q - \frac{1}{p}\frac{1}{|u(x(t))|^p}\bigg)\bigg]x'(t)\,dt\bigg|.
\end{aligned}
$$

Then there exist constants $D_1 > 0$, $D_2 > 0$, and $D_3 > 0$ such that

$$
\begin{aligned}
&\bigg|\frac{1}{2}\int_0^{2\pi}\big(\big(\Delta^2 u(x(t)) + c\Delta u(x(t))\big)\cdot v(x(t))\big)x'(t)\,dt\bigg| \\
&\le \bigg|\frac{1}{2}\int_0^{2\pi}\big(\big(\Delta u(x(t)) - c\nabla u(x(t))\big)\cdot\big(\Delta v(x(t)) - c\nabla v(x(t))\big)\big)x'(t)\,dt\bigg| \\
&\le \frac{1}{2}\int_0^{2\pi}\big|\big(\Delta u(x(t)) - c\nabla u(x(t))\big)\cdot\big(\Delta v(x(t)) - c\nabla v(x(t))\big)\big|x'(t)\,dt \\
&\le \frac{1}{2}\int_0^{2\pi}\big|\big(\Delta u(x(t)) - c\nabla u(x(t))\big)\big|\big|\big(\Delta v(x(t)) - c\nabla v(x(t))\big)\big|x'(t)\,dt \\
&\le \frac{1}{2}\int_0^{2\pi}\big|\big(\Delta u(x(t)) - c\nabla u(x(t))\big)\big|x'(t)\,dt\int_0^{2\pi}\big|\big(\Delta v(x(t)) - c\nabla v(x(t))\big)\big|x'(t)\,dt \\
&\le \frac{1}{2}\bigg(\int_0^{2\pi}\big|\big(\Delta u(x(t)) - c\nabla u(x(t))\big)\big|^q x'(t)\,dt\bigg)^{\frac{1}{q}}
\end{aligned}
$$

$$\begin{aligned}
&\cdot \left(\int_0^{2\pi} \big|\big(\Delta v(x(t)) - c\nabla v(x(t))\big)\big|^q x'(t)\,dt\right)^{\frac{1}{q}} \\
&\le D_1 \|u\circ x\|_H \|v\circ x\|_H = O\big(\|v\circ x\|_H\big), \\
&\left|\frac{1}{2}\int_0^{2\pi} \big[\big(\Delta^2 v(x(t)) + c\Delta v(x(t))\big)\cdot u(x(t))\big] x'(t)\,dt\right| \\
&\le \left|\frac{1}{2}\int_0^{2\pi} \big[\big(\Delta v(x(t)) - c\nabla v(x(t))\big)\cdot\big(\Delta u(x(t)) - c\nabla u(x(t))\big)\big] x'(t)\,dt\right| \\
&\le \frac{1}{2}\int_0^{2\pi} \big|\big(\Delta v(x(t)) - c\nabla v(x(t))\big)\cdot\big(\Delta u(x(t)) - c\nabla u(x(t))\big)\big| x'(t)\,dt \\
&\le \frac{1}{2}\int_0^{2\pi} \big|\big(\Delta v(x(t)) - c\nabla v(x(t))\big)\big|\big|\big(\Delta u(x(t)) - c\nabla u(x(t))\big)\big| x'(t)\,dt \\
&\le \frac{1}{2}\int_0^{2\pi} \big|\big(\Delta v(x(t)) - c\nabla v(x(t))\big)\big| x'(t)\,dt \int_0^{2\pi} \big|\big(\Delta u(x(t)) - c\nabla u(x(t))\big)\big| x'(t)\,dt \\
&\le \frac{1}{2}\left(\int_0^{2\pi} \big|\big(\Delta v(x(t)) - c\nabla v(x(t))\big)\big|^q x'(t)\,dt\right)^{\frac{1}{q}} \\
&\quad\cdot \left(\int_0^{2\pi} \big|\big(\Delta u(x(t)) - c\nabla u(x(t))\big)\big|^q x'(t)\,dt\right)^{\frac{1}{q}} \\
&\le D_2 \|u\circ x\|_H \|v\circ x\|_H = O\big(\|v\circ x\|_H\big), \\
&\left|\frac{1}{2}\int_0^{2\pi} \big[\big(\Delta^2 v(x(t)) + c\Delta v(x(t))\big)\cdot v(x(t))\big] x'(t)\,dt\right| \\
&\le \left|\frac{1}{2}\int_0^{2\pi} \big[\big(\Delta v(x(t)) - c\nabla v(x(t))\big)\cdot\big(\Delta v(x(t)) - c\nabla v(x(t))\big)\big] x'(t)\,dt\right| \\
&\le \frac{1}{2}\int_0^{2\pi} \big|\big(\Delta v(x(t)) - c\nabla v(x(t))\big)\cdot\big(\Delta v(x(t)) - c\nabla v(x(t))\big)\big| x'(t)\,dt \\
&\le \frac{1}{2}\int_0^{2\pi} \big|\big(\Delta v(x(t)) - c\nabla v(x(t))\big)\big|\big|\big(\Delta v(x(t)) - c\nabla v(x(t))\big)\big| x'(t)\,dt \\
&\le \frac{1}{2}\int_0^{2\pi} \big|\big(\Delta v(x(t)) - c\nabla v(x(t))\big)\big| x'(t)\,dt \int_0^{2\pi} \big|\big(\Delta v(x(t)) - c\nabla v(x(t))\big)\big| x'(t)\,dt \\
&\le \frac{1}{2}\left(\int_0^{2\pi} \big|\big(\Delta v(x(t)) - c\nabla v(x(t))\big)\big|^q x'(t)\,dt\right)^{\frac{1}{q}} \\
&\quad\cdot \left(\int_0^{2\pi} \big|\big(\Delta v(x(t)) - c\nabla v(x(t))\big)\big|^q x'(t)\,dt\right)^{\frac{1}{q}} \\
&\le D_3 \|v\circ x\|_H \|v\circ x\|_H = O\big(\|v\circ x\|_H\big).
\end{aligned}$$

Since $u(x(t)), v(x(t)) \in D$, we have that $u(x(t)) > 0$, $v(x(t)) > 0$ and $u(x(t)) + v(x(t)) > 0$. Thus, $\frac{1}{|u(x(t))|^p}$ and $\frac{1}{|u(x(t))+v(x(t))|^p}$ are well defined, continuous, and C^1. By the mean value theorem we have

$$\begin{aligned}
&\left(\frac{1}{q}\big|u(x(t)) + v(x(t))\big|^q - \frac{1}{p}\frac{1}{|u(x(t)) + v(x(t))|^p}\right) - \left(\frac{1}{q}\big|u(x(t))\big|^q - \frac{1}{p}\frac{1}{|u(x(t))|^p}\right) \\
&= \left(\big|u(x(t))\big|^{q-1} + \frac{1}{|u(x(t))|^{p+1}}\right)\cdot v(x(t)) + O\big(\|v\circ x\|_H\big). \qquad (2.3)
\end{aligned}$$

