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Global structure of positive solutions for second-order difference equation with nonlinear boundary value condition

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Abstract

This paper is devoted to the study of the global structure of the positive solution of a second-order nonlinear difference equation coupled with a nonlinear boundary value condition. The main result is based on Dancer's bifurcation theorem.

MSC: 34B15; 39A12

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1 Introduction

The development in numerical analysis has propelled interest in difference equations and their relationship to their differential counterparts. The theory of discrete nonlinear boundary value problems has often been connected (*e.g.* Gaines [1]) to the study of corresponding topics in differential equations and the investigation of the differences between the two approaches. This spirit remains in the recent publications (see *e.g.* Kelley and Peterson [2], Agarwal [3] or Bereanu and Mawhin [4]). This paper can be seen as a part of this research stream. We investigate the nonlinear discrete Sturm-Liouville problems coupled with a nonlinear boundary value condition, transform it into the equivalent operator equation, and use Dancer's bifurcation theorem to obtain the existence of a positive solution.

It is well known that the discrete Sturm-Liouville boundary value problem

$$\begin{aligned} &-\Delta\big[p(k-1)\Delta y(k-1)\big]+q(k)y(k)=f\big(k,y(k)\big), \quad k\in\{1,\dots,N\}=:I,\\ &a_{11}y(0)-a_{12}\Delta y(0)=0, \qquad a_{21}y(N+1)+a_{22}\Delta y(N)=0, \end{aligned} \tag{1.1}$$

has been studied by many authors; see [2–12] and the references therein. Here $\Delta y(k) = y(k+1)-y(k)$ for all $k\in\mathbb{Z}$, $p,q:I\to\mathbb{R}$ are functions, $f:I\times\mathbb{R}\to\mathbb{R}$ is a continuous function.

In 1998, Agarwal and O'Regan [5] studied the existence of solutions of (1.1) by fixed point theorem whenever $p(\cdot)\equiv 1$, $q(\cdot)=0$. In 2000, Atici [6] obtained the existence of positive solutions of (1.1) by the fixed point theorem in cones. Cabada and Otero-Espinar [7], Rodríguez [8, 9], Rodríguez and Abernathy [10], Ma [11], Henderson *et al.* [12], and Anderson *et al.* [13] also studied the discrete Sturm-Liouville problems by various methods. It is worth to point out that Rodríguez and Abernathy [10] studied the existence of solutions of the following boundary value problem of the difference equation:

$$\begin{aligned} &\Delta\big[p(k-1)\Delta x(k-1)\big] + q(k)x(k) + \psi\big(x(k)\big) = G\big(x(k)\big), \quad k \in \{a+1,\dots,b+1\}, \\ &\alpha x(a) + \beta\Delta x(a) + \eta_1(x) = \varphi_1(x), \qquad \gamma x(b+1) + \delta\Delta x(b) + \eta_2(x) = \varphi_2(x), \end{aligned} \tag{1.2}$$

where a and b are integers, $\alpha^2 + \beta^2 \neq 0$, $\gamma^2 + \delta^2 \neq 0$, $\alpha \neq \beta$, $\gamma \neq \delta$, $p : \{a,\dots,b+1\} \to (0,\infty)$, $q : \{a+1,\dots,b+1\} \to \mathbb{R}$, $\psi : \mathbb{R} \to \mathbb{R}$ is continuously differentiable, $G : X \to Y$, $\varphi_1 : X \to \mathbb{R}$ and $\varphi_2 : X \to \mathbb{R}$ are all continuous, $\eta_1, \eta_2 : X \to \mathbb{R}$ are continuously Fréchet differentiable; here X is the set of real-valued functions defined on $\{a,\dots,b+2\}$, Y is the set of real-valued functions defined on $\{a+1,\dots,b+1\}$. Under some hypotheses, they showed that (1.2) has a solution by the Brouwer fixed point theorem.

