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A result on three solutions theorem and its application to p-Laplacian systems with singular weights

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Abstract
In this paper, we consider p-Laplacian systems with singular weights. Exploiting Amann type three solutions theorem for a singular system, we prove the existence, nonexistence, and multiplicity of positive solutions when nonlinear terms have a combined sublinear effect at ∞.

MSC: 35J55; 34B18

Keywords: p-Laplacian system; singular weight; upper solution; lower solution; three solutions theorem

1 Introduction

In this paper, we study one-dimensional p-Laplacian system with singular weights of the form

$$\begin{cases} \varphi_p\big(u'(t)\big)' + \lambda h_1(t) f\big(v(t)\big) = 0, & t \in (0,1), \\ \varphi_p\big(v'(t)\big)' + \lambda h_2(t) g\big(u(t)\big) = 0, & t \in (0,1), \\ u(0) = 0, \quad v(0) = 0, \quad u(1) = 0, \quad v(1) = 0, \end{cases} \qquad (P_\lambda)$$

where $\varphi_p(u) = |u|^{p-2}u$, λ is a nonnegative parameter, h_i, $i = 1,2$ is a nonnegative measurable function on $(0,1)$, $h_i \not\equiv 0$ on any open subinterval in $(0,1)$ and $f, g \in C(\mathbb{R}_+, \mathbb{R}_+)$ with $\mathbb{R}_+ = [0,\infty)$. In particular, h_i may be singular at the boundary or may not be in $L^1(0,1)$. It is easy to see that if $h_i \in L^1(0,1)$, then all solutions of (P_λ) are in $C^1[0,1]$. On the other hand, if $h_i \notin L^1(0,1)$, then this regularity of solutions is not true in general; for example, even for scalar case, if we take $h(t) = (p-1)t^{-1}|1+\ln t|^{p-2}$, $p > 2$ and $\lambda = 1$, $f \equiv 1$, then $h \notin L^1(0,1)$, and the solution u for corresponding scalar problem of (P_λ) is given by $u(t) = -t \ln t$ which is not in $C^1[0,1]$.

For more precise description, let us introduce the following two classes of weights;

$$\mathcal{A} \equiv \left\{ h \in L^1_{loc}(0,1) : \int_0^{\frac{1}{2}} \varphi_p^{-1}\left(\int_s^{\frac{1}{2}} h(\tau)\,d\tau \right) ds + \int_{\frac{1}{2}}^1 \varphi_p^{-1}\left(\int_{\frac{1}{2}}^s h(\tau)\,d\tau \right) ds < \infty \right\},$$

$$\mathcal{B} \equiv \left\{ h \in L^1_{loc}(0,1) : \int_0^1 s^{p-1}(1-s)^{p-1} h(s)\,ds < \infty \right\}.$$

We note that h given in the above example satisfies $h \in \mathcal{A} \cap \mathcal{B}$ but $h \notin L^1(0,1)$. The main interest of this paper is to establish Amann type three solutions theorem [4] when $h_i \in \mathcal{A} \cap \mathcal{B}$ with possibility of $h \notin L^1(0,1)$. The theorem generally describes that two pairs of lower and upper solutions with an ordering condition imply the existence of three solutions. Recently, Ben Naoum and De Coster [6] have proved the theorem for scalar one-dimensional p-Laplacian problems with L^1-Caratheodory condition which corresponds to case $h \in L^1(0,1)$; Henderson and Thompson [18] as well as Lü, O'Regan, and Agarwal [23] - for scalar second order ODEs and one-dimensional p-Laplacian with the derivative-dependent nonlinearity respectively; and De Coster and Nicaise [11] - for semilinear elliptic problems in nonsmooth domains. For noncooperative elliptic systems ($p = 2$) with $k_i \equiv 1$ and Ω bounded, one may refer to Ali, Shivaji, and Ramaswamy [3]. Specially, for subsuper solutions which are not completely ordered, this type of three solutions result was studied in [26].

The three solutions theorem for our system (P_λ) or even for corresponding scalar p-Laplacian problems is not obviously extended from previous works mainly by the possibility $h \notin L^1(0,1)$. Caused by the delicacy of Leray-Schauder degree computation, the crucial step for the proof is to guarantee C^1 regularity of solutions, but with condition $h \in \mathcal{A} \cap \mathcal{B}$, C^1 regularity is not known yet. Due to the singularity of weights on the boundary, the C^1 regularity heavily depends on the shape of nonlinear terms f and g. Therefore, the first step is to investigate certain conditions on f and g to guarantee C^1 regularity of solutions. Another difficulty is to show that a corresponding integral operator is bounded on the set of functions between upper and lower solutions in $C_0^1[0,1]$. To overcome this difficulty, we give some restrictions on upper and lower solutions such that their boundary values are zero. As far as the authors know, our three solutions theorem (Theorem 1.1 in Section 2) is new and first for singular p-Laplacian systems with weights of $\mathcal{A} \cap \mathcal{B}$ class.

To cover a larger class of differential system, we consider the systems of the form

$$\begin{cases} \varphi_p\big(u'(t)\big)' + F\big(t, v(t)\big) = 0, & t \in (0,1), \\ \varphi_p\big(v'(t)\big)' + G\big(t, u(t)\big) = 0, & t \in (0,1), \\ u(0) = 0, \quad v(0) = 0, \quad u(1) = 0, \quad v(1) = 0, \end{cases} \tag{P}$$

where $F, G : (0,1) \times \mathbb{R} \to \mathbb{R}$ are continuous. We give more conditions on F and G as follows:

(F_1) For each $t \in (0,1)$, $F(t,u)$ and $G(t,u)$ are nondecreasing in u.

(H) There exist $h_1, h_2 \in \mathcal{A} \cap \mathcal{B}$ and $f, g \in C(\mathbb{R}, \mathbb{R}^+)$ such that

$$0 \leq \lim_{s \to 0} \frac{f(s)}{\varphi_p(|s|)} < \infty, \qquad 0 \leq \lim_{s \to 0} \frac{g(s)}{\varphi_p(|s|)} < \infty$$

and

$$\big|F(t,u)\big| \leq h_1(t) f(u), \qquad \big|G(t,u)\big| \leq h_2(t) g(u),$$

for all $t \in (0,1)$ and $u \in \mathbb{R}$.

(F_2) $F(t,u)u > 0$ and $G(t,u)u > 0$, for all $(t,u) \in (0,1) \times \mathbb{R}$.

We now state our first main result related to three solutions theorem as follows. See for more details in Section 2.

Theorem 1.1 *Assume (H), (F_1) and (F_2). Let $(\alpha_1,\bar{\alpha}_1)$, $(\beta_2,\bar{\beta}_2)$ be a lower solution and an upper solution and $(\alpha_2,\bar{\alpha}_2)$, $(\beta_1,\bar{\beta}_1)$ be a strict lower solution and a strict upper solution of problem (P) respectively. Also, assume that all of them are contained in $C_0^1[0,1] \times C_0^1[0,1]$ and satisfy $(\alpha_1,\bar{\alpha}_1) \le (\beta_1,\bar{\beta}_1) \le (\beta_2,\bar{\beta}_2)$, $(\alpha_1,\bar{\alpha}_1) \le (\alpha_2,\bar{\alpha}_2) \le (\beta_2,\bar{\beta}_2)$, $(\alpha_2,\bar{\alpha}_2) \nleq (\beta_1,\bar{\beta}_1)$. Then problem (P) has at least three solutions (u_1,v_1), (u_2,v_2) and (u_3,v_3) such that $(\alpha_1,\bar{\alpha}_1) \le (u_1,v_1) \prec (\beta_1,\bar{\beta}_1)$, $(\alpha_2,\bar{\alpha}_2) \prec (u_2,v_2) \le (\beta_2,\bar{\beta}_2)$, $(\alpha_1,\bar{\alpha}_1) \le (u_3,v_3) \le (\beta_2,\bar{\beta}_2)$ and $(u_3,v_3) \nleq (\beta_1,\bar{\beta}_1)$, $(u_3,v_3) \ngeq (\alpha_2,\bar{\alpha}_2)$.*

As an application of Theorem 1.1, we study the existence, nonexistence, and multiplicity of positive radial solutions for the following quasilinear system on an exterior domain:

$$\begin{cases} -\Delta_p z = \lambda k_1(|x|) f(w) & \text{in } \Omega, \\ -\Delta_p w = \lambda k_2(|x|) g(z) & \text{in } \Omega, \\ z(x) = 0, \quad w(x) = 0 & \text{if } |x| = r_0, \\ z(x) \to 0, \quad w(x) \to 0 & \text{if } |x| \to \infty, \end{cases} \tag{P_E}$$

where $\Omega = \{x \in \mathbb{R}^N : |x| > r_0\}$, $r_0 > 0$, $1 < p < N$, $\Delta_p z = \operatorname{div}(|\nabla z|^{p-2}\nabla z)$, $k_i \in C([r_0,\infty), (0,\infty))$, $i = 1,2$ and $f,g \in C(\mathbb{R}_+,\mathbb{R}_+)$ with $\mathbb{R}_+ = [0,\infty)$.

In recent years, the existence of positive solutions for such systems has been widely studied, for example, in [1] and [27] for second order ODE systems, in [3, 7, 9, 10, 13, 14, 16] and [8] for semilinear elliptic systems on a bounded domain and in [5, 15, 17] and [2] for p-Laplacian systems on a bounded domain.