Thus, there exist constants $D_4 > 0$, $D_5 > 0$, $D_6 > 0$, and $D_7 > 0$ such that

$$\begin{aligned}
&\bigg|\int_0^{2\pi}\bigg[\bigg(\frac{1}{2}\Lambda\big(u\big(x(t)+v\big(x(t)\big)\big)\big)^2+\frac{1}{q}\big|u\big(x(t)\big)+v\big(x(t)\big)\big|^q-\frac{1}{p}\frac{1}{|u(x(t))+v(x(t))|^p}\bigg)\\
&\quad-\bigg(\frac{1}{2}\Lambda u\big(x(t)\big)^2+\frac{1}{q}\big|u\big(x(t)\big)\big|^q-\frac{1}{p}\frac{1}{|u(x(t))|^p}\bigg)\bigg]x'(t)\,dt\bigg|\\
&=\bigg|\int_0^{2\pi}\bigg[\Lambda\bigg(u\big(x(t)\big)\cdot v\big(x(t)\big)+\frac{1}{2}\Lambda v\big(x(t)\big)\bigg)^2\bigg]x'(t)\,dt\\
&\quad+\int_0^{2\pi}\bigg[\bigg(\big|u\big(x(t)\big)\big|^{q-1}+\frac{1}{|u(x(t))|^{p+1}}\bigg)\cdot v\big(x(t)\big)+O\big(\|v\circ x\|_H\big)\bigg]x'(t)\,dt\bigg|\\
&\le\bigg|\int_0^{2\pi}\bigg[\Lambda\bigg(u\big(x(t)\big)\cdot v\big(x(t)\big)+\frac{1}{2}\Lambda v\big(x(t)\big)\bigg)^2\bigg]x'(t)\,dt\bigg|\\
&\quad+\bigg|\int_0^{2\pi}\bigg[\bigg(\big|u\big(x(t)\big)\big|^{q-1}+\frac{1}{|u(x(t))|^{p+1}}\bigg)\cdot v\big(x(t)\big)+O\big(\|v\circ x\|_H\big)\bigg]x'(t)\,dt\bigg|\\
&\le\int_0^{2\pi}\bigg|\Lambda\bigg(u\big(x(t)\big)\cdot v\big(x(t)\big)+\frac{1}{2}\Lambda v\big(x(t)\big)\bigg)^2\bigg|x'(t)\,dt\\
&\quad+\int_0^{2\pi}\bigg|\bigg(\big|u\big(x(t)\big)\big|^{q-1}+\frac{1}{|u(x(t))|^{p+1}}\bigg)\cdot v\big(x(t)\big)+O\big(\|v\circ x\|_H\big)\bigg|x'(t)\,dt\\
&\le\int_0^{2\pi}\big|\Lambda\big(u\big(x(t)\big)\cdot v\big(x(t)\big)\big)\big|x'(t)\,dt+\frac{1}{2}\int_0^{2\pi}\big|\Lambda v\big(x(t)\big)^2\big|x'(t)\,dt\\
&\quad+\int_0^{2\pi}\bigg|\big|u\big(x(t)\big)\big|^{q-1}+\frac{1}{|u(x(t))|^{p+1}}\bigg|\big|v\big(x(t)\big)\big|x'(t)\,dt+2\pi O\big(\|v\circ x\|\big)_H\\
&\le\int_0^{2\pi}\big|\Lambda\big(u\big(x(t)\big)\big)\big|x'(t)\,dt\int_0^{2\pi}\big|v\big(x(t)\big)\big|x'(t)\,dt+\frac{1}{2}\int_0^{2\pi}\big|\Lambda v\big(x(t)\big)^2\big|x'(t)\,dt\\
&\quad+\int_0^{2\pi}\bigg|\big|u\big(x(t)\big)\big|^{q-1}+\frac{1}{|u(x(t))|^{p+1}}\bigg|x'(t)\,dt\int_0^{2\pi}\big|v\big(x(t)\big)\big|x'(t)\,dt\\
&\quad+2\pi O\big(\|v\circ x\|_H\big)\\
&\le D_4\|u\circ x\|_H\|v\circ x\|_H+D_5\|v\circ x\|_H^2\\
&\quad+\bigg(\int_0^{2\pi}\bigg|\big|u\big(x(t)\big)\big|^{q-1}+\frac{1}{|u(x(t))|^{p+1}}\bigg|^q x'(t)\,dt\bigg)^{\frac{1}{q}}\bigg(\int_0^{2\pi}\big|v\big(x(t)\big)\big|^q x'(t)\,dt\bigg)^{\frac{1}{q}}\\
&\quad+2\pi O\big(\|v\circ x\|_H\big)\\
&\le D_4\|u\circ x\|_H\|v\circ x\|_H+D_5\|v\circ x\|_H^2+D_6\|u\circ x\|_H^q\|v\circ x\|_H\\
&\quad+D_7\bigg\|\frac{1}{|u\circ x|^{p+1}}\bigg\|_H\|v\circ x\|_H+2\pi O\big(\|v\circ x\|_H\big)=O\big(\|v\circ x\|_H\big).
\end{aligned}$$

Thus, we have

$$\big|J(u\circ x+v\circ x)-J(u\circ x)\big|=O\big(\|v\circ x\|_H^2\big).$$

Next, we shall prove that $J(u\circ x)$ is Fréchet differentiable. Let $u\circ x, v\circ x\in H$. Then since $u(x(t)), v(x(t))\in D$, it follows that $u(x(t))>0$, $v(x(t))>0$, and $u(x(t))+v(x(t))>0$. Thus,