However, as far as we know, there is very little work to study the existence of positive solutions of second-order difference equation with nonlinear boundary value condition. Motivated by the above works [5–12], we study the global structure of positive solutions of the following discrete boundary value problem:

$$\begin{aligned} &-\Delta\big[p(k-1)\Delta y(k-1)\big] + q(k)y(k) = \lambda a(k) f\big(y(k)\big), \quad k \in I, \\ &-\Delta y(0) + \alpha g\big(y(0)\big) = 0, \qquad \Delta y(N) + \beta g\big(y(N+1)\big) = 0, \end{aligned} \tag{1.3}$$

where $\alpha, \beta \geq 0$ are constants, the functions $p : \{0,1,\dots,N\} \to (0,\infty)$, $q, a : I \to [0,\infty)$ with $a(k) > 0$ on $k \in I$ and functions f, g satisfy the following:

(H1) $f \in C([0,\infty),[0,\infty))$ with $f(s) > 0$ for $s > 0$ and there exist constants $f_0, f_\infty \in (0,\infty)$ and functions $\xi, \zeta \in C([0,\infty))$ such that

$$\begin{aligned} &f(s) = f_0 s + \xi(s), \qquad \xi(s) = o\big(|s|\big) \quad \text{as } s \to 0^+, \\ &f(s) = f_\infty s + \zeta(s), \qquad \zeta(s) = o\big(|s|\big) \quad \text{as } s \to +\infty. \end{aligned}$$

(H2) $g \in C([0,\infty),[0,\infty))$ with $g(s) > 0$ for $s > 0$ and there exist constants $g_0, g_\infty \in (0,\infty)$ and functions $\varrho, \eta \in C([0,\infty))$, such that

$$\begin{aligned} &g(s) = g_0 s + \varrho(s), \qquad \varrho(s) = o\big(|s|\big) \quad \text{as } s \to 0^+, \\ &g(s) = g_\infty s + \eta(s), \qquad \eta(s) = o\big(|s|\big) \quad \text{as } s \to +\infty. \end{aligned}$$

Through careful analysis we have found that the boundary condition in (1.3) is nonlinear but it can be linearized and this makes it possible to establish existence results for positive solutions of (1.3) in terms of the principal eigenvalue of the corresponding linearized problem. Notice that this condition is different from those given in [5, 6].

Let $\hat{I} := \{0,1,\dots,N,N+1\}$, and define $E = \{y \mid y : \hat{I} \to \mathbb{R}\}$ to be the space of all maps from $\hat{I}$ into $\mathbb{R}$. Then it is a Banach space with the norm $\|y\| = \max_{k\in\hat{I}} |y(k)|$.

Let $P := \{y \in E \mid y(k) \geq 0, k \in \hat{I}\}$. Then P is a cone which is normal and has a nonempty interior and $E = \overline{P - P}$.

By the constant λ_1^0 we denote the first eigenvalue of the eigenvalue problem,

$$\begin{aligned} &-\Delta\big[p(k-1)\Delta y(k-1)\big] + q(k)y(k) = \lambda a(k) f_0 y(k), \quad k \in I, \\ &-\Delta y(0) + \alpha g_0 y(0) = 0, \qquad \Delta y(N) + \beta g_0 y(N+1) = 0. \end{aligned} \tag{1.4}$$

By a constant λ_1^∞ we denote the first eigenvalue of the eigenvalue problem,

$$\begin{aligned} &-\Delta\big[p(k-1)\Delta y(k-1)\big] + q(k)y(k) = \lambda a(k) f_\infty y(k), \quad k \in I, \\ &-\Delta y(0) + \alpha g_\infty y(0) = 0, \qquad \Delta y(N) + \beta g_\infty y(N+1) = 0. \end{aligned} \tag{1.5}$$

It is well known (*cf.* Kelly and Peterson [2]) that for $\nu \in \{0, \infty\}$, λ_1^ν is positive and simple, and that it is a unique eigenvalue with positive eigenfunction $\varphi_1^\nu \in E$.

Let $\mathscr{C}$ be the closure of the set

$$\big\{(\lambda, u) \in (0,\infty) \times E \mid (\lambda, u) \text{ is a positive solution of (1.3) in } \mathbb{R} \times E\big\}.$$

