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Oscillation criteria for nonlinear functional differential equations of neutral type

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Abstract

In this paper, some new sufficient conditions for the oscillation of all solutions of nonlinear neutral delay differential equations are established aiming at extending and/or improving some well known results in the literature. Our main results are obtained by employing the Riccati transformation aiming to transfer the neutral equation to a nonneutral type and then using some inequality techniques. Some illustrative examples are also included.

MSC: 54H25; 54C60; 54E50

Keywords: oscillations; neutral delay differential equations; nonlinear

1 Introduction

When delays appear in additional terms involving the highest order derivative of the unknown function in a differential equation, we are dealing with a neutral type differential equation. The study of the asymptotic and oscillatory behavior of solutions of neutral differential equations is of importance in applications. This is due to the fact that such equations appear in various phenomena including networks containing lossless transmission lines (as in high-speed computers where such lines are used to interconnect switching circuits), in the study of vibrating masses attached to an elastic bar, as the Euler equations for the minimization of functionals involving a time delay in some variational problems and in the theory of automatic control (see Hale [1], Driver [2] and Boe and Chang [3] and references cited therein). The construction of these models using delays is complemented by the mathematical investigation of nonlinear equations.

In this paper, we mainly consider the nonlinear neutral delay differential equations of the form

$$\left[r(t)\big(a(t)x(t) + p(t)x(t-\tau)\big)\right]' + q(t)f\big(x(t-\sigma)\big) = 0, \tag{1.1}$$

where

$$r(t), a(t), p(t), q(t) \in C\big[[t_0, \infty), \mathbb{R}^+\big], \quad \tau, \sigma \in \mathbb{R}^+, \tag{1.2}$$

and f satisfies

$$f \in C[\mathbb{R}, \mathbb{R}], \quad uf(u) > 0 \quad \text{for } u \neq 0, \tag{1.3}$$

and there exists a positive constant k_0 such that

$$\frac{f(u)}{u} \geq k_0 > 0. \tag{1.4}$$

Many authors have considered linear neutral delay differential equations and established sufficient conditions for oscillation of all solutions. We refer to Ahmed *et al.* [4], Karpuz and Ocalan [5], Gopalsamy *et al.* [6] and Zhou [7].

In 2002, Saker and Kubiaczyk [8] considered the nonlinear neutral equation

$$\big(a(t)x(t) + p(t)x(t-\tau)\big)' + q(t)f\big(x(t-\sigma)\big) = 0, \tag{1.5}$$

where the function f satisfies condition (1.3) with $\lim_{u \to 0} \frac{f(u)}{u} = k_1$ exists and there exists $t^* \geq t_0$ such that

$$\frac{p(t^* + n^*\tau)}{a(t^* + (n^* - 1)\tau)} \leq 1 \quad \text{for } n^* = 0, 1, 2, \dots. \tag{1.6}$$

In 2003, Kubiaczyk *et al.* [9] studied the nonlinear equations of the form

$$\big(x(t) + p(t)x(t-\tau)\big)' + q(t) \prod_{i=1}^{n} \big|x(t-\sigma_i)\big|^{\alpha_i} \operatorname{sign} x(t-\sigma_i) = 0, \tag{1.7}$$

where $\alpha_i > 0$ and $\sum_{i=1}^{n} \alpha_i = 1$, and they have given some sharp sufficient conditions for the oscillation of all solutions of (1.7).

In 2004, Graef *et al.* [10] considered (1.1) when $a(t) \equiv 1$, $r(t) \equiv 1$ and developed some sufficient conditions for the oscillation of all solutions. For further results on the oscillation of various classes of neutral differential equations one can see [11–17].

A primary purpose of this paper is to establish new integral conditions that guarantee the oscillation of all solutions of (1.1) when $p(t)$ is constant and equal p_0. In some sense, the obtained results here extend and generalize several of well known results in the literature.

Let $m = \max\{\tau, \sigma\}$. By a solution of (1.1) we mean a function $x \in C[[t_1 - m, \infty), \mathbb{R}]$ for some $t_1 \geq t_0$ such that $a(t)x(t) + p(t)x(t-\tau)$ is continuously differentiable for $t \geq t_1$ and such that (1.1) is satisfied for $t \geq t_1$.