$\frac{1}{|u(x(t))|^p}$ and $\frac{1}{|u(x(t))+v(x(t))|^p}$ are well defined, continuous, and C^1. Thus, we have

$$
\begin{aligned}
&\bigl|J(u\circ x+v\circ x)-J(u\circ x)-DJ(u\circ x)\cdot(v\circ x)\bigr| \\
&=\biggl|\frac{1}{2}\int_0^{2\pi}\bigl(\Delta^2\bigl(u\bigl(x(t)\bigr)+v\bigl(x(t)\bigr)\bigr)+c\Delta\bigl(u\bigl(x(t)\bigr)+v\bigl(x(t)\bigr)\bigr)\bigr)\cdot\bigl(u\bigl(x(t)\bigr)+v\bigl(x(t)\bigr)\bigr)x'(t)\,dt \\
&\quad-\int_0^{2\pi}\biggl[\frac{1}{2}\Lambda\bigl(u\bigl(x(t)\bigr)+v\bigl(x(t)\bigr)\bigr)^2+\frac{1}{q}\bigl|u\bigl(x(t)\bigr)+v\bigl(x(t)\bigr)\bigr|^q \\
&\quad-\frac{1}{p}\frac{1}{|u(x(t))+v(x(t))|^p}\biggr]x'(t)\,dt \\
&\quad-\frac{1}{2}\int_0^{2\pi}\bigl(\Delta^2u\bigl(x(t)\bigr)+c\Delta u\bigl(x(t)\bigr)\bigr)\cdot u\bigl(x(t)\bigr)x'(t)\,dt \\
&\quad+\int_0^{2\pi}\biggl[\frac{1}{2}\Lambda u\bigl(x(t)\bigr)^2+\frac{1}{q}\bigl|u\bigl(x(t)\bigr)\bigr|^q-\frac{1}{p}\frac{1}{|u(x(t))|^p}\biggr]x'(t)\,dt \\
&\quad-\int_0^{2\pi}\biggl[\bigl(\Delta^2u\bigl(x(t)\bigr)+c\Delta u\bigl(x(t)\bigr)\bigr)\cdot v\bigl(x(t)\bigr) \\
&\quad-\biggl(\Lambda u\bigl(x(t)\bigr)+\bigl|u\bigl(x(t)\bigr)\bigr|^{q-1}+\frac{1}{|u(x(t))|^{p+1}}\biggr)\cdot v\bigl(x(t)\bigr)\biggr]x'(t)\,dt\biggr| \\
&=\biggl|\frac{1}{2}\int_0^{2\pi}\bigl[\bigl(\Delta^2v\bigl(x(t)\bigr)+c\Delta v\bigl(x(t)\bigr)\bigr)\cdot u\bigl(x(t)\bigr) \\
&\quad+\bigl(\Delta^2v\bigl(x(t)\bigr)+c\Delta v\bigl(x(t)\bigr)\bigr)\cdot v\bigl(x(t)\bigr)\bigr]x'(t)\,dt \\
&\quad-\int_0^{2\pi}\biggl[\biggl(\biggl(\frac{1}{2}\Lambda\bigl(u\bigl(x(t)\bigr)+v\bigl(x(t)\bigr)\bigr)^2+\frac{1}{q}\bigl|u\bigl(x(t)\bigr)+v\bigl(x(t)\bigr)\bigr|^q \\
&\quad-\frac{1}{p}\frac{1}{|u(x(t))+v(x(t))|^p}\biggr) \\
&\quad-\biggl(\frac{1}{2}\Lambda u\bigl(x(t)\bigr)^2+\frac{1}{q}\bigl|u\bigl(x(t)\bigr)\bigr|^q-\frac{1}{p}\frac{1}{|u(x(t))|^p}\biggr) \\
&\quad-\biggl(\Lambda u\bigl(x(t)\bigr)+\bigl|u\bigl(x(t)\bigr)\bigr|^{q-1}+\frac{1}{|u(x(t))|^{p+1}}\biggr)\biggr)\cdot v\bigl(x(t)\bigr)\biggr]x'(t)\,dt\biggr|.
\end{aligned}
$$

By (2.3) and the same arguments as in the proof of the continuity of $J(u\circ x)$ we have

$$\bigl|J(u\circ x+v\circ x)-J(u\circ x)-DJ(u\circ x)\cdot(v\circ x)\bigr|=O\bigl(\|v\circ x\|_H\bigr).$$

Thus, $J\in C^1$. □

Lemma 2.3 *Assume that* $\lambda_k < c < \lambda_{k+1}$, $2 < q < p$, $q < \frac{2n}{n-2}$, $\lambda_{k+m}(\lambda_{k+m}-c) < \Lambda < \lambda_{k+m+1}(\lambda_{k+m+1}-c)$, $k\geq 1$, $m\geq 1$, *and that conditions* (A1) *and* (A2) *hold. Then for any sequence* $(u_n\circ x)_n\subset H$ *such that* $u_n\circ x\rightharpoonup u\circ x$ *weakly in* H *with* $u\circ x\in\partial H$, *we have* $J(u_n\circ x)\to\infty$.

Proof We claim that

$$\int_0^{2\pi}\biggl[-\frac{1}{2}\Lambda u_n\bigl(x(t)\bigr)^2-\frac{1}{q}\bigl|u_n\bigl(x(t)\bigr)\bigr|^q+\frac{1}{p}\frac{1}{|u_n(x(t))|^p}\biggr]x'(t)\,dt\longrightarrow+\infty.$$

By (A2) there exists a neighborhood Z of C in R such that for any $u(x(t)) \in Z$, $\{-\frac{1}{2}\Lambda \times u_n(x(t))^2 - \frac{1}{q}|u_n(x(t))|^q + \frac{1}{p}\frac{1}{|u_n(x(t))|^p}\}$ is bounded from below. Thus, it suffices to prove that there exists an interval $[t_1, t_2] \subset [0, 2\pi]$ such that

$$\int_{t_1}^{t_2} \left[-\frac{1}{2}\Lambda u_n\bigl(x(t)\bigr)^2 - \frac{1}{q}\bigl|u_n\bigl(x(t)\bigr)\bigr|^q + \frac{1}{p}\frac{1}{|u_n(x(t))|^p}\right]x'(t)\,dt \longrightarrow +\infty.$$

Since $u \circ x \in \partial H$, there exists $t^* \in [0, 2\pi]$ such that $u(x(t^*)) \in \partial D$. By (A2) there exist constants $A > 0$ and $B > 0$ such that

$$-\frac{1}{2}\Lambda u\bigl(x(t)\bigr)^2 - \frac{1}{q}\bigl|u\bigl(x(t)\bigr)\bigr|^q + \frac{1}{p}\frac{1}{|u(x(t))|^p} \geq \frac{A}{d^2(u(x(t)), C)} - B.$$

Thus, we have

$$\begin{aligned}
&\int_{t^*}^{t^*+\delta} \left[-\frac{1}{2}\Lambda u\bigl(x(t)\bigr)^2 - \frac{1}{q}\bigl|u\bigl(x(t)\bigr)\bigr|^q + \frac{1}{p}\frac{1}{|u(x(t))|^p}\right]x'(t)\,dt \\
&\quad\geq \int_{t^*}^{t^*+\delta} \left[\frac{A}{d^2(u(x(t)), C)} - B\right]x'(t)\,dt
\end{aligned}$$

for all $\delta > 0$. On the other hand, we have

$$\begin{aligned}
\bigl|u\bigl(x(t)\bigr) - u\bigl(x\bigl(t^*\bigr)\bigr)\bigr| &\leq \bigl|t - t^*\bigr|^{\frac{1}{2}}\left(\int_0^{2\pi} \bigl|u'\bigl(x(t)\bigr)\bigr|^2 x'(t)\,dt\right)^{\frac{1}{2}} \\
&\leq \delta^{\frac{1}{2}}\left(\int_0^{2\pi} \bigl|u'\bigl(x(t)\bigr)\bigr|^2 x'(t)\,dt\right)^{\frac{1}{2}}.
\end{aligned} \tag{2.4}$$