For a precise description, let us give the list of assumptions that we consider.

(k) $k_i \in \mathcal{K}_{\mathcal{A}} \cap \mathcal{K}_{\mathcal{B}}$, where

$$\mathcal{K}_{\mathcal{A}} = \left\{ k \in C\big([r_0,\infty),(0,\infty)\big) : \int_{r_0}^{\infty} \varphi_p^{-1}\left(\tau^{1-N} \int_{r_0}^{\tau} r^{N-1} k(r)\,dr\right) d\tau < \infty \right\},$$
$$\mathcal{K}_{\mathcal{B}} = \left\{ k \in C\big([r_0,\infty),(0,\infty)\big) : \int_{r_0}^{\infty} r^{p-1} k(r)\,dr < \infty \right\},$$

(f_1) $f_0 = \lim_{s\to 0^+} \frac{f(s)}{s^{p-1}} = 0$ and $g_0 = \lim_{s\to 0^+} \frac{g(s)}{s^{p-1}} = 0$,
(f_2) $\lim_{s\to\infty} \frac{f(\rho(g(s))^{\frac{1}{p-1}})}{s^{p-1}} = 0$ for all $\rho > 0$,
(f_3) f and g are nondecreasing.

Condition (f_2) is sometimes called a combined sublinear effect at ∞ and simple examples satisfying (f_1) $\backsim$ (f_3) can be given as follows:

$$f(w) = \begin{cases} w^r, & w \le 1, \\ w^q, & w \ge 1, \end{cases} \qquad g(z) = \begin{cases} z^\gamma, & z \le 1, \\ z^\delta, & z \ge 1, \end{cases}$$

where $r, \gamma > p-1$ and $q\delta < (p-1)^2$, and also

$$\begin{cases} f(z) = \arctan(z^r), \\ g(w) = w^q, \end{cases}$$

where $r, q > p-1$.

Among the reference works mentioned above, Hai and Shivaji [17] and Ali and Shivaji [2] (with more general nonlinearities) considered problem (P_E) with case $k_i \equiv 1$ and Ω bounded. For C^1 monotone functions f and g with $\lim_{s\to\infty} f(s) = \infty = \lim_{s\to\infty} g(s)$ and satisfying condition (f_2), they proved that there exists $\lambda^* > 0$ such that the problem has at least one positive solution for $\lambda > \lambda^*$.

We first transform (P_E) into one-dimensional p-Laplacian systems (P_λ) with change of variables $z(r) = z(|x|)$, $w(r) = w(|x|)$, $u(t) = z((\frac{r}{r_0})^{\frac{-N+p}{p-1}})$ and $v(t) = w((\frac{r}{r_0})^{\frac{-N+p}{p-1}})$ where h_i is given by

$$h_i(t) = \left(\frac{p-1}{N-p}\right)^p r_0^p t^{\frac{-p(N-1)}{N-p}} k_i\left(r_0 t^{\frac{-(p-1)}{N-p}}\right).$$

It is not hard to see that if k_i in (P_E) satisfies (k), then h_i in (P_λ) satisfies $h_i \in \mathcal{A} \cap \mathcal{B}$, for $i = 1, 2$. Mainly by making use of Theorem 1.1, we prove the following existence result for problem (P_λ)

Theorem 1.2 *Assume $h_i \in \mathcal{A} \cap \mathcal{B}$, $i = 1, 2$, (f_1), (f_2) and (f_3). Then there exists $\lambda^* > 0$ such that (P_λ) has no positive solution for $\lambda < \lambda^*$, at least one positive solution at $\lambda = \lambda^*$ and at least two positive solutions for $\lambda > \lambda^*$.*

As a corollary, we obtain our second main result as follows.

Corollary 1.3 *Assume (k), (f_1), (f_2) and (f_3). Then there exists $\lambda^* > 0$ such that (P_E) has no positive radial solution for $\lambda < \lambda^*$, at least one positive radial solution at $\lambda = \lambda^*$ and at least two positive radial solutions for $\lambda > \lambda^*$.*

We finally notice that the first eigenfunctions of

$$\begin{cases} \varphi_p(u'(t))' + \mu h_i(t)\varphi_p(u(t)) = 0, \quad t \in (0,1), \\ u(0) = 0, \qquad u(1) = 0, \qquad i = 1, 2 \end{cases} \tag{E}$$

make an important role to construct upper solutions in the proofs of Theorem 1.2 and Theorem 1.1. This is possible due to a recent work of Kajikiya, Lee, and Sim [19] which exploits the existence of discrete eigenvalues and the properties of corresponding eigenfunctions for problem (E) with $h_i \in \mathcal{A} \cap \mathcal{B}$.

This paper is organized as follows. In Section 2, we state a C^1-regularity result and a three solutions theorem for singular p-Laplacian systems. In addition, we introduce definitions of (strict) upper and lower solutions, a related theorem and a fixed point theorem for later use. In Section 3, we prove Theorem 1.2.

2 Three solutions theorem

In this section, we give definitions of upper and lower solutions and prove three solutions theorem for the following singular system

$$\begin{cases} \varphi_p\big(u'(t)\big)' + F\big(t, v(t)\big) = 0, & t \in (0,1), \\ \varphi_p\big(v'(t)\big)' + G\big(t, u(t)\big) = 0, & t \in (0,1), \\ u(0) = 0, \quad v(0) = 0, \quad u(1) = 0, \quad v(1) = 0, \end{cases} \tag{P}$$

where $F, G : (0,1) \times \mathbb{R} \to \mathbb{R}$ are continuous.

We call (u,v) a solution of (P) if $(u,v) \in (C[0,1] \times C[0,1]) \cap (C^1(0,1) \times C^1(0,1))$, $(\varphi_p(u'(t)), \varphi_p(v'(t))) \in C^1(0,1) \times C^1(0,1)$ and (u,v) satisfies (P).

Definition 2.1 We say that $(\alpha, \bar{\alpha})$ is a *lower solution* of problem (P) if $(\alpha, \bar{\alpha}) \in (C^1(0,1) \times C^1(0,1)) \cap (C[0,1] \times C[0,1])$, $(\varphi_p(\alpha'(t)), \varphi_p(\bar{\alpha}'(t))) \in C^1(0,1) \times C^1(0,1)$ and

$$\begin{cases} \varphi_p\big(\alpha'(t)\big)' + F\big(t, \bar{\alpha}(t)\big) \geq 0, & t \in (0,1), \\ \varphi_p\big(\bar{\alpha}'(t)\big)' + G\big(t, \alpha(t)\big) \geq 0, & t \in (0,1), \\ \alpha(0) \leq 0, \quad \bar{\alpha}(0) \leq 0, \\ \alpha(1) \leq 0, \quad \bar{\alpha}(1) \leq 0. \end{cases}$$

We also say that $(\beta, \bar{\beta})$ is an *upper solution* of problem (P) if $(\beta, \bar{\beta}) \in (C^1(0,1) \times C^1(0,1)) \cap (C[0,1] \times C[0,1])$, $(\varphi_p(\beta'(t)), \varphi_p(\bar{\beta}'(t))) \in C^1(0,1) \times C^1(0,1)$ and it satisfies the reverse of the above inequalities. We say that $(\alpha, \bar{\alpha})$ and $(\beta, \bar{\beta})$ are *strict* lower solution and *strict* upper solution of problem (P), respectively, if $(\alpha, \bar{\alpha})$ and $(\beta, \bar{\beta})$ are lower solution and upper solution of problem (P), respectively and satisfying $\varphi_p(\alpha'(t))' + F(t, \bar{\alpha}(t)) > 0$, $\varphi_p(\bar{\alpha}'(t))' + G(t, \alpha(t)) > 0$, $\varphi_p(\beta'(t))' + F(t, \bar{\beta}(t)) < 0$, $\varphi_p(\bar{\beta}'(t))' + G(t, \beta(t)) < 0$ for $t \in (0,1)$.

We note that the inequality on $\mathbb{R}^2$ can be understood componentwise. Let $D_\alpha^\beta = \{(t,u,v) | (\alpha(t), \bar{\alpha}(t)) \leq (u,v) \leq (\beta(t), \bar{\beta}(t)), t \in (0,1)\}$. Then the fundamental theorem on upper and lower solutions for problem (P) is given as follows. The proof can be done by obvious combination from Lee [20], Lee and Lee [21] and Lü and O'Regan [22].

Theorem 2.2 *Let $(\alpha, \bar{\alpha})$ and $(\beta, \bar{\beta})$ be a lower solution and an upper solution of problem (P) respectively such that*

(a_1) $(\alpha(t), \bar{\alpha}(t)) \leq (\beta(t), \bar{\beta}(t))$, *for all* $t \in [0,1]$.

Assume (F_1). Also assume that there exist $h_F, h_G \in \mathcal{A} \cap \mathcal{B}$ *such that*

(a_2) $|F(t,v)| \leq h_F(t)$, $|G(t,u)| \leq h_G(t)$, *for all* $(t,u,v) \in D_\alpha^\beta$.