Let $t_1 \geq t_0$ be a given initial point and let $\Phi \in C[[t_1 - m, t_1], \mathbb{R}]$ be a given initial function. Then one can show by using the method of steps that (1.1) has a unique solution on $[t_1, \infty)$ satisfying the initial function

$$x(t) = \Phi(t), \quad \text{for } t_1 - m \leq t \leq t_1.$$

As is customary, a solution of (1.1) is said to be oscillatory if it has arbitrarily large zeros. Otherwise the solution is said to be nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

In the sequel, unless otherwise specified, when we write a functional inequality we shall assume that it holds for all sufficiently large values of t.

2 Auxiliary lemmas

We give here some useful lemmas which will play an important role in the study of the oscillation of (1.1).

Lemma 2.1 [18] *Assume that $P_i(t)$, $\tau_i(t) \in C[[t_0, \infty), [0, \infty)]$, $i = 1, 2, \ldots$.*
Then the differential inequality

$$x'(t) + \sum_{i=1}^{n} P_i(t)x\big(t - \tau_i(t)\big) \le 0 \tag{2.1}$$

has an eventually positive solution if and only if the equation

$$x'(t) + \sum_{i=1}^{n} P_i(t)x\big(t - \tau_i(t)\big) = 0 \tag{2.2}$$

has an eventually positive solution.

Lemma 2.2 [19] *Assume that*

$$\limsup_{t\to\infty} \int_{t}^{t+\tau_i} p_i(s)\, ds > 0, \quad \text{for some } i. \tag{2.3}$$

If $x(t)$ is an eventually positive solution of the delay differential equation

$$x'(t) + \sum_{i=1}^{n} p_i(t)x(t - \tau_i) = 0, \tag{2.4}$$

then, for the same i,

$$\liminf_{t\to\infty} \frac{x(t - \tau_i)}{x(t)} < \infty.$$

Lemma 2.3 *Assume that*

$$a(t) \le 1, \qquad \sigma > \tau, \quad \text{and} \quad p_0 > 1, \tag{2.5}$$

$$\limsup_{t\to\infty} \int_{t}^{t+\sigma-\tau} q(s)\, ds > 0. \tag{2.6}$$

Let $x(t)$ be an eventually positive solution of the equation

$$\big(a(t)x(t) + p_0 x(t - \tau)\big)' + q(t)f\big(x(t - \sigma)\big) = 0. \tag{2.7}$$

Set

$$z(t) = a(t)x(t) + p_0 x(t - \tau), \tag{2.8}$$

then

$$\liminf_{t\to\infty} \frac{z(t - \sigma + \tau)}{z(t)} < \infty. \tag{2.9}$$

Proof In view of (2.7) and our hypothesis, we see that $z(t) > 0$ eventually. From (2.8) we have

$$z(t + \tau) = a(t + \tau)x(t + \tau) + p_0 x(t) \tag{2.10}$$

and

$$p_0 x(t - \tau) = z(t) - a(t)x(t) \geq z(t) - x(t). \tag{2.11}$$

Since $z'(t) < 0$, we have $z(t) > z(t + \tau)$, which implies with (2.10) that

$$z(t) \geq p_0 x(t) \quad \Rightarrow \quad x(t) \leq \frac{z(t)}{p_0}.$$

Substituting in (2.11), we obtain

$$p_0 x(t - \tau) \geq z(t) - \frac{z(t)}{p_0},$$

or

$$p_0^2 x(t - \tau) \geq p_0 z(t) - z(t).$$

Hence

$$x(t - \sigma) \geq \frac{p_0 - 1}{p_0^2} z(t + \tau - \sigma). \tag{2.12}$$

From (2.7) and (2.12), we get

$$z'(t) + \frac{k_0(p_0 - 1)}{p_0^2} q(t) z(t + \tau - \sigma) \leq 0. \tag{2.13}$$

By Lemma 2.1, we find that

$$z'(t) + \frac{k_0(p_0 - 1)}{p_0^2} q(t) z(t + \tau - \sigma) = 0 \tag{2.14}$$

has an eventually positive solution as well.