It follows from (2.4) that

$$\begin{aligned}
&\int_{t^*}^{t^*+\delta} \left[-\frac{1}{2}\Lambda u\bigl(x(t)\bigr)^2 - \frac{1}{q}\bigl|u\bigl(x(t)\bigr)\bigr|^q + \frac{1}{p}\frac{1}{|u(x(t))|^p}\right]x'(t)\,dt \\
&\quad\geq \int_{t^*}^{t^*+\delta} \left[\frac{A}{\delta \int_0^{2\pi} |u'(x(t))|^2 x'(t)\,dt} - B\right]x'(t)\,dt \longrightarrow +\infty \quad \text{as } \delta \to 0.
\end{aligned}$$

Thus,

$$\int_{t^*}^{t^*+\delta} \left[-\frac{1}{2}\Lambda u\bigl(x(t)\bigr)^2 - \frac{1}{q}\bigl|u\bigl(x(t)\bigr)\bigr|^q + \frac{1}{p}\frac{1}{|u(x(t))|^p}\right]x'(t)\,dt \longrightarrow \infty$$

as $\delta \to 0$. Since the embedding $H(S^1, R) \hookrightarrow C(S^1, R)$ is compact, we have

$$\max\bigl\{\bigl|u\bigl(x(t)\bigr) - u_n\bigl(x(t)\bigr)\bigr| \mid \forall t \in S^1\bigr\} \longrightarrow 0 \quad \text{as } n \to \infty.$$

By Fatou's lemma we have

$$\begin{aligned}
&\liminf \int_{t^*}^{t^*+\delta} \left[-\frac{1}{2}\Lambda u_n\bigl(x(t)\bigr)^2 - \frac{1}{q}\bigl|u_n\bigl(x(t)\bigr)\bigr|^q + \frac{1}{p}\frac{1}{|u_n(x(t))|^p}\right]x'(t)\,dt \\
&\quad\geq \int_{t^*}^{t^*+\delta} \liminf\left[-\frac{1}{2}\Lambda u_n\bigl(x(t)\bigr)^2 - \frac{1}{q}\bigl|u_n\bigl(x(t)\bigr)\bigr|^q + \frac{1}{p}\frac{1}{|u_n(x(t))|^p}\right]x'(t)\,dt \\
&\quad= \int_{t^*}^{t^*+\delta} \left[-\frac{1}{2}\Lambda u\bigl(x(t)\bigr)^2 - \frac{1}{q}\bigl|u\bigl(x(t)\bigr)\bigr|^q + \frac{1}{p}\frac{1}{|u(x(t))|^p}\right]x'(t)\,dt \longrightarrow \infty
\end{aligned}$$

as $\delta \to 0$. Thus,

$$\liminf \int_{t^*}^{t^*+\delta} \left[-\frac{1}{2}\Lambda u_n\big(x(t)\big)^2 - \frac{1}{q}\big|u_n\big(x(t)\big)\big|^q + \frac{1}{p}\frac{1}{|u_n(x(t))|^p} \right] x'(t)\,dt = +\infty,$$

so that $J(u \circ x) \longrightarrow +\infty$, and the lemma is proved. □

Lemma 2.4 *Assume that* $\lambda_k < c < \lambda_{k+1}$, $2 < q < p$, $q < \frac{2n}{n-2}$, $\lambda_{k+m}(\lambda_{k+m} - c) < \Lambda < \lambda_{k+m+1}(\lambda_{k+m+1} - c)$, $k \geq 1$, $m \geq 1$, *and that conditions* (A1) *and* (A2) *hold. Then if* $\|u_n \circ x\|_H \to \infty$ *and* $(u_n \circ x)_n$ *is a sequence in* H *such that*

$$\frac{\int_0^{2\pi} [(|u_n(x(t))|^{q-1} + \frac{1}{|u_n(x(t))|^{p+1}}) \cdot u_n(x(t)) - (\frac{2}{q}|u_n(x(t))|^q - \frac{2}{p}\frac{1}{|u_n(x(t))|^p})]x'(t)\,dt}{\|u_n \circ x\|_H} \longrightarrow 0,$$

then there exist $(u_{h_n} \circ x)_n$ *and* $z \circ x$ *in* H *such that*

$$\frac{|u_{h_n} \circ x|^{q-1} + \frac{1}{|u_{h_n} \circ x|^{p+1}}}{\|u_n \circ x\|_H} \to z \circ x \in H, \qquad \frac{u_{h_n} \circ x}{\|u_{h_n} \circ x\|_H} \to 0.$$

Proof Since

$$\frac{\int_0^{2\pi} [(|u_n(x(t))|^{q-1} + \frac{1}{|u_n(x(t))|^{p+1}}) \cdot u_n(x(t)) - (\frac{2}{q}|u_n(x(t))|^q - \frac{2}{p}\frac{1}{|u_n(x(t))|^p})]x'(t)\,dt}{\|u_n \circ x\|_H} \longrightarrow 0,$$

the sequence $\left(\frac{\int_0^{2\pi} [(|u_n(x(t))|^{q-1} + \frac{1}{|u_n(x(t))|^{p+1}}) \cdot u_n(x(t)) - (\frac{2}{q}|u_n(x(t))|^q - \frac{2}{p}\frac{1}{|u_n(x(t))|^p})]x'(t)\,dt}{\|u_n \circ x\|_H}\right)_n$ is bounded, and there exists a constant $C_1 > 0$ such that

$$\left\| \frac{\int_0^{2\pi} [(|u_n(x(t))|^{q-1} + \frac{1}{|u_n(x(t))|^{p+1}}) \cdot u_n(x(t)) - (\frac{2}{q}|u_n(x(t))|^q - \frac{2}{p}\frac{1}{|u_n(x(t))|^p})]x'(t)\,dt}{\|u_n \circ x\|_H} \right\|_H \leq C_1.$$