Theorem 1.1 *Let* (H1)-(H2) *hold. Then there exists an unbounded, closed, and connected component* $\mathcal{C} \subset (0,\infty) \times E$ *in* $\mathscr{C}$, *which joins* $(\lambda_1^0, 0)$ *with* $(\lambda_1^\infty, \infty)$. *Moreover, if*

$$\lambda_1^\infty < \lambda < \lambda_1^0 \quad \text{or} \quad \lambda_1^0 < \lambda < \lambda_1^\infty \tag{1.6}$$

hold. Then (1.3) *has at least one positive solution.*

Corollary 1.2 *Let* (H1)-(H2) *hold. If*

$$\lambda_1^\infty < 1 < \lambda_1^0 \quad \text{or} \quad \lambda_1^0 < 1 < \lambda_1^\infty$$

hold. Then the problem

$$\begin{aligned} &-\Delta\big[p(k-1)\Delta y(k-1)\big] + q(k)y(k) = a(k) f\big(y(k)\big), \quad k \in I, \\ &-\Delta y(0) + \alpha g\big(y(0)\big) = 0, \qquad \Delta y(N) + \beta g\big(y(N+1)\big) = 0 \end{aligned}$$

has at least one positive solution.

Remark 1.1 Compared with references [5, 6], Theorem 1.1 gives the sharp condition (1.6) for the existence of a positive solution of (1.3). In fact, let us consider the function

$$f(s) = \lambda_1^0 s + \arctan\left(\frac{s^2}{1+s^2}\right),$$

which satisfies $f_0 = f_\infty = \lambda_1^0$, then the nonlinear boundary value problem

$$\begin{aligned} &-\Delta\big[p(k-1)\Delta y(k-1)\big] + q(k)y(k) = f\big(y(k)\big), \quad k \in I, \\ &-\Delta y(0) + \alpha y(0) = 0, \qquad \Delta y(N) + \beta y(N+1) = 0 \end{aligned}$$

has no positive solution.

The rest of this paper is organized as follows. In Section 2, we state some preliminary results and Dancer's bifurcation theorem. It is worth to note that the proof of the main result is based upon Dancer's bifurcation theorem, which is different from the topological degree arguments used in [5, 6, 12, 13]. In Section 3, we reduce (1.3) to a compact operator equation and prove Theorem 1.1 and Corollary 1.2.

2 Preliminaries and Dancer's global bifurcation theorem

Let $\phi(k)$, $\psi(k)$ be the solution of the initial value problem

$$\begin{aligned} &-\Delta\big[p(k-1)\Delta\phi(k-1)\big]+q(k)\phi(k)=0 \quad \text{for } k\in I,\\ &\phi(0)=1, \qquad \Delta\phi(0)=\bar{\alpha}, \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} &-\Delta\big[p(k-1)\Delta\psi(k-1)\big]+q(k)\psi(k)=0 \quad \text{for } k\in I,\\ &\psi(N+1)=1, \qquad \Delta\psi(N)=-\bar{\beta}, \end{aligned} \tag{2.2}$$

respectively, where $\bar{\alpha},\bar{\beta}\in[0,\infty)$. It is easy to compute and show that

(i) $\phi(k)=1+\bar{\alpha}\sum_{s=0}^{k-1}\frac{p(0)}{p(s)}+\sum_{s=1}^{k-1}(\sum_{j=s}^{k-1}\frac{1}{p(j)})q(s)\phi(s)>0$, and ϕ is increasing on $\hat{I}$;

(ii) $\psi(k)=1+\bar{\beta}\sum_{s=k}^{N}\frac{p(N)}{p(s)}+\sum_{s=k+1}^{N}(\sum_{j=k+1}^{N-1}\frac{1}{p(j)})q(s)\psi(s)>0$, and ψ is decreasing on $\hat{I}$.

Lemma 2.1 *Let $h: I\to\mathbb{R}$. Then the linear boundary value problem*

$$\begin{aligned} &-\Delta\big[p(k-1)\Delta y(k-1)\big]+q(k)y(k)=h(k), \quad k\in I,\\ &-\Delta y(0)+\bar{\alpha}y(0)=0, \qquad \Delta y(N)+\bar{\beta}y(N+1)=0 \end{aligned} \tag{2.3}$$

has a solution

$$y(k)=\sum_{s=1}^{N}G(k,s)h(s), \quad k\in\hat{I}, \tag{2.4}$$

where

$$G(k,s)=\begin{cases}\phi(s)\psi(k), & 1\le s\le k\le T+1,\\ \phi(k)\psi(s), & 0\le k\le s\le T.\end{cases} \tag{2.5}$$

Moreover, if $h(k)\ge 0$ and $h\not\equiv 0$ on I, then $y(k)>0$ on $\hat{I}$.