As a result, by Lemma 2.2 and (2.6), we have

$$\liminf_{t \to \infty} \frac{z(t - \sigma + \tau)}{z(t)} < \infty,$$

which is the desired result. The proof is complete. □

Lemma 2.4 *Assume that conditions* (2.5) *hold. If* (2.7) *has an eventually positive solution, then*

$$\int_t^{t+\sigma-\tau} q(s)\, ds \leq \frac{p_0^2}{k_0(p_0 - 1)} \tag{2.15}$$

for all sufficiently large t.

Proof Proceeding as in the proof of Lemma 2.3, we again obtain the inequality (2.13). Integrating (2.13) from t to $t + \sigma - \tau$, it follows that

$$z(t - \tau + \sigma) - z(t) + \frac{k_0(p_0 - 1)}{p_0^2} \int_t^{t+\sigma-\tau} q(s)z(s - \sigma + \tau) \, ds \leq 0.$$

Using Bonnet's theorem, it follows that

$$z(t - \tau + \sigma) - z(t) + \frac{k_0(p_0 - 1)}{p_0^2} z(t) \int_t^{t+\sigma-\tau} q(s) \, ds \leq 0.$$

Then

$$z(t - \tau + \sigma) + \left(\frac{k_0(p_0 - 1)}{p_0^2} \int_t^{t+\sigma-\tau} q(s) \, ds - 1 \right) z(t) \leq 0. \tag{2.16}$$

Since $z(t) > 0$, (2.16) implies

$$\frac{k_0(p_0 - 1)}{p_0^2} \int_t^{t+\sigma-\tau} q(s) \, ds - 1 \leq 0.$$

Hence, for all sufficiently large t, we have

$$\int_t^{t+\sigma-\tau} q(s) \, ds \leq \frac{p_0^2}{k_0(p_0 - 1)},$$

which is the desired result. The proof is complete. □

Remark 2.5 Lemma 2.3 and Lemma 2.4 extend results of Graef *et al.* [10].

3 Oscillation of solutions

Theorem 3.1 *Assume that conditions* (2.5) *and* (2.6) *hold. If*

$$\int_{t_0}^{\infty} q(t) \ln \left(\frac{e k_0 (p_0 - 1)}{p_0^2} \int_t^{t+\sigma-\tau} q(s) \, ds \right) dt = \infty. \tag{3.1}$$

Then every solution of (2.7) *oscillates.*

Proof For the sake of obtaining a contradiction, assume that there is an eventually positive solution $x(t)$ of (2.7). Set $z(t)$ as in (2.8). Then $z(t)$ is eventually positive and decreasing and satisfies the inequality (2.13). That is,

$$z'(t) + \frac{k_0(p_0 - 1)}{p_0^2} q(t)z(t + \tau - \sigma) \leq 0.$$

Let

$$\psi(t) = -\frac{z'(t)}{z(t)}.$$

Then $\psi(t)$ is continuous and nonnegative. So, there exists $t_1 \geq t_0$ with $z(t_1) > 0$ such that

$$z(t) = z(t_1) \exp\left(-\int_{t_1}^{t} \psi(s)\,ds\right).$$

Moreover, $\psi(t)$ satisfies

$$\psi(t) \geq \frac{k_0(p_0 - 1)}{p_0^2} q(t) \exp\left(\int_{t+\tau-\sigma}^{t} \psi(s)\,ds\right). \tag{3.2}$$