Then problem (P) has at least one solution (u,v) *such that*

$$\big(\alpha(t), \bar{\alpha}(t)\big) \leq \big(u(t), v(t)\big) \leq \big(\beta(t), \bar{\beta}(t)\big), \quad \textit{for all } t \in [0,1].$$

Remark 2.3 It is not hard to see that condition (H) implies the following condition;

For each $M > 0$, there exists $C_M > 0$ such that

$$|F(t,u)| \le C_M h_1(t)\varphi_p(|u|), \qquad |G(t,u)| \le C_M h_2(t)\varphi_p(|u|),$$

for $t \in (0,1)$ and $|u| \le M$.

Lemma 2.4 *Assume (H) and (F_2). Let (u,v) be a nontrivial solution of (P). Then there exists $a > 0$ such that both u and v have no interior zeros in $(0,a] \cup [1-a,1)$.*

Proof Let (u,v) be a nontrivial solution of (P). Suppose, on the contrary, that there exist sequences (t_n), (s_n) of interior zeros of u and v respectively with $t_n, s_n \to 0$. We note that both sequences should exist simultaneously. Indeed, if one of the sequences say, (t_n), does not exist, then assuming without loss of generality, $u > 0$ on $(0,a]$ for some $a > 0$, we get $\varphi_p(v'(s))' = G(t,u(t)) > 0$ for $t \in (0,a]$ by (F_2). From the monotonicity of φ_p, we know that v is concave on the interval. Thus v should have at most one interior zero in $(0,a]$, a contradiction. With this concave-convex argument, we know that $(t_n,t_{n-1}) \cap (s_n,s_{n-1}) \ne \emptyset$, $uv \ge 0$ on $(t_n,t_{n-1}) \cap (s_n,s_{n-1})$ and if t_n^* and s_n^* are local extremal points of u and v on (t_n,t_{n-1}) and (s_n,s_{n-1}) respectively, thus both t_n^* and s_n^* are in $(t_n,t_{n-1}) \cap (s_n,s_{n-1})$. We consider the case that $t_n \le s_n$, $t_n^* \le s_n^*$ and $u,v > 0$ in $(t_n,t_{n-1}) \cap (s_n,s_{n-1})$. All other cases can be explained by the same argument. If $M = \max\{\|u\|_\infty, \|v\|_\infty\}$, then by using Remark 2.3, we have

$$\begin{aligned} u(t_n^*) &= \int_{t_n}^{t_n^*} \varphi_p^{-1}\left(\int_s^{t_n^*} F(r,v(r))\,dr\right) ds \\ &\le \int_{t_n}^{t_n^*} \varphi_p^{-1}\left(\int_{s_n}^{t_n^*} F(r,v(r))\,dr\right) ds \\ &\le C_M \int_{t_n}^{t_n^*} \varphi_p^{-1}\left(\int_{s_n}^{t_n^*} h_1(r)v(r)^{p-1}\,dr\right) ds \\ &\le C_M \left(\int_{t_n}^{t_n^*} \varphi_p^{-1}\left(\int_{s_n}^{t_n^*} h_1(r)\,dr\right) ds\right) v(t_n^*) \end{aligned} \tag{2.1}$$

and similarly,

$$v(t_n^*) \le C_M \left(\int_{s_n}^{t_n^*} \varphi_p^{-1}\left(\int_s^{s_n^*} h_2(r)\,dr\right) ds\right) u(t_n^*). \tag{2.2}$$

Therefore, it follows from plugging (2.2) into (2.1) that

$$u(t_n^*) \le (C_M)^2 \left(\int_{t_n}^{t_n^*} \varphi_p^{-1}\left(\int_{s_n}^{t_n^*} h_1(r)\,dr\right) ds\right)\left(\int_{s_n}^{t_n^*} \varphi_p^{-1}\left(\int_s^{s_n^*} h_2(r)\,dr\right) ds\right) u(t_n^*). \tag{2.3}$$

Since $h_i \in \mathcal{A}$, for sufficiently large n, we obtain

$$(C_M)^2 \left(\int_{t_n}^{t_n^*} \varphi_p^{-1}\left(\int_{s_n}^{t_n^*} h_1(r)\,dr\right) ds\right)\left(\int_{s_n}^{t_n^*} \varphi_p^{-1}\left(\int_s^{s_n^*} h_2(r)\,dr\right) ds\right) < 1/2.$$

This contradicts (2.3) and the proof is done. □

Theorem 2.5 *Assume (H) and (F_2). If (u,v) is a solution of (P), then $(u,v) \in C_0^1[0,1] \times C_0^1[0,1]$.*

Proof Let (u,v) be a nontrivial solution of (P). Then $u,v \in C_0[0,1] \cap C^1(0,1)$ so that it is enough to show

$$|u'(0^+)| < \infty, \qquad |u'(1^-)| < \infty, \qquad |v'(0^+)| < \infty, \qquad |v'(1^-)| < \infty.$$

We will show $|u'(0^+)| < \infty$. Other facts can be proved by the same manner. Suppose $|u'(0^+)| = \infty$. By Lemma 2.4 and the concave-convex argument, we may assume without loss of generality that there exists $a \in (0,1)$ such that $u, v, u', v' > 0$ on $(0,a]$. Then for given $\varepsilon > 0$, by the fact $h_i \in \mathcal{B}$, $i = 1,2$, there exists $\delta \in (0,a)$ such that

$$\int_0^\delta t^{p-1} h_i(t)\, dt < \varepsilon, \quad i = 1,2.$$

Let $M = \max\{\|u\|_\infty, \|v\|_\infty\}$. Then integrating (P) over $(s,t) \subset (0,\delta)$ and using Remark 2.3, we have

$$\begin{aligned} u'(s)^{p-1} &\le u'(t)^{p-1} + C_M \int_s^t h_1(\tau)\left(\frac{v(\tau)}{\tau}\right)^{p-1} \tau^{p-1}\, d\tau \\ &\le u'(t)^{p-1} + C_M \left(\frac{v(s)}{s}\right)^{p-1} \int_s^t h_1(\tau)\tau^{p-1}\, d\tau \\ &\le u'(t)^{p-1} + C_M \varepsilon \left(\frac{v(s)}{s}\right)^{p-1}, \end{aligned} \tag{2.4}$$

where we use the fact that $(\frac{v(\tau)}{\tau})^{p-1}$ is decreasing since v is concave. From $u'(0^+) = \infty$ and (2.4), we know $\lim_{s\to 0^+} (\frac{v(s)}{s})^{p-1} = \infty$. This implies that conditions $u'(0^+) = \infty$ and $v'(0^+) = \infty$ are equivalent. From (2.4), we have

$$\left(\frac{su'(s)}{v(s)}\right)^{p-1} \le \left(\frac{s}{v(s)}\right)^{p-1} u'(t)^{p-1} + C_M \varepsilon.$$

Thus we have

$$\limsup_{s\to 0^+} \left(\frac{su'(s)}{v(s)}\right)^{p-1} \le C_M \varepsilon.$$

Since ε is arbitrary, we have

$$\limsup_{s\to 0^+} \left(\frac{su'(s)}{v(s)}\right)^{p-1} = 0. \tag{2.5}$$

Using the fact $v'(0^+) = \infty$, with same argument, we have

$$\limsup_{s\to 0^+} \left(\frac{sv'(s)}{u(s)}\right)^{p-1} = 0. \tag{2.6}$$

On the other hand, we observe the inequality

$$(\alpha+\beta)^{\frac{1}{p-1}} \le C_p\big(\alpha^{\frac{1}{p-1}}+\beta^{\frac{1}{p-1}}\big), \quad \text{for } \alpha,\beta \ge 0, \tag{2.7}$$

where

$$C_p = \begin{cases} 1 & \text{if } p \ge 2, \\ 2^{\frac{2-p}{p-1}} & \text{if } 1 < p < 2. \end{cases}$$

Since $h_i \in \mathcal{A}$, we may choose $b \in (0, \min\{a, \frac{1}{2}\})$ such that

$$(C_M)^{\frac{1}{p-1}} C_p \int_0^b \left(\int_s^{\frac{1}{2}} h_i(\tau)\,d\tau\right)^{\frac{1}{p-1}} ds < \frac{1}{2}. \tag{2.8}$$

Integrating (P) over (s,t) with $0 < s < t < b$ and using Remark 2.3, we get

$$u'(s)^{p-1} \le u'(t)^{p-1} + C_M v(t)^{p-1} \int_s^t h_1(\tau)\,d\tau,$$

here we use the fact that $v(t)$ is increasing in $(0,b)$. Using (2.7), we have

$$u'(s) \le C_p u'(t) + (C_M)^{\frac{1}{p-1}} C_p v(t) \left(\int_s^{\frac{1}{2}} h_1(\tau)\,d\tau\right)^{\frac{1}{p-1}}. \tag{2.9}$$

Integrating (2.9) over $(0,t)$ with respect to s and using (2.8), we have

$$\begin{aligned} u(t) &\le C_p t u'(t) + (C_M)^{\frac{1}{p-1}} C_p v(t) \int_0^t \left(\int_s^{\frac{1}{2}} h_1(\tau)\,d\tau\right)^{\frac{1}{p-1}} ds \\ &\le C_p t u'(t) + \frac{1}{2} v(t). \end{aligned} \tag{2.10}$$