Proof It is a direct consequence of Atici [6, Section 2], so we omit it. □

Let $\omega_1,\omega_2\in(0,\infty)$. Then the linear boundary value problem

$$\begin{aligned} &-\Delta\big[p(k-1)\Delta y(k-1)\big]+q(k)y(k)=0, \quad k\in I,\\ &-\Delta y(0)+\bar{\alpha}y(0)=w_1, \qquad \Delta y(N)+\bar{\beta}y(N+1)=w_2 \end{aligned} \tag{2.6}$$

has a solution

$$y(k)=\frac{\omega_2}{(1+\bar{\beta})\phi(N+1)-\phi(N)}\phi(k)+\frac{\omega_1}{(1+\bar{\alpha})\psi(0)-\psi(1)}\psi(k), \quad k\in\hat{I}. \tag{2.7}$$

From the properties of $\phi(k)$, $\psi(k)$, it follows that

$$y(k)>0, \quad k\in\hat{I}.$$

Let $T : E \to E$ be defined as follows:

$$T[h](k) = \sum_{s=1}^{N} G(k,s)h(s), \quad k \in \hat{I}.$$

By a standard compact operator argument, it is easy to show that T is a compact operator and it is strongly positive, meaning that $Th > 0$ on $\hat{I}$ for any $h \in E$ with the condition that $h \geq 0$ and $h \not\equiv 0$ on I; see [5, 6].

Let $R[\omega_1, \omega_2] : \mathbb{R}^2 \to E$ be defined as

$$R[\omega_1,\omega_2](k) = \frac{\omega_2}{(1+\bar{\beta})\phi(N+1) - \phi(N)}\phi(k) + \frac{\omega_1}{(1+\bar{\alpha})\psi(0) - \psi(1)}\psi(k), \quad k \in \hat{I}.$$

Then $R[\omega_1, \omega_2]$ is a linear bounded function in E.

Suppose that E is a real Banach space with norm $\|\cdot\|$. Let K be a cone in E. A nonlinear mapping $A : [0,\infty) \times K \to E$ is said to be positive if $A([0,\infty) \times K) \subset K$. It is said to be K-completely continuous if A is continuous and maps bounded subsets of $[0,\infty) \times K$ to a precompact subset of E. Finally, a positive linear operator V on E is said to be a linear minorant for A if $A(\lambda, u) \geq \mu V(u)$ for $(\lambda, u) \in [0,\infty) \times K$. If B is a continuous linear operator on E, denote by $r(B)$ the spectrum radius of B. Define

$$C_K(B) = \bigl\{\lambda \in [0,\infty) \mid \text{there exists } u \in K \text{ with } \|u\| = 1 \text{ and } u = \lambda Bu\bigr\}. \tag{2.8}$$

The following lemma will play a very important role in the proof of our main results, which is essentially a consequence of Dancer [14, Theorem 2].

Lemma 2.2 *Assume that*

(i) *K has a nonempty interior and $E = \overline{K - K}$;*

(ii) *$A : [0,\infty) \times K \to E$ is K-completely continuous and positive, $A(\lambda, 0) = 0$ for $\lambda \geq 0$, $A(0,u) = 0$ for $u \in K$ and*

$$A(\lambda, u) = \lambda Bu + F(\lambda, u),$$

where $B : E \to E$ is a strongly positive linear compact operator on E with $r(B) > 0$, $F : [0,\infty) \times K \to E$ satisfies $\|F(\lambda,u)\| = o(\|u\|)$ as $\|u\| \to 0$ locally uniformly in λ.

Then there exists an unbounded connected subset $\mathcal{C}$ of

$$D_K(A) = \bigl\{(\lambda,u) \in [0,\infty) \times K \mid u = A(\lambda,u), u \neq 0\bigr\} \cup \bigl\{\bigl(r(B)^{-1}, 0\bigr)\bigr\}$$

such that $(r(B)^{-1}, 0) \in \mathcal{C}$.

Moreover, if A has a linear minorant V and there exists a $(\mu, y) \in (0,\infty) \times K$ such that $\|y\| = 1$ and $\mu V(y) \geq y$, then $\mathcal{C}$ can be chosen in $D_K(A) \cap ([0,\mu] \times K)$.

3 The proof of the main result

To prove Theorem 1.1, we begin with the reduction of (1.3) to a suitable equation for a compact operator.