By using the inequality (*cf.* Erbe *et al.* [20], p.32)

$$e^{rx} \geq x + \frac{\ln(er)}{r}, \quad x, r > 0,$$

we have from (3.2)

$$\psi(t) \geq \frac{k_0(p_0 - 1)}{p_0^2} q(t) \exp\left(\frac{A(t)}{A(t)} \int_{t+\tau-\sigma}^{t} \psi(s)\,ds\right)$$

$$\geq \frac{k_0(p_0 - 1)}{p_0^2} q(t) \exp\left(\frac{1}{A(t)} \int_{t+\tau-\sigma}^{t} \psi(s)\,ds + \frac{\ln(eA(t))}{A(t)}\right),$$

where

$$A(t) = \frac{k_0(p_0 - 1)}{p_0^2} \int_{t}^{t+\sigma-\tau} q(s)\,ds.$$

Therefore,

$$\psi(t) \int_{t}^{t+\sigma-\tau} q(s)\,ds - q(t) \int_{t+\tau-\sigma}^{t} \psi(s)\,ds \geq q(t) \ln\left(\frac{ek_0(p_0 - 1)}{p_0^2} \int_{t}^{t+\sigma-\tau} q(s)\,ds\right).$$

Hence, for $\xi > T + \sigma - \tau$,

$$\int_{T}^{\xi} \psi(t)\left(\int_{t}^{t+\sigma-\tau} q(s)\,ds\right)dt - \int_{T}^{\xi} q(t)\left(\int_{t+\tau-\sigma}^{t} \psi(s)\,ds\right)dt$$

$$\geq \int_{T}^{\xi} q(t) \ln\left(\frac{ek_0(p_0 - 1)}{p_0^2} \int_{t}^{t+\sigma-\tau} q(s)\,ds\right)dt. \tag{3.3}$$

By interchanging the order of integration, we have

$$\int_{T}^{\xi} q(t)\left(\int_{t+\tau-\sigma}^{t} \psi(s)\,ds\right)dt \geq \int_{T}^{\xi+\tau-\sigma} \psi(t)\left(\int_{t}^{t+\sigma-\tau} q(s)\,ds\right)dt. \tag{3.4}$$

Combining (3.3) and (3.4), leads to

$$\int_{\xi+\tau-\sigma}^{\xi} \psi(t)\left(\int_{t}^{t+\tau-\sigma} q(s)\,ds\right)dt \geq \int_{T}^{\xi} q(t) \ln\left(\frac{ek_0(p_0 - 1)}{p_0^2} \int_{t}^{t+\sigma-\tau} q(s)\,ds\right)dt. \tag{3.5}$$

Using (2.15) of Lemma 2.4 in (3.5), we obtain

$$\int_{\xi+\tau-\sigma}^{\xi} \psi(t)\,dt \geq \frac{k_0(p_0 - 1)}{p_0^2} \int_{T}^{\xi} q(t) \ln\left(\frac{ek_0(p_0 - 1)}{p_0^2} \int_{t}^{t+\sigma-\tau} q(s)\,ds\right)dt,$$

i.e.,

$$\ln \frac{z(\xi + \tau - \sigma)}{z(\xi)} \geq \frac{k_0(p_0 - 1)}{p_0^2} \int_T^\xi q(t) \ln\left(\frac{ek_0(p_0 - 1)}{p_0^2} \int_t^{t+\sigma-\tau} q(s)\, ds \right) dt.$$

This result along with condition (3.1) leads to

$$\lim_{t\to\infty} \frac{z(t + \tau - \sigma)}{z(t)} = \infty,$$

which contradicts (2.9) and completes the proof. □

Example 3.2 Consider the equation

$$\left(\frac{1}{t} x(t) + 3x(t-1) \right)' + \frac{9}{2e}\left(1 + \frac{1}{t} \right) x(t-2)\left(1 + x^2(t-2)\right) = 0, \quad t \geq e. \tag{3.6}$$