Then we have

$$\begin{aligned} &\left\| \frac{\int_0^{2\pi} [(|u_n(x(t))|^{q-1} + \frac{1}{|u_n(x(t))|^{p+1}}) \cdot u_n(x(t))]x'(t)\,dt}{\|u_n \circ x\|_H} \right\|_H \\ &\quad \leq \left\| \frac{\int_0^{2\pi} [(|u_n(x(t))|^{q-1} + \frac{1}{|u_n(x(t))|^{p+1}}) \cdot u_n(x(t)) - (\frac{2}{q}|u_n(x(t))|^q - \frac{2}{p}\frac{1}{|u_n(x(t))|^p})]x'(t)\,dt}{\|u_n \circ x\|_H} \right\|_H \\ &\qquad + \left\| \frac{\int_0^{2\pi} [\frac{2}{q}|u_n(x(t))|^q - \frac{2}{p}\frac{1}{|u_n(x(t))|^p}]x'(t)\,dt}{\|u_n \circ x\|_H} \right\|_H \\ &\quad \leq C_1 + \left\| \frac{\int_0^{2\pi} [\frac{2}{q}|u_n(x(t))|^q - \frac{2}{p}\frac{1}{|u_n(x(t))|^p}]x'(t)\,dt}{\|u_n \circ x\|_H} \right\|_H. \end{aligned} \tag{2.5}$$

We note that

$$\int_0^{2\pi} \left[\frac{2}{q}\big|u_n\big(x(t)\big)\big|^q - \frac{2}{p}\frac{1}{|u_n(x(t))|^p} \right] x'(t)\,dt \leq \frac{2}{q}\|u_n \circ x\|^q_{L^q(S^1,R)} + \frac{2}{p}\left\| \frac{1}{|u_n \circ x|} \right\|^p_{L^q(S^1,R)}.$$

Then there exist constants $C_2 > 0$ and $C_3 > 0$ such that

$$\begin{aligned}
&\left\| \frac{\int_0^{2\pi} [\frac{2}{q}|u_n(x(t))|^q - \frac{2}{p}\frac{1}{|u_n(x(t))|^p}]x'(t)\,dt}{\|u_n \circ x\|_H} \right\|_H \\
&\quad \le C_2 \left(\frac{\|u_n \circ x\|^q_{L^q(S^1,R)}}{\|u_n \circ x\|_H} \right)^{\frac{q-1}{q}} \|u_n \circ x\|^l_H + C_3 \frac{\|\frac{1}{|u_n\circ x|}\|^p_{L^q(S^1,R)}}{\|u_n \circ x\|_H},
\end{aligned} \tag{2.6}$$

where $l = -1 + \frac{q-1}{q} = -\frac{1}{q} < 0$. Note that since $(\frac{\|u_n\circ x\|^q_{L^q(S^1,R)}}{\|u_n\circ x\|_H})^{\frac{q-1}{q}}$ is bounded, it follows from $l < 0$ that the right-hand side of (2.6) is bounded from above and

$$\left\| \frac{\int_0^{2\pi} [\frac{2}{q}|u_n(x(t))|^q - \frac{2}{p}\frac{1}{|u_n(x(t))|^p}]x'(t)\,dt}{\|u_n \circ x\|_H} \right\|_H \longrightarrow 0 \quad \text{as } n \to \infty.$$

Thus, by (2.5),

$$\left\| \frac{\int_0^{2\pi} [(|u_n(x(t))|^{q-1} + \frac{1}{|u_n(x(t))|^{p+1}}) \cdot u_n(x(t))]x'(t)\,dt}{\|u_n \circ x\|_H} \right\|_H \quad \text{is bounded from above}$$

and

$$\lim_{n\to\infty} \frac{\int_0^{2\pi} [(|u_n(x(t))|^{q-1} + \frac{1}{|u_n(x(t))|^{p+1}}) \cdot u_n(x(t))]x'(t)\,dt}{\|u_n \circ x\|_H} = 0.$$

Thus, the sequence $(\frac{\int_0^{2\pi} [(|u_n(x(t))|^{q-1} + \frac{1}{|u_n(x(t))|^{p+1}}) \cdot u_n(x(t))]x'(t)\,dt}{\|u_n\circ x\|_H})_n$ is bounded. When $2 < q < \frac{2n}{n-2}$, the embedding $H \hookrightarrow L^q(\Omega)$ is compact, Thus there exists a subsequence $(u_{h_n} \circ x)_n$ such that

$$\begin{aligned}
&\lim_{n\to\infty} \frac{\int_0^{2\pi} [(|u_{h_n}(x(t))|^{q-1} + \frac{1}{|u_{h_n}(x(t))|^{p+1}}) \cdot u_{h_n}(x(t))]x'(t)\,dt}{\|u_{h_n} \circ x\|_H} \\
&\quad = \lim_{n\to\infty} \int_0^{2\pi} \left(\left|u_{h_n}\big(x(t)\big)\right|^{q-1} + \frac{1}{|u_{h_n}(x(t))|^{p+1}} \right) \cdot \frac{u_{h_n}(x(t))}{\|u_{h_n} \circ x\|_H} x'(t)\,dt = 0.
\end{aligned} \tag{2.7}$$

We note that $0 < |u_{h_n}(x(t))|^{q-1} + \frac{1}{|u_{h_n}(x(t))|^{p+1}} \le \|u_{h_n}(x(t))\|^{q-1}_{L^q(S^1,R)} + \|\frac{1}{|u_{h_n}(x(t))|^{p+1}}\|_{L^q(S^1,R)} < \infty$. It follows from (2.7) that there exists $z \circ x$ in H such that

$$\frac{|u_{h_n} \circ x|^{q-1} + \frac{1}{|u_{h_n}\circ x|^{p+1}}}{\|u_n \circ x\|_H} \to z \circ x \in H, \qquad \frac{u_{h_n} \circ x}{\|u_{h_n} \circ x\|_H} \to 0.$$

Thus, the lemma is proved. □

Let us set

$$\begin{aligned}
H^-\big(S^1, R\big) &= \bigoplus_{\Lambda_i < 0} H_{\lambda_i(\lambda_i - c)}\big(S^1, R\big), \\
H^+\big(S^1, R\big) &= \bigoplus_{\Lambda_i > 0} H_{\lambda_i(\lambda_i - c)}\big(S^1, R\big),
\end{aligned}$$

$$H^0(S^1,R) = \bigoplus_{\Lambda_i=0} H_{\lambda_i(\lambda_i-c)}(S^1,R).$$

Then

$$H = H^-(S^1,R) \oplus H^+(S^1,R) \oplus H^0(S^1,R).$$

We note that if $\lambda_{k+m}(\lambda_{k+m}-c) < \Lambda < \lambda_{k+m+1}(\lambda_{k+m+1}-c)$, $k \geq 1$, $m \geq 1$, then

$$H^-(S^1,R) = \bigoplus_{\Lambda_1 \leq \Lambda_i \leq \Lambda_{k+m}} H_{\lambda_i(\lambda_i-c)}(S^1,R),$$
$$H^+(S^1,R) = \bigoplus_{\Lambda_i \geq \Lambda_{k+m+1}} H_{\lambda_i(\lambda_i-c)}(S^1,R),$$

$\dim H^-(S^1,R) < \infty$, $H^0(S^1,R) = \emptyset$, and

$$H = H^-(S^1,R) \oplus H^+(S^1,R).$$

Now, we shall prove that $J(u \circ x)$ satisfies $(P.S.)_c$ condition for $c \in R$.