Similarly, we have

$$v(t) \le C_p t v'(t) + \frac{1}{2} u(t). \tag{2.11}$$

Adding (2.10) and (2.11), we have

$$0 < \frac{1}{2C_p} < \frac{tu'(t) + tv'(t)}{u(t) + v(t)} \le \frac{tu'(t)}{v(t)} + \frac{tv'(t)}{u(t)}, \tag{2.12}$$

on $(0,b)$. From (2.5) and (2.6), we see that the right-hand side of (2.12) goes to zero as $t \to 0$. This is a contradiction and the proof is complete. □

Now, we consider the three solutions theorem for singular p-Laplacian system (P). For $\nu \in L^1(0,1)$, if

$$\zeta(x) = \int_0^1 \varphi_p^{-1}\left(x - \int_0^s \nu(\tau)\,d\tau\right) ds,$$

then the zero of $\zeta(x)$, denoted by $\xi(\nu)$ is uniquely determined by ν. Define $A : L^1(0,1) \to C_0^1[0,1]$ by taking

$$A(\nu)(t) = \int_0^t \varphi_p^{-1}\left(\xi(\nu) - \int_0^s \nu(\tau)\,d\tau\right) ds.$$

It is known that A is completely continuous [24]. Define $X \triangleq C_0^1[0,1] \times C_0^1[0,1]$ with norm $\|(u,v)\|_X = \|u'\|_\infty + \|v'\|_\infty$. We note that

$$\left|u(t)\right| \le 2t(1-t)\left\|u'\right\|_\infty, \quad \text{for all } u \in C_0^1[0,1]. \tag{2.13}$$

If F and G satisfy condition (H), then for $(u,v) \in X$, from Remark 2.3 and (2.13), we get

$$\begin{aligned}\int_0^1 \left|F\big(t,v(t)\big)\right| dt &\le \int_0^1 h_1(t) f\big(v(t)\big)\,dt \\ &\le \int_0^1 h_1(t) C_0 \left|v(t)\right|^{p-1} dt \\ &\le 2^{p-1} C_0 \left\|v'\right\|_\infty^{p-1} \int_0^1 t^{p-1}(1-t)^{p-1} h_1(t)\,dt.\end{aligned}$$

This implies $F(\cdot, v(\cdot)) \in L^1(0,1)$ and by similar computation, we also get $G(\cdot, u(\cdot)) \in L^1(0,1)$. This fact enables us to define the integral operator for problem (P) and the regularity of solutions (Theorem 2.5) is crucial in this argument. Now, define an operator T by

$$T(u,v) = \Big(A\big(F\big(t,v(t)\big)\big), A\big(G\big(t,u(t)\big)\big)\Big),$$

then we see that $T : X \to X$ and completely continuous.

Lemma 2.6 *Assume (H), (F_1) and (F_2). Let* $(\alpha,\bar\alpha)$ *and* $(\beta,\bar\beta)$ *be a strict lower solution and a strict upper solution of problem (P) respectively such that* $(\alpha,\bar\alpha) \in X$, $(\beta,\bar\beta) \in X$ *and* $(\alpha,\bar\alpha) \prec (\beta,\bar\beta)$. *Then problem (P) has at least one solution* $(u,v) \in X$ *such that*

$$(\alpha,\bar\alpha) \prec (u,v) \prec (\beta,\bar\beta).$$

Moreover, for $R > 0$ *large enough,*

$$\deg(I - T, \Omega, 0) = 1,$$

where $\Omega = \{(u,v) \in X \mid (\alpha,\bar\alpha) \prec (u,v) \prec (\beta,\bar\beta), \|(u,v)\|_X < R\}$.

Proof Define $\gamma : [0,1] \times \mathbb{R} \to \mathbb{R}$ given by

$$\gamma(t,u) = \begin{cases} \beta(t) & \text{if } u > \beta(t), \\ u & \text{if } \alpha(t) \le u \le \beta(t), \\ \alpha(t) & \text{if } u < \alpha(t), \end{cases}$$

$$\bar{\gamma}(t,v) = \begin{cases} \bar{\beta}(t) & \text{if } v > \bar{\beta}(t), \\ v & \text{if } \bar{\alpha}(t) \le v \le \bar{\beta}(t), \\ \bar{\alpha}(t) & \text{if } v < \bar{\alpha}(t), \end{cases}$$

and also define

$$F^*(t,v(t)) = F(t,\bar{\gamma}(t,v(t))), \qquad G^*(t,u(t)) = G(t,\gamma(t,u(t))).$$

Let us consider the following modified problem

$$\begin{cases} \varphi_p(u'(t))' + F^*(t,v(t)) = 0, & t \in (0,1), \\ \varphi_p(v'(t))' + G^*(t,u(t)) = 0, & t \in (0,1), \\ u(0) = 0, \quad v(0) = 0, \quad u(1) = 0, \quad v(1) = 0. \end{cases} \tag{$\bar{P}$}$$

We first show that there exists a constant $R > 0$ such that if (u,v) is a solution of $(\bar{P})$, then $(u,v) \in \Omega$. In fact, every solution (u,v) of $(\bar{P})$ satisfies $(\alpha,\bar{\alpha}) \le (u,v) \le (\beta,\bar{\beta})$ on $[0,1]$. From (H), (F_1) and the fact that $(\alpha,\bar{\alpha}) \in X$, $(\beta,\bar{\beta}) \in X$, we get

$$\begin{aligned} \left|\varphi_p(u'(t))\right| &= \left|\int_{t_0}^{t} F^*(\tau,v(\tau))\,d\tau\right| \le \int_{t_0}^{t} \max\left\{\left|F(\tau,\bar{\alpha}(\tau))\right|, \left|F(\tau,\bar{\beta}(\tau))\right|\right\} d\tau \\ &\le \int_0^1 h_1(t) \max_{t\in[0,1]} \left\{f(\bar{\alpha}(t)), f(\bar{\beta}(t))\right\} dt \\ &\le \int_0^1 c_1 h_1(t) \max_{t\in[0,1]} \left\{\left|\bar{\alpha}(t)\right|^{p-1}, \left|\bar{\beta}(t)\right|^{p-1}\right\} dt \\ &\le 2^{p-1} c_1 \max\left\{\left\|\overline{\alpha}'\right\|_\infty^{p-1}, \left\|\overline{\beta}'\right\|_\infty^{p-1}\right\} \int_0^1 t^{p-1}(1-t)^{p-1} h_1(t)\,dt < \infty. \end{aligned}$$

Similarly, we see that $\|v'\|_\infty$ is bounded. Therefore, $\|(u,v)\|_X < R$, for some $R > 0$. Thus it is enough to show that

$$(\alpha,\bar{\alpha}) \prec (u,v) \prec (\beta,\bar{\beta}).$$

Assume, on the contrary, that there exists $t_0 \in (0,1)$ such that

$$\min(u(t) - \alpha(t)) = u(t_0) - \alpha(t_0) = 0.$$

Then choosing $t_1 \in (t_0,1)$ with $(u-\alpha)'(t_1) \ge 0$, we get the following contradiction:

$$\begin{aligned} 0 &\le \left[\varphi_p(u'(t_1)) - \varphi_p(\alpha'(t_1))\right] - \left[\varphi_p(u'(t_0)) - \varphi_p(\alpha'(t_0))\right] \\ &= \int_{t_0}^{t_1} -F^*(t,v(t)) - \varphi_p(\alpha'(t))'\,dt \le \int_{t_0}^{t_1} -F(t,\bar{\alpha}(t)) - \varphi_p(\alpha'(t))'\,dt < 0. \end{aligned}$$

Now, assume $u'(0) = \alpha'(0)$. Since $u(t) > \alpha(t)$ on $t \in (0,1)$ and $u(0) = \alpha(0) = 0$, there exists $t_2 \in (0,1)$ such that $u'(t_2) \ge \alpha'(t_2)$ and we get the same contradiction from the above calculation by using 0 instead of t_0. For $u'(1) = \alpha'(1)$ case, we also get the same contradiction.

Consequently, we get $\alpha \prec u$. The other cases can be proved by the same manner. Taking $\Omega = \{(u,v) \in X | (\alpha, \bar{\alpha}) \prec (u,v) \prec (\beta, \bar{\beta}), \|(u,v)\|_X < R\}$, we see that every solution of $(\bar{P})$ is contained in Ω. We now compute $\deg(I - T, \Omega, 0)$. For this purpose, let us consider the operator $\bar{T} : X \to X$ defined by

$$\bar{T}(u,v)(t) = \bigl(A\bigl(F^*\bigl(t, v(t)\bigr)\bigr), A\bigl(G^*\bigl(t, u(t)\bigr)\bigr)\bigr).$$