Here we have

$$a(t) = \frac{1}{t}, \qquad q(t) = \frac{9}{2e}\left(1 + \frac{1}{t} \right), \qquad p_0 = 3, \qquad \tau = 1, \quad \text{and} \quad \sigma = 2,$$

$$f\left(x(t-\sigma)\right) = x(t-2)\left(1 + x^2(t-2)\right).$$

Then we have

$$\frac{f(u)}{u} = \frac{x(t-2)(1 + x^2(t-2))}{x(t-2)} = 1 + x^2(t-2)) \geq 1.$$

Let $k_0 = 1$,

$$\int_{t_0}^\infty q(t) \ln\left(\frac{ek_0(p_0 - 1)}{p_0^2} \int_t^{t+\sigma-\tau} q(s)\, ds \right) dt$$

$$= \int_e^\infty \frac{9}{2e}\left(1 + \frac{1}{t} \right) \ln\left(\int_{t+1}^{t+2} \left(1 + \frac{1}{s} \right) ds \right) dt$$

$$= \frac{9}{2e} \int_e^\infty \left(1 + \frac{1}{t} \right)\left(1 + \ln\left(1 + \frac{1}{t} \right) \right)$$

$$\geq \frac{9}{2e} \int_e^\infty \ln\left(1 + \ln\left(1 + \frac{1}{t} \right) \right) = \infty.$$

Hence, according to Theorem 3.1, all solutions of (3.6) oscillate.

Remark 3.3 Theorem 3.1 extends results of Graef *et al.* ([10], Theorem 2.1 and Theorem 2.2), where $a(t) \equiv 1$.

Theorem 3.4 *Assume that conditions* (2.5) *hold. If*

$$0 < c \leq \liminf_{t\to\infty} \int_t^{t-\tau+\sigma} \frac{q(s)}{r(s + \tau - \sigma)}\, ds \tag{3.7}$$

and

$$\int_{t_0}^{\infty} \frac{q(t)}{r(t + \tau - \sigma)} \ln\left(\frac{ek_0(p_0 - 1)}{p_0^2} \int_t^{t+\sigma-\tau} \frac{q(s)}{r(s + \tau - \sigma)} \, ds \right) dt = \infty. \tag{3.8}$$

Then every solution of (1.1) *oscillates.*

Proof For the sake of obtaining a contradiction, assume that there is an eventually positive solution $x(t)$ of (1.1). Set $z(t)$ as in (2.8). Then $z(t)$ is eventually positive and decreasing. From (2.12) and (1.1), we have

$$\left(r(t)z(t) \right)' + \frac{k_0(p_0 - 1)}{p_0^2} q(t)z(+\tau - \sigma) \le 0. \tag{3.9}$$

Dividing (3.9) by $r(t) > 0$, we obtain

$$z'(t) + \frac{r'(t)}{r(t)} z(t) + \frac{k_0(p_0 - 1)}{p_0^2} \frac{q(t)}{r(t)} z(+\tau - \sigma) \le 0. \tag{3.10}$$

Let

$$z(t) = \exp\left(- \int_{t_0}^t \frac{r'(s)}{r(s)} \, ds \right) y(t), \tag{3.11}$$

which implies that $y(t) > 0$. Substituting in (3.10) yields for all $t \ge t_0$,

$$y'(t) + \frac{k_0(p_0 - 1)}{p_0^2} \frac{q(t)}{r(t + \tau - \sigma)} y(+\tau - \sigma) \le 0. \tag{3.12}$$

Set

$$\lambda(t) = -\frac{y'(t)}{y(t)}.$$

Then $\lambda(t)$ is continuous and positive. So, there exists $t_1 \ge t_0$ with $y(t_1) > 0$ such that

$$y(t) = y(t_1) \exp\left(\int_{t_1}^t \lambda(s) \, ds \right).$$

Moreover, $\lambda(t)$ satisfies

$$\lambda(t) \ge \frac{k_0(p_0 - 1)}{p_0^2} \frac{q(t)}{r(t + \tau - \sigma)} \exp\left(\int_{t+\tau-\sigma}^t \lambda(s) \, ds \right). \tag{3.13}$$