Lemma 2.5 (Palais-Smale condition) *Assume that* $\lambda_k < c < \lambda_{k+1}$, $2 < q < p$, $q < \frac{2n}{n-2}$, $\lambda_{k+m}(\lambda_{k+m}-c) < \Lambda < \lambda_{k+m+1}(\lambda_{k+m+1}-c)$, $k \geq 1$, $m \geq 1$, *and that conditions* (A1) *and* (A2) *hold. Then* $J(u \circ x)$ *satisfies* $(P.S.)_c$ *condition for any* $c \in R$: *if* $(u_n \circ x)_n \in H$ *is any sequence such that* $J(u_n \circ x) \to c$ *and* $DJ(u_n \circ x) \to 0$, *then* $(u_n \circ x)_n$ *has a convergent subsequence* $(u_{n_i} \circ x)$ *such that*

$$u_{n_i} \circ x \to u \circ x \in H.$$

Proof Let $c \in R$, and let $(u_n \circ x)_n \subset H$ be a sequence such that $J(u_n \circ x) \to c$ and

$$DJ(u_n \circ x) = \Delta^2(u_n \circ x) + c\Delta(u_n \circ x) - \left(\Lambda(u_n \circ x) + |u_n \circ x|^{q-1} + \frac{1}{|u_n \circ x|^{p+1}}\right) \longrightarrow 0 \quad \text{in } H$$

or, equivalently,

$$u_n \circ x - \left(\Delta^2 + c\Delta\right)^{-1}\left(\Lambda(u_n \circ x) + |u_n \circ x|^{q-1} + \frac{1}{|u_n \circ x|^{p+1}}\right) \longrightarrow 0, \tag{2.8}$$

where $(\Delta^2 + c\Delta)^{-1}$ is a compact operator. We shall show that $(u_n \circ x)_n$ has a convergent subsequence. We claim that $\{u_n \circ x\}$ is bounded in H. By contradiction we suppose that $\|u_n \circ x\|_H \to \infty$ and set $w_n \circ x = \frac{u_n \circ x}{\|u_n \circ x\|_H}$. Since $(w_n \circ x)_n$ is bounded, up to a subsequence, $(w_n \circ x)_n$ converges weakly to some $w_0 \circ x$ in H. Since $J(u_n \circ x) \to c$ and $DJ(u_n \circ x) \to 0$, we have

$$\frac{DJ(u_n \circ x) \cdot (u_n \circ x)}{\|u_n \circ x\|_H}$$
$$= \frac{2J(u_n \circ x)}{\|u_n \circ x\|_H}$$

$$- \frac{\int_0^{2\pi} [(|u_n(x(t))|^{q-1} + \frac{1}{|u_n(x(t))|^{p+1}}) \cdot u_n(x(t)) - (\frac{2}{q}|u_n(x(t))|^q - \frac{2}{p}\frac{1}{|u_n(x(t))|^p})]x'(t)\,dt}{\|u_n \circ x\|_H}$$
$$\longrightarrow 0.$$

Thus, we have

$$\frac{\int_0^{2\pi} [(|u_n(x(t))|^{q-1} + \frac{1}{|u(x(t))|^{p+1}}) \cdot u_n(x(t)) - (\frac{2}{q}|u_n(x(t))|^q - \frac{2}{p}\frac{1}{|u_n(x(t))|^p})]x'(t)\,dt}{\|u_n \circ x\|_H} \longrightarrow 0.$$

By Lemma 2.4 and (2.8) there exist $(u_{h_n} \circ x)_n$ and $z \circ x$ in H such that

$$\frac{|u_{h_n} \circ x|^{q-1} + \frac{1}{|u_{h_n} \circ x|^{p+1}}}{\|u_n \circ x\|_H} \to z \circ x \in H, \qquad \frac{u_{h_n} \circ x}{\|u_{h_n} \circ x\|_H} \to 0.$$

Thus, we have $w_0 \circ x = 0$, which is absurd because $\|w_0 \circ x\|_H = 1$. Thus, $\{u_n \circ x\}$ is bounded in H. Thus, $(u_n \circ x)_n$ has a convergent subsequence converging weakly to some $u \circ x$ in H. We claim that this subsequence of $(u_n \circ x)_n$ converges strongly to $u \circ x$. Since $DJ(u_n \circ x) \to 0$, we have

$$DJ(u_n \circ x) = (\Delta^2 + c\Delta - \Lambda)(u_n \circ x) - \left(|u_n \circ x|^{q-1} + \frac{1}{|u_n \circ x|^{p+1}}\right) \longrightarrow 0.$$

We claim that the mapping $u_n \circ x \to\mapsto (|u_n \circ x|^{q-1} + \frac{1}{|u_n \circ x|^{p+1}})_n$ is compact. Since the embedding $H \hookrightarrow L^q(S^1, R)$ is compact for $2 < q < \frac{2n}{n-2}$, the mapping $H \to L^q(S^1, R)$: $u_n \circ x \mapsto \int_0^{2\pi} (|u_n(x(t))|^{q-1} + \frac{1}{|u_n(x(t))|^{p+1}})u_n(x(t))x'(t)\,dt$ is compact. Thus, the sequence $(\int_0^{2\pi} (|u_n(x(t))|^{q-1} + \frac{1}{|u_n(x(t))|^{p+1}})u_n(x(t))x'(t)\,dt)_n$ has a convergent subsequence that converges to $\int_0^{2\pi} (|u(x(t))|^{q-1} + \frac{1}{|u(x(t))|^{p+1}})u(x(t))x'(t)\,dt$. Because $\{u_n \circ x\}$ is bounded and the subsequence of $(u_n \circ x)_n$ converges weakly to some $u \circ x$ in H, $(|u_n \circ x|^{q-1} + \frac{1}{|u_n \circ x|^{p+1}})_n$ has a convergent subsequence. Since $(|u_n \circ x|^{q-1} + \frac{1}{|u_n \circ x|^{p+1}})_n$ has a convergent subsequence, the subsequence of $(\Delta^2 + c\Delta - \Lambda)(u_n \circ x)$ converges. Since $(\Delta^2 + c\Delta - \Lambda)^{-1}$ is compact, the sequence $(u_n \circ x)_n$ has a subsequence converging strongly to $u \circ x$ in H. □

3 Proof of Theorem 1.1

Let us set again

$$H^-(S^1, R) = \bigoplus_{\Lambda_i < 0} H_{\lambda_i(\lambda_i - c)}(S^1, R),$$
$$H^+(S^1, R) = \bigoplus_{\Lambda_i > 0} H_{\lambda_i(\lambda_i - c)}(S^1, R),$$
$$H^0(S^1, R) = \bigoplus_{\Lambda_i = 0} H_{\lambda_i(\lambda_i - c)}(S^1, R).$$