Then it is obvious that $\bar{T}$ is completely continuous. We show that there exists $\bar{R} > 0$ such that $\bar{R} > R$ and $\bar{T}(X) \subset B(0, \bar{R})$. Indeed, since $A(F^*(0, v(0))) = 0 = A(F^*(1, v(1)))$, there is $\tilde{t} \in (0,1)$ such that $\frac{d}{dt}A(F^*(t, v(t)))|_{t=\tilde{t}} = 0$. By integrating

$$\frac{d}{dt}\varphi_p\left(\frac{d}{dt}A\bigl(F^*\bigl(t, v(t)\bigr)\bigr)\right) = F^*\bigl(t, v(t)\bigr)$$

from $\tilde{t}$ to t, we have

$$\begin{aligned}
\left|\varphi_p\left(\frac{d}{dt}A\bigl(F^*\bigl(t, v(t)\bigr)\bigr)\right)\right| &= \left|\int_{\tilde{t}}^{t} F^*\bigl(\tau, v(\tau)\bigr)\,d\tau\right| \\
&\le \int_0^1 h_1(t) f\bigl(\bar{\gamma}\bigl(t, v(t)\bigr)\bigr)\,dt \le \int_0^1 h_1(t) C_1 \bigl|\bar{\gamma}\bigl(t, v(t)\bigr)\bigr|^{p-1}\,dt \\
&\le \int_0^1 h_1(t) C_1 \max\bigl\{\bigl|\bar{\beta}(t)\bigr|^{p-1}, \bigl|\bar{\alpha}(t)\bigr|^{p-1}\bigr\}\,dt \\
&\le C_2 \max\bigl\{\bigl\|\bar{\beta}'\bigr\|_\infty^{p-1}, \bigl\|\bar{\alpha}'\bigr\|_\infty^{p-1}\bigr\} \int_0^1 t^{p-1}(1-t)^{p-1} h_1(t)\,dt.
\end{aligned}$$

Similarly, we see that $\frac{d}{dt}A(G^*(t, u(t)))$ is bounded. Therefore, we get

$$\deg\bigl(I - \bar{T}, B(0, \bar{R}), 0\bigr) = 1.$$

Since every solution of $(\bar{P})$ is contained in Ω, the excision property implies that

$$\deg(I - \bar{T}, \Omega, 0) = \deg\bigl(I - \bar{T}, B(0, \bar{R}), 0\bigr) = 1.$$

Since $\bar{T} = T$ on Ω, we finally get

$$\deg(I - T, \Omega, 0) = \deg(I - \bar{T}, \Omega, 0) = 1.$$

This completes the proof. □

We now prove three solutions theorem for (P).

Proof of Theorem 1.1

Define

$$\gamma_1(t,u) = \begin{cases} \beta_2(t) & \text{if } u > \beta_2(t), \\ u & \text{if } \alpha_1(t) \le u \le \beta_2(t), \\ \alpha_1(t) & \text{if } u < \alpha_1(t), \end{cases}$$

$$\bar{\gamma}_1(t,v) = \begin{cases} \bar{\beta}_2(t) & \text{if } v > \bar{\beta}_2(t), \\ v & \text{if } \bar{\alpha}_1(t) \le v \le \bar{\beta}_2(t), \\ \bar{\alpha}_1(t) & \text{if } v < \bar{\alpha}_1(t), \end{cases}$$

and let us consider

$$\begin{cases} \varphi_p(u'(t))' + F(t, \bar{\gamma}_1(t, v(t))) = 0, & t \in (0,1), \\ \varphi_p(v'(t))' + G(t, \gamma_1(t, u(t))) = 0, & t \in (0,1), \\ u(0) = 0, \quad v(0) = 0, \quad u(1) = 0, \quad v(1) = 0. \end{cases} \tag{$\tilde{P}$}$$

Then noting that every solution (u,v) of $(\tilde{P})$ satisfies $(\alpha_1, \bar{\alpha}_1) \le (u,v) \le (\beta_2, \bar{\beta}_2)$, we may choose $K_1, K_2 > 0$, by (H) such that

$$\begin{aligned} &|f(v)| \le K_1 \varphi_p(|v|) \quad \text{for all } |v| \le \max\{\|\bar{\beta}_2\|, \|\bar{\alpha}_1\|\}, \\ &|g(u)| \le K_2 \varphi_p(|u|) \quad \text{for all } |u| \le \max\{\|\beta_2\|, \|\alpha_1\|\}. \end{aligned}$$

Let λ_1 and μ_1 be the first eigenvalues of

$$\begin{cases} \varphi_p(u'(t))' + \mu h_i(t) \varphi_p(u(t)) = 0, & t \in (0,1), \\ u(0) = 0, \quad u(1) = 0, \end{cases} \tag{E}$$

for $i = 1,2$ respectively and let e_1 and e_2 be corresponding eigenfunctions with $\|e_1\|_\infty = \|e_2\|_\infty = 1$. Since $e_1, e_2 \in C_0^1[0,1]$ are positive and concave [19], we may choose $M_1, M_2 > 0$ such that $(M_1e_1, M_2e_2) \succ (\beta_2, \bar{\beta}_2)$, $(-M_1e_1, -M_2e_2) \prec (\alpha_1, \bar{\alpha}_1)$ and for $t \in (0,1)$,

$$\begin{aligned} &K_1 \max\{\varphi_p(|\bar{\beta}_2(t)|), \varphi_p(|\bar{\alpha}_1(t)|)\} < \lambda_1 \varphi_p(M_1 e_1(t)), \\ &K_2 \max\{\varphi_p(|\alpha_1(t)|), \varphi_p(|\beta_2(t)|)\} < \mu_1 \varphi_p(M_2 e_2(t)). \end{aligned}$$

We show that (M_1e_1, M_2e_2) and $(-M_1e_1, -M_2e_2)$ are a strict upper solution and a strict lower solution of $(\tilde{P})$ respectively. Indeed,

$$\begin{aligned} \varphi_p(M_1e_1'(t))' + F(t, \bar{\gamma}_1(t, M_2e_2(t))) &= \varphi_p(M_1e_1'(t))' + F(t, \bar{\beta}_2(t)) \\ &\le -\lambda_1 h_1(t) \varphi_p(M_1e_1(t)) + h_1(t) f(\bar{\beta}_2(t)) \\ &\le -\lambda_1 h_1(t) \varphi_p(M_1e_1(t)) + h_1(t) K_1 \varphi_p(|\bar{\beta}_2(t)|) < 0. \end{aligned}$$

Similarly, we get

$$\varphi_p(M_2e_2'(t))' + G(t, \gamma_1(t, M_1e_1(t))) < 0.$$

Moreover,

$$\begin{aligned} &\varphi_p(-M_2e_2'(t))' + G(t, \gamma_1(t, -M_1e_1(t))) \\ &\quad = -\varphi_p(M_2e_2'(t))' + G(t, \alpha_1(t)) \end{aligned}$$

$$\geq \mu_1 h_2(t)\varphi_p\big(M_2 e_2(t)\big) - h_2(t)g\big(\alpha_1(t)\big)$$
$$\geq \mu_1 h_2(t)\varphi_p\big(M_2 e_2(t)\big) - h_2(t)K_2\varphi_p\big(\big|\alpha_1(t)\big|\big) > 0.$$

Similarly, we also get

$$\varphi_p\big(-M_2 e_2'(t)\big)' + F\big(t, \bar{\gamma}_1\big(t, -M_2 e_2(t)\big)\big) > 0.$$

For $R > 0$, large enough, define

$$\Omega_1 = \big\{(u,v) \in X | (-M_1 e_1, -M_2 e_2) \prec (u,v) \prec (\beta_1, \bar{\beta}_1), \big\|(u,v)\big\|_X < R\big\},$$
$$\Omega_2 = \big\{(u,v) \in X | (\alpha_2, \bar{\alpha}_2) \prec (u,v) \prec (M_1 e_1, M_2 e_2), \big\|(u,v)\big\|_X < R\big\},$$
$$\Omega_3 = \big\{(u,v) \in X | (-M_1 e_1, -M_2 e_2) \prec (u,v) \prec (M_1 e_1, M_2 e_2), \big\|(u,v)\big\|_X < R\big\}.$$

Then by Theorem 2.2, there exist two solutions (u_1, v_1) and (u_2, v_2) of (P) satisfying $(\alpha_1, \bar{\alpha}_1) \preceq (u_1, v_1) \prec (\beta_1, \bar{\beta}_1)$ and $(\alpha_2, \bar{\alpha}_2) \prec (u_2, v_2) \preceq (\beta_2, \bar{\beta}_2)$. Therefore, by Lemma 2.6, we get

$$\deg(I - \tilde{T}, \Omega_1, 0) = \deg(I - \tilde{T}, \Omega_2, 0) = \deg(I - \tilde{T}, \Omega_3, 0) = 1,$$

and by the excision property, we have

$$\deg\big(I - \tilde{T}, \Omega_3 \setminus (\Omega_1 \cup \Omega_2), 0\big) = -1.$$

This completes the proof.