From Lemma 2.1 and the compactness of T, let $T_0 : E \to E$ denote the inverse operator of the linear boundary value problem

$$-\Delta\bigl[p(k-1)y(k-1)\bigr] + q(k)y(k) = h(k), \quad k \in I,$$
$$-\Delta y(0) + \alpha g_0 y(0) = 0, \qquad \Delta y(N) + \beta g_0 y(N+1) = 0.$$

Taking into account $\bar{\alpha} = \alpha g_0$, $\bar{\beta} = \beta g_0$, one can repeat the argument of the operator T with some minor changes, and it follows that T_0 is a linear mapping of E compactly into E and it is strongly positive.

Let R_0 be the solution of the linear boundary value problem

$$-\Delta\bigl[p(k-1)y(k-1)\bigr] + q(k)y(k) = 0, \quad k \in I,$$
$$-\Delta y(0) + \alpha g_0 y(0) = -\varrho\bigl(y(0)\bigr), \qquad \Delta y(N) + \beta g_0 y(N+1) = -\varrho\bigl(y(N+1)\bigr).$$

Repeating the argument of $R[\omega_1, \omega_2]$ with some minor changes, it follows that $R_0 : \mathbb{R}^2 \to E$ is a linear, bounded mapping and

$$R_0\bigl[\tau\bigl(-\varrho(y)\bigr)\bigr](k) = \frac{-\varrho(y(N+1))\phi_0(k)}{(1+\beta g_0)\phi_0(N+1) - \phi_0(N)} + \frac{-\varrho(y(0))\psi_0(k)}{(1+\alpha g_0)\psi_0(0) - \psi_0(1)}, \quad k \in \hat{I},$$

here $\tau : \{y(0), y(1), \ldots, y(N+1)\} \to \{y(0), y(N+1)\}$ is the trace operator and $\phi_0(k)$, $\psi_0(k)$ satisfies (2.1) and (2.2) with $\bar{\alpha} = \alpha g_0$, $\bar{\beta} = \beta g_0$, respectively. Then the problem (1.3) is equivalent to the operator equation

$$y(k) = \lambda T_0\bigl[af_0 y + \xi(y)\bigr](k) + R_0\bigl[\tau\bigl(-\varrho(y)\bigr)\bigr](k), \quad k \in \hat{I}. \tag{3.1}$$

Similarly, let $T_\infty : E \to E$ denote the inverse operator of the linear boundary value problem

$$-\Delta\bigl[p(k-1)y(k-1)\bigr] + q(k)y(k) = h(k), \quad k \in I,$$
$$-\Delta y(0) + \alpha g_\infty y(0) = 0, \qquad \Delta y(N) + \beta g_\infty y(N+1) = 0.$$

Then T_∞ is a linear mapping of E compactly into E and it is strongly positive. Let R_∞ be the solution of the linear boundary value problem

$$-\Delta\bigl[p(k-1)y(k-1)\bigr] + q(k)y(k) = 0, \quad k \in I,$$
$$-\Delta y(0) + \alpha g_\infty y(0) = -\eta\bigl(y(0)\bigr), \qquad \Delta y(N) + \beta g_\infty y(N+1) = -\eta\bigl(y(N+1)\bigr).$$

Then $R_\infty : \mathbb{R}^2 \to E$ is a linear mapping bounded mapping and

$$R_\infty\bigl[\tau\bigl(-\eta(y)\bigr)\bigr](k) = \frac{-\eta(y(N+1))\phi_\infty(k)}{(1+\beta g_\infty)\phi_\infty(N+1) - \phi_\infty(N)} + \frac{-\eta(y(0))\psi_\infty(k)}{(1+\alpha g_\infty)\psi_\infty(0) - \psi_\infty(1)}, \quad k \in \hat{I},$$

here $\phi_\infty(k)$, $\psi_\infty(k)$ satisfies (2.1) and (2.2) with $\bar{\alpha} = \alpha g_\infty$, $\bar{\beta} = \beta g_\infty$, respectively. Furthermore, the problem (1.3) is also equivalent to the operator equation

$$y(k) = \lambda T_\infty\bigl[af_\infty y + \zeta(y)\bigr](k) + R_\infty\bigl[\tau\bigl(-\eta(y)\bigr)\bigr](k), \quad k \in \hat{I}. \tag{3.2}$$