Then

$$H = H^-(S^1, R) \oplus H^+(S^1, R) \oplus H^0(S^1, R).$$

We note that if $\lambda_{k+m}(\lambda_{k+m}-c) < \Lambda < \lambda_{k+m+1}(\lambda_{k+m+1}-c)$, $k \geq 1$, $m \geq 1$, then

$$H^-(S^1,R) = \bigoplus_{\Lambda_1 \leq \Lambda_i \leq \Lambda_{k+m}} H_{\lambda_i(\lambda_i-c)}(S^1,R),$$
$$H^+(S^1,R) = \bigoplus_{\Lambda_i \geq \Lambda_{k+m+1}} H_{\lambda_i(\lambda_i-c)}(S^1,R),$$

$\dim H^-(S^1,R) < \infty$, $H^0(S^1,R) = \emptyset$, and

$$H = H^-(S^1,R) \oplus H^+(S^1,R).$$

Let us set

$$\begin{aligned}
B_r &= \{u \circ x \in H(S^1,R) \mid \|u \circ x\|_H \leq r, u(x(t)) \in D = R^+ \setminus C, x(t) \in \chi \subset \Omega, \forall t \in S^1\},\\
S_r &= \partial B_r\\
&= \{u \circ x \in H(S^1,R) \mid \|u \circ x\|_H = r, u(x(t)) \in D = R^+ \setminus C,\\
&\quad \forall x(t) \in \chi \subset \Omega, \forall t \in S^1\},\\
Q &= \bar{B_R} \cap H^-(S^1,R) \oplus \{r(e \circ x) \mid e \circ x \in \partial B_1 \cap H_{\lambda_{k+m+1}(\lambda_{k+m+1}-c)}(S^1,R)\\
&\quad \subset \partial B_1 \cap H^+(S^1,R), 0 < r < R\}.
\end{aligned}$$

Let us define

$$\Gamma = \{\gamma \in C(\bar{Q}, H(S^1,R)) \mid \gamma = \mathrm{id} \text{ on } \partial Q\}.$$

Lemma 3.1 *Under the assumptions of Theorem* 1.1, *there exists a large number* $R > 0$ *such that if* $e \circ x \in \partial B_1 \cap H_{\lambda_{k+m+1}(\lambda_{k+m+1}-c)}(S^1,R) \subset \partial B_1 \cap H^+(S^1,R)$ *and* $u \circ x \in \partial Q = \partial(\bar{B_R} \cap H^-(S^1,R) \oplus \{r(e \circ x) \mid 0 < r < R\})$, *then*

$$\sup_{u \circ x \in \partial Q} J(u \circ x) < 0, \qquad \sup_{u \circ x \in Q} J(u \circ x) < \infty.$$

Proof Let us choose elements $e \circ x \in \partial B_1 \cap H_{\lambda_{k+m+1}(\lambda_{k+m+1}-c)}(S^1,R) \subset \partial B_1 \cap H^+(S^1,R)$ and $u \circ x \in H^-(S^1,R) \oplus \{r(e \circ x) \mid r > 0\}$. Then we have

$$\begin{aligned}
J(u \circ x) &= \frac{1}{2}\int_0^{2\pi} \left[\left(\Delta^2 u(x(t)) + c\Delta u(x(t))\right) \cdot u(x(t)) - \Lambda u(x(t))^2\right] x'(t)\,dt\\
&\quad - \int_0^{2\pi} \frac{1}{q}\left|u(x(t))\right|^q dx + \int_\Omega \frac{1}{p}\frac{1}{|u(x(t))|^p} x'(t)\,dt\\
&\leq \frac{1}{2}\Lambda_{k+m+1}\|u \circ x\|^2_{L^q(S^1,R)}\\
&\quad - \frac{1}{q}\|u \circ x\|^q_{L^q(S^1,R)} + \int_0^{2\pi} \frac{1}{p}\frac{1}{|u(x(t))|^p} x'(t)\,dt.
\end{aligned}$$

If $u \circ x \in \partial Q$, then since $2 < p$, there exists a constant $\bar{C}$ such that $\int_0^{2\pi} \frac{1}{p}\frac{1}{|u(x(t))|^p} x'(t)\,dt < \bar{C}$. Thus, we have

$$J(u \circ x) \leq \frac{1}{2}\Lambda_{k+m+1}\|u \circ x\|^2_{L^q(S^1,R)} - \frac{1}{q}\|u \circ x\|^q_{L^q(S^1,R)} + \bar{C}.$$

Since $2 < q$, there exists a large number $R > 0$ such that if $u \circ x \in \partial Q$, then $J(u \circ x) < 0$. Thus, we have $\sup_{u\circ x\in\partial Q} J(u \circ x) < 0$. Moreover, if $u \circ x \in Q$, then $J(u \circ x) \leq \frac{1}{2}\Lambda_{k+m+1}\|u \circ x\|^2_{L^q(S^1,R)} + \bar{C} < \infty$. □

Lemma 3.2 *Under the assumptions of Theorem* 1.1, *there exists a small number* $r > 0$ *such that*

$$\inf_{u\circ x\in\partial B_r\cap H^+(S^1,R)} J(u \circ x) > 0, \qquad \inf_{u\circ x\in B_r\cap H^+(S^1,R)} J(u \circ x) > -\infty.$$

Proof Let $u \circ x \in \partial B_r \cap H^+(S^1, R)$. Then we have

$$\begin{aligned} J(u \circ x) &= \frac{1}{2}\int_0^{2\pi} \bigl[\Delta^2 u\bigl(x(t)\bigr) + c\Delta u\bigl(x(t)\bigr) \cdot u\bigl(x(t)\bigr) - \Lambda u\bigl(x(t)\bigr)^2\bigr]x'(t)\,dt \\ &\quad - \int_0^{2\pi} \frac{1}{q}\bigl|u\bigl(x(t)\bigr)\bigr|^q x'(t)\,dt + \int_\Omega \frac{1}{p}\frac{1}{|u(x(t))|^p}x'(t)\,dt \\ &\geq \frac{1}{2}\int_0^{2\pi} \bigl[\Delta^2 u\bigl(x(t)\bigr) + c\Delta u\bigl(x(t)\bigr) \cdot u\bigl(x(t)\bigr) - \Lambda u\bigl(x(t)\bigr)^2\bigr]x'(t)\,dt \\ &\quad - \int_0^{2\pi} \frac{1}{q}\bigl|u\bigl(x(t)\bigr)\bigr|^q x'(t)\,dt \\ &\geq \frac{1}{2}\Lambda_{k+m+1}\|u \circ x\|^2_{L^q(S^1,R)} - \frac{1}{q}\bigl\|u\bigl(x(t)\bigr)\bigr\|^q_{L^q(S^1,R)}. \end{aligned}$$