3 Application

In this section, we prove the existence, nonexistence, and multiplicity of positive solutions for (P_λ) by using three solutions theorem in Section 2. Let us define a cone

$$K = \big\{u \in C[0,1] | u \text{ are concave and } u(0) = 0 = u(1)\big\},$$

and define $A_\lambda, B_\lambda : K \to C[0,1]$ by taking

$$A_\lambda(v)(t) = \begin{cases} \int_0^t \varphi_p^{-1}\left(\int_s^{\sigma_v} \lambda h_1(\tau) f\big(v(\tau)\big)\, d\tau\right) ds, \\ \int_t^1 \varphi_p^{-1}\left(\int_{\sigma_v}^s \lambda h_1(\tau) f\big(v(\tau)\big)\, d\tau\right) ds, \end{cases}$$
$$B_\lambda(u)(t) = \begin{cases} \int_0^t \varphi_p^{-1}\left(\int_s^{\sigma_u} \lambda h_2(\tau) g\big(u(\tau)\big)\, d\tau\right) ds, \\ \int_t^1 \varphi_p^{-1}\left(\int_{\sigma_u}^s \lambda h_2(\tau) g\big(u(\tau)\big)\, d\tau\right) ds, \end{cases}$$

where σ_v and σ_u are unique zeros of

$$x_v(t) = \int_0^t \varphi_p^{-1}\left(\int_s^t \lambda h_1(\tau) f\big(v(\tau)\big)\, d\tau\right) ds - \int_t^1 \varphi_p^{-1}\left(\int_t^s \lambda h_1(\tau) f\big(v(\tau)\big)\, d\tau\right) ds,$$

$$y_u(t) = \int_0^t \varphi_p^{-1}\left(\int_s^t \lambda h_2(\tau)g\big(u(\tau)\big)\,d\tau\right)ds - \int_t^1 \varphi_p^{-1}\left(\int_t^s \lambda h_2(\tau)g\big(u(\tau)\big)\,d\tau\right)ds,$$

respectively. And define $T_\lambda : K \times K \to C[0,1] \times C[0,1]$ by

$$T_\lambda(u,v) = \big(A_\lambda(v), B_\lambda(u)\big).$$

Then it is known that $T_\lambda : K \times K \to K \times K$ is completely continuous [25] and $(u,v) = T_\lambda(u,v)$ in $K \times K$ is equivalent to the fact that (u,v) is a positive solution of (P_λ). We know from Theorem 2.5 that under assumptions $h_i \in \mathcal{A} \cap \mathcal{B}$, $i = 1,2$ and (f_1), any solution (u,v) of problem (P_λ) is in $C_0^1[0,1] \times C_0^1[0,1]$.

Remark 3.1 If (u,v) is a solution of (P_λ), then $u = A_\lambda(B_\lambda(u))$ and $v = B_\lambda(A_\lambda(v))$.

For later use, we introduce the following well-known result. See [12] for proof and details.

Proposition 3.2 *Let X be a Banach space, $\mathcal{K}$ an order cone in X. Assume that Ω_1 and Ω_2 are bounded open subsets in X with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subseteq \Omega_2$. Let $A : \mathcal{K} \cap (\overline{\Omega}_2 \setminus \Omega_1) \to \mathcal{K}$ be a completely continuous operator such that either*

(i) $\|Au\| \le \|u\|$, $u \in \mathcal{K} \cap \partial\Omega_1$, and $\|Au\| \ge \|u\|$, $u \in \mathcal{K} \cap \partial\Omega_2$

or

(ii) $\|Au\| \ge \|u\|$, $u \in \mathcal{K} \cap \partial\Omega_1$, and $\|Au\| \le \|u\|$, $u \in \mathcal{K} \cap \partial\Omega_2$.

Then A has a fixed point in $\mathcal{K} \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Lemma 3.3 *Assume $h_i \in \mathcal{A} \cap \mathcal{B}$, $i = 1,2$, (f_2) and (f_3). Let $\mathcal{R}$ be a compact subset of $(0,\infty)$. Then there exists a constant $b_{\mathcal{R}} > 0$ such that for all $\lambda \in \mathcal{R}$ and all possible positive solutions (u,v) of (P_λ), one has $\|(u,v)\|_\infty \le b_{\mathcal{R}}$.*

Proof If it is not true, then there exist $\{\lambda_n\} \subset \mathcal{R}$ and solutions $\{(u_n,v_n)\}$ of (P_{λ_n}) such that $\|(u_n,v_n)\|_\infty \to \infty$. We note that

$$\begin{aligned}
\|u_n\|_\infty &= \big\|A_{\lambda_n}(v_n)\big\|_\infty \le \Lambda Q_1 \varphi_p^{-1}\big(f\big(\|v_n\|_\infty\big)\big),\\
\|v_n\|_\infty &= \big\|B_{\lambda_n}(u_n)\big\|_\infty \le \Lambda Q_2 \varphi_p^{-1}\big(g\big(\|u_n\|_\infty\big)\big),
\end{aligned}$$

where $\Lambda = \max\{\lambda^{\frac{1}{p-1}} \mid \lambda \in \mathcal{R}\}$ and

$$Q_i = \int_0^{\frac{1}{2}} \varphi_p^{-1}\left(\int_s^{\frac{1}{2}} h_i(\tau)\,d\tau\right)ds + \int_{\frac{1}{2}}^1 \varphi_p^{-1}\left(\int_{\frac{1}{2}}^s h_i(\tau)\,d\tau\right)ds, \quad i = 1,2.$$

This implies both $\|u_n\|_\infty \to \infty$ and $\|v_n\|_\infty \to \infty$. Moreover, by the above estimation,

$$\|u_n\|_\infty \le \Lambda Q_1 \varphi_p^{-1}\big(f\big(\Lambda Q_2 \varphi_p^{-1}\big(g\big(\|u_n\|_\infty\big)\big)\big)\big).$$

Thus we get

$$\frac{1}{\varphi_p(\Lambda Q_1)} \le \frac{f(\Lambda Q_2 \varphi_p^{-1}(g(\|u_n\|_\infty)))}{\varphi_p(\|u_n\|_\infty)} \to 0$$

as $\|u_n\|_\infty \to \infty$ and this contradiction completes the proof. □

Lemma 3.4 *Assume $h_i \in \mathcal{A} \cap \mathcal{B}$, $i = 1,2$, (f_2) and (f_3). If (P_λ) has a lower solution $(\alpha, \bar{\alpha}) \in C_0^1[0,1] \times C_0^1[0,1]$ for some $\lambda > 0$, then (P_λ) has a solution (u, v) such that $(\alpha, \bar{\alpha}) \le (u, v)$.*

Proof It suffices to show the existence of an upper solution $(\beta, \bar{\beta})$ of (P_λ) satisfying $(\alpha, \bar{\alpha}) \le (\beta, \bar{\beta})$. Let ϕ_i and $i = 1,2$ be positive solutions of

$$\begin{cases} \varphi_p(u'(t))' + \lambda h_i(t) = 0, & t \in (0,1), \\ u(0) = 0, \quad u(1) = 0. \end{cases}$$

(Case I) Both f and g are bounded.

Since ϕ_i ($i = 1,2$) are positive concave functions and $(\alpha, \bar{\alpha}) \in C_0^1[0,1] \times C_0^1[0,1]$, we may choose $M > 0$ such that $M > \max\{\|f\|_\infty^{\frac{1}{p-1}}, \|g\|_\infty^{\frac{1}{p-1}}\}$ and $(M\phi_1, M\phi_2) \ge (\alpha, \bar{\alpha})$. We now show that $(\beta, \bar{\beta}) = (M\phi_1, M\phi_2)$ is an upper solution of (P_λ). In fact,

$$\begin{aligned} \varphi_p(\beta'(t))' + \lambda h_1(t) f(\bar{\beta}(t)) &= M^{p-1}\varphi_p(\phi_1'(t))' + \lambda h_1(t) f(M\phi_2(t)) \\ &= \lambda h_1(t)\left[f(M\phi_2(t)) - M^{p-1}\right] \\ &\le \lambda h_1(t)\left[\|f\|_\infty - M^{p-1}\right] \le 0. \end{aligned}$$

Similarly,

$$\varphi_p(\bar{\beta}'(t))' + \lambda h_2(t) g(\beta(t)) \le \lambda h_2(t)\left[\|g\|_\infty - M^{p-1}\right] \le 0.$$

(Case II) $g(u) \to \infty$ as $u \to \infty$.

Using (f_2), choose $M > 0$ such that $(g(M\|\phi_1\|_\infty))^{\frac{1}{p-1}}\phi_2 \ge \bar{\alpha}$, $M\phi_1 \ge \alpha$ and

$$\frac{f(\|\phi_2\|_\infty (g(M\|\phi_1\|_\infty))^{\frac{1}{p-1}})}{(M\|\phi_1\|_\infty)^{p-1}} \le \frac{1}{\|\phi_1\|_\infty^{p-1}}.$$

Let $(\beta, \bar{\beta}) = (M\phi_1, (g(M\|\phi_1\|_\infty))^{\frac{1}{p-1}}\phi_2)$. Then

$$\begin{aligned} \varphi_p(\beta'(t))' + \lambda h_1(t) f(\bar{\beta}(t)) &= M^{p-1}\varphi_p(\phi_1'(t))' + \lambda h_1(t) f\left(\left(g(M\|\phi_1\|_\infty)\right)^{\frac{1}{p-1}}\phi_2(t)\right) \\ &= \lambda h_1(t)\left[f\left(\left(g(M\|\phi_1\|_\infty)\right)^{\frac{1}{p-1}}\phi_2(t)\right) - M^{p-1}\right] \\ &\le \lambda h_1(t)\left[f\left(\|\phi_2\|_\infty \left(g(M\|\phi_1\|_\infty)\right)^{\frac{1}{p-1}}\right) - M^{p-1}\right] \le 0. \end{aligned}$$

And

$$\begin{aligned} \varphi_p(\bar{\beta}'(t))' + \lambda h_2(t) g(\beta(t)) &= g(M\|\phi_1\|_\infty)\varphi_p(\phi_2'(t))' + \lambda h_2(t) g(M\phi_1(t)) \\ &= \lambda h_2(t)\left[g(M\phi_1(t)) - g(M\|\phi_1\|_\infty)\right] \\ &\le \lambda h_2(t)\left[g(M\|\phi_1\|_\infty) - g(M\|\phi_1\|_\infty)\right] = 0. \end{aligned}$$

Thus $(\beta, \bar{\beta})$ is an upper solution of (P_λ).