From (H1) and (H2), it follows that

$$\lim_{|s|\to 0}\frac{\xi(s)}{s}=0,\qquad \lim_{|s|\to 0}\frac{\varrho(s)}{s}=0, \tag{3.3}$$

$$\lim_{|s|\to \infty}\frac{\zeta(s)}{s}=0,\qquad \lim_{|s|\to \infty}\frac{\eta(s)}{s}=0. \tag{3.4}$$

Let $\bar{\zeta}(r)=\max\{|\zeta(s)| \mid 0\le s\le r\}$, $\bar{\eta}(r)=\max\{|\eta(s)| \mid 0\le s\le r\}$. Then $\bar{\zeta}$ and $\bar{\eta}$ are nondecreasing and satisfy

$$\lim_{|s|\to \infty}\frac{\bar{\zeta}(s)}{s}=\lim_{|s|\to \infty}\frac{\bar{\eta}(s)}{s}=0.$$

Let us consider

$$y=\lambda T_0\bigl[af_0y+\xi(y)\bigr]+R_0\bigl[\tau\bigl(-\varrho(y)\bigr)\bigr]=:A(\lambda,y) \tag{3.5}$$

as a bifurcation problem from the trivial solution $u\equiv 0$.

Define the linear operator B

$$By(k):=T_0[af_0y](k),\quad k\in\hat{I}.$$

It is easy to verify that $B:P\to P$ is completely continuous and strongly positive on E. From [15, Theorem 19.3], it follows that $\lambda_1^0=[r(B)]^{-1}$. Define $F:[0,\infty)\times E\to E$ by

$$F(\lambda,y):=\lambda T_0\bigl[\xi(y)\bigr]+R_0\bigl[\tau\bigl(-\varrho(y)\bigr)\bigr],$$

then we have from (3.3)

$$\bigl\|F(\lambda,y)\bigr\|=o\bigl(\|y\|\bigr)\quad\text{locally uniform in }\lambda.$$

So, we imply that if (λ,u) with $\lambda>0$ is a nontrivial solution of (3.5), then $y\in\operatorname{int}P$. Combining this with Lemma 2.2, we conclude that there exists an unbounded connected subset $\mathcal{C}$ of the set

$$\bigl\{(\lambda,y)\in[0,\infty)\times P\mid y=A(\lambda,y),y\in\operatorname{int}P\bigr\}\cup\bigl\{\bigl(\lambda_1^0,0\bigr)\bigr\}$$

such that $(\lambda_1^0,0)\in\mathcal{C}$.

Proof of Theorem 1.1 It is clear that any solution of (3.5) of the form (λ,y) yields a solution y of (1.3). We will show that $\mathcal{C}$ joins $(\lambda_1^0,0)$ to $(\lambda_1^\infty,\infty)$.

Let $(\mu_n,y_n)\in\mathcal{C}$ satisfy

$$|\mu_n|+\|y_n\|\to\infty,\quad n\to\infty.$$

Then $\mu_n>0$ for all $n\in\mathbb{N}$ since $y=0$ is the only solution for (3.5) (*i.e.* (3.2), since (3.2) and (3.5) are equivalent to (1.3)) for $\lambda=0$.

In fact, suppose on the contrary that y is a nontrivial solution of the problem

$$-\Delta\big[p(k-1)\Delta y(k-1)\big]+q(k)y(k)=0,\quad k\in I,$$
$$-\Delta y(0)+\alpha g\big(y(0)\big)=0,\qquad \Delta y(N)+\beta g\big(y(N+1)\big)=0,$$

then y satisfies the linear boundary value problem

$$-\Delta\big[p(k-1)\Delta y(k-1)\big]+q(k)y(k)=0,\quad k\in I,$$
$$-\Delta y(0)+\tilde{\alpha}y(0)=0,\qquad \Delta y(N)+\tilde{\beta}y(N+1)=0,$$

here $\tilde{\alpha}=\alpha\frac{g(y(0))}{y(0)}$, $\tilde{\beta}=\beta\frac{g(y(N+1))}{y(N+1)}$. This together with (H2) and [6, Lemma 2.2] implies that $y\equiv 0$, which is a contradiction. Therefore, (3.5) with $\lambda=0$ has only a trivial solution.

Case 1 $\lambda_1^\infty<\lambda<\lambda_1^0$.

In this case, we show that

$$\big(\lambda_1^\infty,\lambda_1^0\big)\subseteq\big\{\lambda\in\mathbb{R}\mid\exists(\lambda,y)\in\mathcal{C}\big\}.$$

We divide the proof into two steps.