Since $2 < q$, there exists a small number $r > 0$ such that if $u \circ x \in \partial B_r \cap H^+(S^1, R)$, then $J(u \circ x) > 0$. Thus, $\inf_{u\circ x\in\partial B_r\cap H^+(S^1,R)} J(u \circ x) > 0$. Moreover, if $u \circ x \in B_r \cap H^+(S^1, R)$, then $J(u \circ x) \geq -\frac{1}{q}\|u \circ x\|^q_{L^q(S^1,R)} > -\infty$. Thus, $\inf_{u\circ x\in B_r\cap H^+(S^1,R)} J(u \circ x) > -\infty$. So the lemma is proved. □

Let us define

$$c = \inf_{h\in\Gamma}\sup_{u\circ x\in Q} J\bigl(h(u \circ x)\bigr).$$

Lemma 3.3 *Under the assumptions of Theorem* 1.1, *we have*

$$0 < \inf_{u\circ x\in\partial B_r\cap H^+(S^1,R)} J(u \circ x) \leq c = \inf_{h\in\Gamma}\sup_{u\circ x\in Q} J\bigl(h(u \circ x)\bigr) \leq \sup_{u\circ x\in Q} J(u \circ x) < \infty.$$

Proof By Lemma 3.1 we have

$$\inf_{h\in\Gamma}\sup_{u\circ x\in Q} J\bigl(h(u \circ x)\bigr) \leq \sup_{u\circ x\in Q} J(u \circ x) < \infty.$$

By Lemma 3.2 we have

$$\inf_{h\in\Gamma}\sup_{u\circ x\in Q} J\bigl(h(u \circ x)\bigr) \geq \inf_{u\circ x\in\partial B_r\cap H^+(S^1,R)} J(u \circ x) > 0.$$

Thus, the lemma is proved. □

Proof of Theorem 1.1 Assume that $\lambda_k < c < \lambda_{k+1}$, $2 < q < p$, $q < \frac{2n}{n-2}$, $\lambda_{k+m}(\lambda_{k+m} - c) < \Lambda < \lambda_{k+m+1}(\lambda_{k+m+1} - c)$, $k \geq 1$, $m \geq 1$, and that conditions (A1) and (A2) hold. Note that $J(u \circ x)$ is continuous and Fréchet differentiable in H and $DJ \in C$. By Lemma 2.5, $J(u \circ x)$ satisfies $(P.S.)_c$ condition for $c \in R$. We claim that $c = \inf_{h \in \Gamma} \sup_{u \circ x \in Q} J(h(u \circ x)) > 0$ is a critical value of $J(u \circ x)$, that is, $J(u \circ x)$ has a critical point $u_0 \circ x$ such that

$$J(u_0 \circ x) = c,$$
$$DJ(u_0 \circ x) = 0.$$

In fact, by contradiction we suppose that $c > 0$ is not a critical value of $J(u \circ x)$. Then by Theorem A.4 in [6], for any $\bar{\epsilon} \in (0, c) > 0$, there exist a constant $\epsilon \in (0, \bar{\epsilon})$ and a deformation $\eta \in C([0, 1] \times H, H)$ such that:

(i) $\eta(0, u \circ x) = u \circ x$ for all $u \circ x \in H$,
(ii) $\eta(s, u \circ x) = u \circ x$ for all $s \in [0, 1]$ if $J(u \circ x) \notin [c - \bar{\epsilon}, c + \bar{\epsilon}]$,
(iii) $J(\eta(1, u \circ x)) \leq c - \epsilon$ if $J(u \circ x) \leq c + \epsilon$.

We can choose $h \in \Gamma$ such that

$$\sup_{u \circ x \in Q} J\bigl(h(u \circ x)\bigr) \leq c + \epsilon$$

and

$$J\bigl(h(u \circ x)\bigr) < c - \bar{\epsilon} \quad \text{on } \partial Q.$$

This leads to $J(h(u \circ x)) \notin [c - \bar{\epsilon}, c + \bar{\epsilon}]$. Thus, by (ii),

$$\eta\bigl(1, h(u \circ x)\bigr) = h(u \circ x) \quad \text{on } \partial Q.$$

Hence, $\eta(1, h(u \circ x)) \in \Gamma$. By (iii) and the definition of c,

$$c \leq \sup_{u \circ x \in Q} J\bigl(\eta\bigl(1, h(u \circ x)\bigr)\bigr) = \sup_{u \circ x \in Q} J\bigl(h(u \circ x)\bigr) \leq c - \epsilon,$$

which is a contradiction. Thus, c is a critical value of $J(u \circ x)$. So $J(u \circ x)$ has a critical point $u_0 \circ x$ with a critical value

$$c = J(u_0 \circ x)$$

such that

$$0 < \inf_{u \circ x \in \partial B_r \cap H^+(S^1, R)} J(u \circ x) \leq c \leq \sup_{u \circ x \in Q} J(u \circ x) < \infty.$$

By Lemma 2.3,

$$u_0\bigl(x(t)\bigr) \neq 0.$$

Thus, (1.1) has at least one nontrivial solution u_0 such that $u_0(x(t)) \neq 0$, and Theorem 1.1 is proved. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and read and approved the final manuscript.

Author details
[1]Department of Mathematics, Kunsan National University, Kunsan, 573-701, Korea. [2]Department of Mathematics Education, Inha University, Incheon, 402-751, Korea.

Acknowledgements
This work was supported by Inha University Research Grant.

Received: 12 July 2015 Accepted: 16 February 2016 Published online: 25 February 2016

References
1. Ghergu, M, Rădulescu, VD: Singular Elliptic Problems: Bifurcation and Asymptotic Analysis. Clarendon, Oxford (2008)
2. Choi, QH, Jung, T: Multiplicity results on nonlinear biharmonic operator. Rocky Mt. J. Math. **29**(1), 141-164 (1999)
3. Jung, TS, Choi, QH: Multiplicity results on a nonlinear biharmonic equation. Nonlinear Anal. TMA **30**(8), 5083-5092 (1997)
4. Tarantello, G: A note on a semilinear elliptic problem. Differ. Integral Equ. **5**(3), 561-565 (1992)
5. Micheletti, AM, Pistoia, A: Multiplicity results for a fourth-order semilinear elliptic problem. Nonlinear Anal. TMA **31**(7), 895-908 (1998)
6. Rabinowitz, PH: Minimax Methods in Critical Point Theory with Applications to Differential Equations. CBMS. Regional Conf. Ser. Math., vol. 65. Am. Math. Soc., Providence (1986)