(Case III) g is bounded and $f(u) \to \infty$ as $u \to \infty$.

Choose $M > 0$ such that $(f(M\|\phi_2\|_\infty))^{\frac{1}{p-1}}\phi_1 \geq \alpha$, $M > \|g\|_\infty^{\frac{1}{p-1}}$ and $M\phi_2 \geq \bar{\alpha}$ and let

$$(\beta, \bar{\beta}) = \left(\left(f\left(M\|\phi_2\|_\infty\right)\right)^{\frac{1}{p-1}}\phi_1, M\phi_2\right).$$

Then

$$\begin{aligned}\varphi_p\left(\beta'(t)\right)' + \lambda h_1(t) f\left(\bar{\beta}(t)\right) &= f\left(M\|\phi_2\|_\infty\right)\varphi_p\left(\phi_1'(t)\right)' + \lambda h_1(t) f\left(M\phi_2(t)\right) \\ &= \lambda h_1(t)\left[f\left(M\phi_2(t)\right) - f\left(M\|\phi_2\|_\infty\right)\right] \\ &\leq \lambda h_1(t)\left[f\left(M\|\phi_2\|_\infty\right) - f\left(M\|\phi_2\|_\infty\right)\right] = 0.\end{aligned}$$

And

$$\begin{aligned}\varphi_p\left(\bar{\beta}'(t)\right)' + \lambda h_2(t) g\left(\beta(t)\right) &= M^{p-1}\varphi_p\left(\phi_2'(t)\right)' + \lambda h_2(t) g\left(\left(f\left(M\|\phi_2\|_\infty\right)\right)^{\frac{1}{p-1}}\phi_1(t)\right) \\ &= \lambda h_2(t)\left[g\left(\left(f\left(M\|\phi_2\|_\infty\right)\right)^{\frac{1}{p-1}}\phi_1(t)\right) - M^{p-1}\right] \\ &\leq \lambda h_2(t)\left[\|g\|_\infty - M^{p-1}\right] \leq 0.\end{aligned}$$

Consequently, by Theorem 2.2, (P_λ) has a solution satisfying

$$(\alpha, \bar{\alpha}) \leq (u, v) \leq (\beta, \bar{\beta}).$$

□

Lemma 3.5 *Assume $h_i \in \mathcal{A} \cap \mathcal{B}$, $i = 1, 2$, (f_1), (f_2) and (f_3). Then there exists $\bar{\lambda} > 0$ such that if (P_λ) has a positive solution (u, v), then $\lambda \geq \bar{\lambda}$.*

Proof Let (u, v) be a positive solution of (P_λ). Without loss of generality, we may assume $\lambda < 1$. From (f_1), we know that

$$\lim_{x \to 0^+} \frac{f(\rho\varphi_p^{-1}(g(u)))}{\varphi_p(u)} = 0, \quad \text{for all } \rho > 0. \tag{3.1}$$

From (3.1) and (f_2), we can choose $M_f > 0$ such that

$$f\left(Q_2\varphi_p^{-1}\left(g(u)\right)\right) \leq M_f\varphi_p(u), \quad \text{for all } u > 0, \tag{3.2}$$

where $Q_i = \int_0^{\frac{1}{2}} \varphi_p^{-1}(\int_s^{\frac{1}{2}} h_i(\tau)\,d\tau)\,ds + \int_{\frac{1}{2}}^1 \varphi_p^{-1}(\int_{\frac{1}{2}}^s h_i(\tau)\,d\tau)\,ds$, $i = 1, 2$. Using (3.2) and (f_3), we have

$$\begin{aligned}\|u\|_\infty &\leq Q_1\varphi_p^{-1}\left(\lambda f\left(\left\|B_\lambda(u)\right\|_\infty\right)\right) \leq Q_1\varphi_p^{-1}\left(\lambda f\left(Q_2\varphi_p^{-1}\left(\lambda g\left(\|u\|_\infty\right)\right)\right)\right) \\ &\leq Q_1\varphi_p^{-1}\left(\lambda f\left(Q_2\varphi_p^{-1}\left(g\left(\|u\|_\infty\right)\right)\right)\right) \leq Q_1\varphi_p^{-1}\left(\lambda M_f\varphi_p\left(\|u\|_\infty\right)\right) \\ &\leq Q_1\varphi_p^{-1}(\lambda M_f)\|u\|_\infty.\end{aligned}$$

Thus we have

$$\bar{\lambda} \triangleq \frac{1}{\varphi_p(Q_1)M_f} \leq \lambda.$$

□

Lemma 3.6 *Assume $h_i \in \mathcal{A} \cap \mathcal{B}$, $i = 1, 2$, (f_1), (f_2) and (f_3). Then for each $R > 0$, there exists $\lambda_R > 0$ such that for $\lambda > \lambda_R$, (P_λ) has a positive solution (u, v) with $\|u\|_\infty > R$ and $\|v\|_\infty > R$.*

Proof We know that if (u, v) satisfies $u = A_\lambda(B_\lambda(u))$ and $v = B_\lambda(u)$, then (u, v) is a solution of (P_λ). Since $A_\lambda, B_\lambda : K \to K$ are completely continuous, $A_\lambda \circ B_\lambda : K \to K$ is also completely continuous. Given $R > 0$, choose

$$\lambda_R = \max\left\{ \varphi_p\left(\frac{2R}{\Gamma_2}\right)\frac{1}{g(\frac{R}{4})}, \varphi_p\left(\frac{2R}{\Gamma_1}\right)\frac{1}{f(\frac{R}{4})} \right\},$$

where $\Gamma_i = \min_{t\in[\frac{1}{4},\frac{3}{4}]}\{\int_{\frac{1}{4}}^{t} \varphi_p^{-1}(\int_s^t h_i(\tau)\,d\tau)\,ds + \int_t^{\frac{3}{4}} \varphi_p^{-1}(\int_t^s h_i(\tau)\,d\tau)\,ds\}$. Let $\Omega_1 = \{u \in C[0,1] \mid \|u\|_\infty < R\}$. If $u \in \partial\Omega_1 \cap K$, then for $t \in [\frac{1}{4}, \frac{3}{4}]$, $u(t) \geq \frac{1}{4}\|u\|_\infty \geq \frac{1}{4}R$. From the definition of $B_{\lambda_R}(u)$, we know that $B_{\lambda_R}(u)(\sigma_u)$ is the maximum value of $B_{\lambda_R}(u)$ on $[0, 1]$. If $\sigma_u \in [\frac{1}{4}, \frac{3}{4}]$, then from the choice of λ_R, we have

$$\begin{aligned}
&\|B_{\lambda_R}(u)\|_\infty \\
&\quad \geq \frac{1}{2}\left[\int_{\frac{1}{4}}^{\sigma_u} \varphi_p^{-1}\left(\int_s^{\sigma_u} \lambda_R h_2(\tau) g(u(\tau))\,d\tau\right) ds + \int_{\sigma_u}^{\frac{3}{4}} \varphi_p^{-1}\left(\int_{\sigma_u}^{s} \lambda_R h_2(\tau) g(u(\tau))\,d\tau\right) ds\right] \\
&\quad \geq \frac{1}{2}\left[\int_{\frac{1}{4}}^{\sigma_u} \varphi_p^{-1}\left(\int_s^{\sigma_u} \lambda_R h_2(\tau) g\left(\frac{R}{4}\right) d\tau\right) ds + \int_{\sigma_u}^{\frac{3}{4}} \varphi_p^{-1}\left(\int_{\sigma_u}^{s} \lambda_R h_2(\tau) g\left(\frac{R}{4}\right) d\tau\right) ds\right] \\
&\quad \geq \frac{1}{2}\Gamma_2 \varphi_p^{-1}\left(\lambda_R g\left(\frac{R}{4}\right)\right) \geq R.
\end{aligned}$$

If $\sigma_u > \frac{3}{4}$, then we have

$$\|B_{\lambda_R}(u)\|_\infty \geq \int_{\frac{1}{4}}^{\frac{3}{4}} \varphi_p^{-1}\left(\int_s^{\frac{3}{4}} \lambda_R h_2(\tau) g\left(\frac{R}{4}\right) d\tau\right) ds \geq \frac{1}{2}\Gamma_2 \varphi_p^{-1}\left(\lambda_R g\left(\frac{R}{4}\right)\right) \geq R.$$

If $\sigma_u < \frac{1}{4}$, then

$$\|B_{\lambda_R}(u)\|_\infty \geq \int_{\frac{1}{4}}^{\frac{3}{4}} \varphi_p^{-1}\left(\int_{\frac{1}{4}}^{s} \lambda_R h_2(\tau) g\left(\frac{R}{4}\right) d\tau\right) ds \geq \frac{1}{2}\Gamma_2 \varphi_p^{-1}\left(\lambda_R g\left(\frac{R}{4}\right)\right) \geq R.$$

By the concavity of $B_{\lambda_R}(u)$, we get for $t \in [\frac{1}{4}, \frac{3}{4}]$,

$$B_{\lambda_R}(u)(t) \geq \frac{1}{4}\|B_{\lambda_R}(u)\|_\infty \geq \frac{1}{4}R. \tag{3.3}$$