Step 1. We show that if there exists a constant number $M>0$ such that

$$\mu_n\subset(0,M],\tag{3.6}$$

then $\mathcal{C}$ joins $(\lambda_1^0,0)$ with $(\lambda_1^\infty,\infty)$.

From (3.6), we have $\|y_n\|\to\infty$ as $n\to\infty$. We divide the equation

$$y_n=\mu_nT_\infty\big[af_\infty y_n+\zeta(y_n)\big]+R_\infty\big[\tau\big(-\eta(y_n)\big)\big]\quad\text{in }\hat{I}$$

by $\|y_n\|$ and let $v_n=\frac{y_n}{\|y_n\|}$. Since v_n is bounded in E, choosing a subsequence and relabeling if necessary, we see that $v_n\to\bar{v}$ for some $\bar{v}\in E$ with $\|\bar{v}\|=1$. Moreover, from (3.4) and the fact that $\bar{\zeta}$ and $\bar{\eta}$ are nondecreasing, we have

$$\lim_{n\to\infty}\frac{|\zeta(y_n)|}{\|y_n\|}=\lim_{n\to\infty}\frac{|\eta(y_n)|}{\|y_n\|}=0,$$

since $\lim_{n\to\infty}\frac{|\zeta(y_n)|}{\|y_n\|}\le\lim_{n\to\infty}\frac{\bar{\zeta}(|y_n|)}{\|y_n\|}\le\lim_{n\to\infty}\frac{\bar{\zeta}(\|y_n\|)}{\|y_n\|}$ and $\lim_{n\to\infty}\frac{|\eta(y_n)|}{\|y_n\|}\le\lim_{n\to\infty}\frac{\bar{\eta}(|y_n|)}{\|y_n\|}\le\lim_{n\to\infty}\frac{\bar{\eta}(\|y_n\|)}{\|y_n\|}$. Thus

$$\bar{v}=\bar{\mu}T_\infty[af_\infty\bar{v}],$$

where $\bar{\mu}=\lim_{n\to\infty}\mu_n$, again choosing a subsequence and relabeling if necessary. So it follows that

$$-\Delta\big[p(k-1)\Delta\bar{v}(k-1)\big]+q(k)\bar{v}(k)=\bar{\mu}af_\infty\bar{v}(k),\quad k\in I,$$
$$-\Delta\bar{v}(0)+\alpha g_\infty\bar{v}(0)=0,\qquad \Delta\bar{v}(N)+\beta g_\infty\bar{v}(N+1)=0.$$

Since $\|\bar{v}\| = 1$, and $\bar{v} \geq 0$, the strong positivity of T_∞ ensures that $\bar{v} > 0$ on $\bar{I}$. Therefore, $\mu = \lambda_1^\infty$, and accordingly, $\mathcal{C}$ joins $(\lambda_1^0, 0)$ to $(\lambda_1^\infty, \infty)$.

Step 2. We show that there exists a constant M such that $\mu_n \in (0, M]$ for all n.

By Lemma 2.2, we only need to show that A has a linear minorant V and there exists a $(\mu, y) \in (0, \infty) \times P$ such that $\|y\| = 1$ and $\mu V(y) \geq y$.

From (H1) and (H2), there exist constants $\kappa_1, \kappa_2 \in (0, \infty)$ such that

$$f(y) \geq \kappa_1 y, \quad \text{and} \quad g(y) \leq \kappa_2 y \quad \text{for any } y \geq 0. \tag{3.7}$$

By the same method as used for defining T_0 and R_0, we may define T^* and R^* as follows: Let $T^* : E \to E$ denote the inverse operator of the linear boundary value problem

$$\begin{aligned} &-\Delta\big[p(k-1)y(k-1)\big] + q(k)y(k) = h(k), \quad k \in I, \\ &-\Delta y(0) + \alpha\kappa_2 y(0) = 0, \qquad \Delta y(N) + \beta\kappa_2 y(N+1) = 0. \end{aligned}$$