Applying the inequality

$$e^{rx} \ge x + \frac{\ln(er)}{r}, \quad x, r > 0$$

to (3.13) yields that

$$\lambda(t) \ge \left(\frac{k_0(p_0 - 1)}{p_0^2} \right) \left(\frac{q(t)}{r(t + \tau - \sigma)} \right) \exp\left(\frac{B(t)}{B(t)} \int_{t+\tau-\sigma}^t \lambda(s) \, ds \right)$$

$$\ge \left(\frac{k_0(p_0 - 1)}{p_0^2} \right) \left(\frac{q(t)}{r(t + \tau - \sigma)} \right) \left(\frac{1}{B(t)} \int_{t+\tau-\sigma}^t \lambda(s) \, ds + \frac{\ln(eB(t))}{B(t)} \right),$$

where

$$B(t) = \frac{k_0(p_0 - 1)}{p_0^2} \int_t^{t-\tau+\sigma} \frac{q(s)}{r(s+\tau-\sigma)} \, ds.$$

Therefore,

$$\lambda(t) \int_t^{t-\tau+\sigma} \frac{q(s)}{r(s+\tau-\sigma)} \, ds - \frac{q(t)}{r(t+\tau-\sigma)} \int_{t+\tau-\sigma}^t \lambda(s) \, ds$$
$$\geq \frac{q(t)}{r(t+\tau-\sigma)} \ln \left(\frac{ek_0(p_0-1)}{p_0^2} \int_t^{t-\tau+\sigma} \frac{q(s)}{r(s+\tau-\sigma)} \, ds \right).$$

Hence, for $\eta > T_1 + \sigma - \tau$

$$\int_{T_1}^\eta \lambda(t) \left(\int_t^{t-\tau+\sigma} \frac{q(s)}{r(s+\tau-\sigma)} \, ds \right) dt - \int_{T_1}^\eta \frac{q(t)}{r(t+\tau-\sigma)} \left(\int_{t+\tau-\sigma}^t \lambda(s) \, ds \right) dt$$
$$\geq \int_{T_1}^\eta \frac{q(t)}{r(t+\tau-\sigma)} \ln \left(\frac{ek_0(p_0-1)}{p_0^2} \int_t^{t-\tau+\sigma} \frac{q(s)}{r(s+\tau-\sigma)} \, ds \right) dt. \tag{3.14}$$

By interchanging the order of integration, we have

$$\int_{T_1}^\eta \frac{q(t)}{r(t+\tau-\sigma)} \left(\int_{t+\tau-\sigma}^t \lambda(s) \, ds \right) dt \geq \int_{T_1}^{\eta+\tau-\sigma} \lambda(t) \left(\int_t^{t-\tau+\sigma} \frac{q(s)}{r(s+\tau-\sigma)} \, ds \right) dt. \tag{3.15}$$

From (3.14) and (3.15), we obtain

$$\int_{\eta+\tau-\sigma}^\eta \lambda(t) \left(\int_t^{t-\tau+\sigma} \frac{q(s)}{r(s+\tau-\sigma)} \, ds \right) dt$$
$$\geq \int_{T_1}^\eta \frac{q(t)}{r(t+\tau-\sigma)} \ln \left(\frac{ek_0(p_0-1)}{p_0^2} \int_t^{t-\tau+\sigma} \frac{q(s)}{r(s+\tau-\sigma)} \, ds \right) dt. \tag{3.16}$$

Employing (2.15) in (3.16), it follows that

$$\int_{\eta+\tau-\sigma}^\eta \lambda(t) \, dt \geq \frac{k_0(p_0-1)}{p_0^2} \int_{T_1}^\eta \frac{q(t)}{r(t+\tau-\sigma)} \ln \left(\frac{ek_0(p_0-1)}{p_0^2} \int_t^{t-\tau+\sigma} \frac{q(s)}{r(s+\tau-\sigma)} \, ds \right) dt$$

or

$$\ln \frac{y(\eta+\tau-\sigma)}{y(\eta)}$$
$$\geq \frac{k_0(p_0-1)}{p_0^2} \int_{T_1}^\eta \frac{q(t)}{r(t+\tau-\sigma)} \ln \left(\frac{ek_0(p_0-1)}{p_0^2} \int_t^{t-\tau+\sigma} \frac{q(s)}{r(s+\tau-\sigma)} \, ds \right) dt. \tag{3.17}$$