By similar argument as the above, with (3.3), we may show that

$$\|A_{\lambda_R}(B_{\lambda_R}(u))\|_\infty \geq \frac{1}{2}\Gamma_1 \varphi_p^{-1}\left(\lambda_R f\left(\frac{R}{4}\right)\right) \geq R = \|u\|_\infty.$$

Let $Q_i = \int_0^{\frac{1}{2}} \varphi_p^{-1}(\int_s^{\frac{1}{2}} h_i(\tau)\,d\tau)\,ds + \int_{\frac{1}{2}}^{1} \varphi_p^{-1}(\int_{\frac{1}{2}}^{s} h_i(\tau)\,d\tau)\,ds$, $i = 1, 2$. For $\varepsilon < \frac{1}{\varphi_p(Q_1)\lambda_R}$, from ($f_2$), we may choose $\tilde{R} > 0$ such that $\tilde{R} > R$ and

$$f\left(Q_2 \varphi_p^{-1}(\lambda_R)\varphi_p^{-1}(g(\tilde{R}))\right) \leq \varepsilon \varphi_p(\tilde{R}).$$

Let $\Omega_2 = \{u \in C[0,1] \mid \|u\|_\infty < \tilde{R}\}$, then $\overline{\Omega_1} \subset \Omega_2$ and for $u \in \partial\Omega_2 \cap K$,

$$\begin{aligned}\|A_{\lambda_R}(B_{\lambda_R}(u))\|_\infty &\le Q_1\varphi_p^{-1}\left(\lambda_R f\left(Q_2\varphi_p^{-1}(\lambda_R)\varphi_p^{-1}\left(g(\tilde{R})\right)\right)\right)\\ &\le Q_1\varphi_p^{-1}\left(\lambda_R\varepsilon\varphi_p(\tilde{R})\right) \le Q_1\varphi_p^{-1}(\lambda_R\varepsilon)\tilde{R} \le \tilde{R} = \|u\|_\infty.\end{aligned}$$

By Proposition 3.2, (P_{λ_R}) has a positive solution (u_R, v_R) such that $\|u_R\|_\infty > R$ and $\|v_R\|_\infty > R$. We know that (u_R, v_R) is a lower solution of (P_λ) for $\lambda > \lambda_R$ and by Lemma 3.4, the proof is complete. □

We now prove one of the main results for this paper.

Proof of Theorem 1.2

From Lemma 3.6 and Lemma 3.5, we know that the set $\mathcal{S} = \{\lambda > 0 \mid (P_\lambda)$ has a positive solution$\}$ is not empty and $\lambda^* = \inf\mathcal{S} > 0$. By Lemma 3.3 and complete continuity of T, there exist sequences $\{\lambda_n\}$ and $\{(u_n, v_n)\}$ such that $\lambda_n \to \lambda^*$ and $(u_n, v_n) \to (u^*, v^*)$ in $K \times K$ with (u^*, v^*) a solution of (P_{λ^*}). We claim that (u^*, v^*) is a nontrivial solution of (P_{λ^*}). Suppose that it is not true, then there exists a sequence of solutions (u_n, v_n) for (P_{λ_n}) such that $(u_n, v_n) \to (0,0)$ and $\lambda_n \to \lambda^*$. As in the proof of Lemma 3.3, we get

$$\frac{1}{\varphi_p(\Lambda Q_1)} \le \frac{f(\Lambda Q_2\varphi_p^{-1}(g(\|u_n\|_\infty)))}{\varphi_p(\|u_n\|_\infty)}.$$

But from (f_1), we have a contradiction to the fact that the right side of the above inequality converges to zero as $\|u_n\| \to 0$. Thus (u^*, v^*) is a nontrivial solution of (P_{λ^*}). According to Lemma 3.4 and the definition of λ^*, we know that (P_λ) has at least one positive solution at $\lambda \ge \lambda^*$ and no positive solution for $\lambda < \lambda^*$. To prove the existence of the second positive solution of (P_λ) for $\lambda > \lambda^*$, we will use Theorem 1.1. Let $\lambda > \lambda^*$. Then we have $(\alpha_1, \bar{\alpha}_1) = (0,0)$ a lower solution of (P_λ) and $(\alpha_2, \bar{\alpha}_2) = (u^*, v^*)$ a strict lower solution of (P_λ) in $C_0^1[0,1] \times C_0^1[0,1]$ satisfying $(\alpha_2, \bar{\alpha}_2) \ge (\alpha_1, \bar{\alpha}_1)$. For upper solutions, let λ_1 and μ_1 be the first eigenvalues of

$$\begin{cases}\varphi_p\left(u'(t)\right)' + \mu h_i(t)\varphi_p\left(u(t)\right) = 0, & t \in (0,1),\\ u(0) = 0, \quad u(1) = 0,\end{cases} \tag{E}$$

for $i = 1, 2$ respectively and let e_1 and e_2 be corresponding eigenfunctions with $\|e_1\|_\infty = \|e_2\|_\infty = 1$. Since e_1 and e_2 are in $C_0^1[0,1]$ and positive [19], we may choose $c_1 > 0$ and $c_2 > 0$ such that

$$\lambda c_1 e_2^{p-1} < \lambda_1 e_1^{p-1} \quad \text{and} \quad \lambda c_2 e_1^{p-1} < \mu_1 e_2^{p-1}.$$

Also by the fact $f_0 = g_0 = 0$, there exists $a > 0$ such that

$$f(u) \le c_1 u^{p-1}, \qquad g(u) \le c_2 u^{p-1},$$

for all $|u| \le a$ and

$$ae_1(t) < \alpha_2(t), \qquad ae_2(t) < \bar{\alpha}_2(t).$$

Let $(\beta_1,\bar{\beta}_1) = (ae_1, ae_2)$. Then $(\beta_1,\bar{\beta}_1) \not\geq (\alpha_2,\bar{\alpha}_2)$ and it is a strict upper solution of (P_λ) in $C_0^1[0,1] \times C_0^1[0,1]$. Indeed,

$$\begin{aligned}\varphi_p\bigl(\beta_1'(t)\bigr)' + \lambda h_1(t) f\bigl(\bar{\beta}_1(t)\bigr) &= a^{p-1}\varphi_p\bigl(e_1'(t)\bigr)' + \lambda h_1(t) f\bigl(ae_2(t)\bigr) \\ &\leq -\lambda_1 h_1(t) a^{p-1}\varphi_p\bigl(e_1(t)\bigr) + \lambda h_1(t) c_1 a^{p-1}\varphi_p\bigl(e_2(t)\bigr) \\ &= a^{p-1} h_1(t)\bigl[\lambda c_1\varphi_p\bigl(e_2(t)\bigr) - \lambda_1\varphi_p\bigl(e_1(t)\bigr)\bigr] < 0\end{aligned}$$

and

$$\begin{aligned}\varphi_p\bigl(\bar{\beta}_1'(t)\bigr)' + \lambda h_2(t) g\bigl(\beta_1(t)\bigr) &= a^{p-1}\varphi_p\bigl(e_2'(t)\bigr)' + \lambda h_2(t) g\bigl(ae_1(t)\bigr) \\ &\leq -\mu_1 h_2(t) a^{p-1}\varphi_p\bigl(e_2(t)\bigr) + \lambda h_2(t) c_2 a^{p-1}\varphi_p\bigl(e_1(t)\bigr) \\ &= a^{p-1} h_2(t)\bigl[\lambda c_2\varphi_p\bigl(e_1(t)\bigr) - \mu_1\varphi_p\bigl(e_2(t)\bigr)\bigr] < 0.\end{aligned}$$

Finally, from Lemma 3.6, there exists $\bar{\lambda} > \lambda$ such that $(P_{\bar{\lambda}})$ has a positive solution $(\bar{u},\bar{v}) \in C_0^1[0,1] \times C_0^1[0,1]$ satisfying $\|\bar{u}\|_\infty > \max\{\|\alpha_2'\|_\infty, \|\beta_1'\|_\infty\}$ and $\|\bar{v}\|_\infty > \max\{\|\bar{\alpha}_2'\|_\infty, \|\bar{\beta}_1'\|_\infty\}$. By using the concavity of solutions, it is easily verified that

$$(\beta_1,\bar{\beta}_1) \leq (\bar{u},\bar{v}) \quad \text{and} \quad (\alpha_2,\bar{\alpha}_2) \leq (\bar{u},\bar{v}).$$

Therefore, $(\beta_2,\bar{\beta}_2) = (\bar{u},\bar{v})$ is an upper solution of (P_λ) in $C_0^1[0,1] \times C_0^1[0,1]$. Now by Theorem 1.1, (P_λ) has at least two positive solutions (u_1,v_1) and (u_2,v_2) such that $(\alpha_2,\bar{\alpha}_2) \prec (u_1,v_1) \leq (\beta_2,\bar{\beta}_2)$ and $(\alpha_1,\bar{\alpha}_1) \leq (u_2,v_2) \leq (\beta_2,\bar{\beta}_2)$ and $(u_2,v_2) \not\leq (\beta_1,\bar{\beta}_1)$, $(u_2,v_2) \not\geq (\alpha_2,\bar{\alpha}_2)$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors have equally contributed in obtaining new results in this article and also read and approved the final manuscript.

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