From (3.17) and (3.9), we have

$$\lim_{t \to \infty} \frac{y(t+\tau-\sigma)}{y(t)} = \infty. \tag{3.18}$$

On the other hand, from condition (3.8), there exists a sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$, and there exists $\mu_n \in (t_n - \sigma, t)$ for every n such that

$$\int_{t_n + \tau - \sigma}^{\mu_n} \frac{q(s)}{r(s + \tau - \sigma)} \, ds \geq \frac{c}{2} \quad \text{and} \quad \int_{\mu_n}^{t_n} \frac{q(s)}{r(s + \tau - \sigma)} \, ds \geq \frac{c}{2}. \tag{3.19}$$

Integrating both sides of (3.12) over the interval $[t_n, \mu_n]$ and $[\mu_n, t_n + \sigma - \tau]$, we have

$$y(\mu_n) - y(t_n) + \frac{k_0(p_0 - 1)}{p_0^2} \int_{t_n}^{\mu_n} \frac{q(s)}{r(s + \tau - \sigma)} y(s + \tau - \sigma) \, ds = 0 \tag{3.20}$$

and

$$y(t_n + \sigma - \tau) - y(\mu_n) + \frac{k_0(p_0 - 1)}{p_0^2} \int_{\mu_n}^{t_n - \tau + \sigma} \frac{q(s)}{r(s + \tau - \sigma)} y(s + \tau - \sigma) \, ds = 0. \tag{3.21}$$

From (3.19), (3.20), and (3.21), we have

$$-y(t_n) + \frac{c}{2} y(\mu_n + \sigma - \tau) \leq 0$$

and

$$-y(\mu_n) + \frac{c}{2} y(t_n) \leq 0.$$

This implies eventually

$$\frac{y(\mu_n - \sigma + \tau)}{y(\mu_n)} \leq \left(\frac{2}{c}\right)^2,$$

which is a contradiction with (3.18). The proof is complete. □

Example 3.5 Consider the equation

$$\left[e^t \big(x(t) + 2x(t - 1)\big)\right]' + \frac{t + 1}{t} e^{t-1} x(t - 2)\big(1 + x^4(t - 2)\big) = 0, \quad t \geq 2. \tag{3.22}$$

Here we have,

$$r(t) = e^t, \qquad a(t) = 1, \qquad q(t) = \frac{t + 1}{t} e^{t-1}, \qquad p_0 = 2, \qquad \tau = 1 \quad \text{and} \quad \sigma = 2,$$

$$f\big(x(t - \sigma)\big) = x(t - 2)\big(1 + x^4(t - 2)\big).$$

Then we have

$$\frac{f(u)}{u} = 1 + x^4(t - 2)) \geq 1.$$

Let $k_0 = 1$; we have

$$\liminf_{t \to \infty} \int_t^{t - \tau + \sigma} \frac{q(s)}{r(s + \tau - \sigma)} \, ds = \liminf_{t \to \infty} \int_t^{t+1} \left[\frac{e^{s-1}\left(\frac{s+1}{s}\right)}{e^{s-1}}\right] ds$$

$$= \liminf_{t \to \infty} \int_t^{t+1} \left(1 + \frac{1}{s}\right) ds \geq 1 > 0$$

and

$$\int_{t_0}^{\infty} \frac{q(t)}{r(t+\tau-\sigma)} \ln\left(\frac{ek_0(p_0-1)}{p_0^2} \int_t^{t+\sigma-\tau} \frac{q(s)}{r(s+\tau-\sigma)}\, ds \right) dt$$

$$= \int_2^{\infty} \left(\frac{t+1}{t} \ln\left(\frac{e}{4} \int_t^{t+1} \frac{s+1}{s}\, ds \right) \right) dt \geq \frac{e}{4} \int_2^{\infty} \ln\left(\ln\left(1+\frac{1}{t} \right) \right) dt = \infty.$$

Hence, according to Theorem 3.1, all solutions of (3.22) oscillate.

Remark 3.6 Theorem 3.4 extends results of Graef *et al.* [10], Ahmed *et al.* [4], Saker and Elabbasy [21] and Saker and Kubiaczyk [8].

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