Then T^* is a linear mapping of E compactly into E and it is strong positive. Let R^* be the solution of the linear boundary value problem

$$\begin{aligned} &-\Delta\big[p(k-1)y(k-1)\big] + q(k)y(k) = 0, \quad k \in I, \\ &-\Delta y(0) + \alpha\kappa_2 y(0) = \chi\big(y(0)\big), \qquad \Delta y(N) + \beta\kappa_2 y(N+1) = \chi\big(y(N+1)\big), \end{aligned}$$

where $\chi(y) = \kappa_2 y - g(y)$. Then $R^* : \mathbb{R}^2 \to E$ is a linear, bounded mapping and

$$R^*\big[\tau\big(\chi(y)\big)\big](k) = \frac{\chi(y(N+1))\phi_*(k)}{(1+\beta\kappa_2)\phi_*(N+1) - \phi_*(N)} + \frac{\chi(y(0))\psi_*(k)}{(1+\alpha\kappa_2)\psi_*(0) - \psi_*(1)}, \quad k \in \hat{I},$$

where $\phi_*(k)$, $\psi_*(k)$ satisfies (2.1) and (2.2) with $\bar{\alpha} = \alpha\kappa_2$, $\bar{\beta} = \beta\kappa_2$, respectively.

Moreover, the problem (1.3) can be rewritten as the operator equation

$$y(k) = \lambda T^*\big[af(y)\big](k) + R^*\big[\tau\big(\chi(y)\big)\big](k), \quad k \in \hat{I}. \tag{3.8}$$

Thus

$$\begin{aligned} A(\lambda, y) &= \lambda T^*\big[af(y)\big] + R^*\big[\tau\big(\chi(y)\big)\big] \\ &\geq \lambda T^*[a\kappa_1 y]. \end{aligned}$$

Choose

$$V(y) := T^*[a\kappa_1 y](k) \quad \text{in } \hat{I}.$$

Then V is a linear minorant of A. Let λ_1^* be the eigenvalue of the linear problem

$$\begin{aligned} &-\Delta\big[p(k-1)y(k-1)\big] + q(k)y(k) = \lambda\kappa_1 y(k), \quad k \in I, \\ &-\Delta y(0) + \alpha\kappa_2 y(0) = 0, \qquad \Delta y(N) + \beta\kappa_2 y(N+1) = 0, \end{aligned}$$

and let $\varphi_1^* \in P$ be the corresponding eigenfunction. Then

$$\lambda_1^* V(\varphi_1^*) = \varphi_1^*.$$

Therefore we have from Lemma 2.2

$$|\mu_n| \le \lambda_1^*.$$

Case 2 $\lambda_1^0 < \lambda < \lambda_1^\infty$.

In this case, if $(\mu_n, y_n) \in \mathcal{C}$ is such that

$$\lim_{n\to\infty} (\mu_n + y_n) = +\infty$$

and $\lim_{n\to\infty} \mu_n = \infty$, then

$$(\lambda_1^0, \lambda_1^\infty) \subseteq \{\lambda \in \mathbb{R} \mid \exists(\lambda, y) \in \mathcal{C}\}$$

and moreover,

$$(\{\lambda\} \times E) \cap \mathcal{C} \ne \emptyset.$$

If there exists $M > 0$, such that for all $n \in \mathbb{N}$, $\mu_n \in (0, M]$. Applying a similar argument to that used in Step 1 of Case 1, after taking a subsequence and relabeling if necessary, it follows that

$$(\mu_n, y_n) \to (\lambda_1^\infty, \infty), \quad \text{as } n \to \infty.$$

Again $\mathcal{C}$ joins $(\lambda_1^0, 0)$ to $(\lambda_1^\infty, \infty)$ and the result follows. □

Proof of Corollary 1.2 It is a direct consequence of Theorem 1.1, so we omit it. □

Example Let us consider the following boundary value problem of the difference equation:

$$\begin{aligned} &-\Delta^2 u(k-1) = \lambda f\big(u(k)\big), \quad k \in \{1,2,3,4,5\}, \\ &-\Delta u(0) + g\big(u(0)\big) = 0, \qquad \Delta u(5) + 2g\big(u(6)\big) = 0, \end{aligned} \tag{3.9}$$

where

$$f(u) = \begin{cases} \frac{1}{4}u + \frac{u^2}{2}, & 0 \le u \le 4, \\ \frac{10u}{u^2-8} + u, & u \ge 4, \end{cases} \quad \text{and} \quad g(u) = \begin{cases} 2u, & 0 \le u \le 1, \\ (3u-2) + \frac{1}{u^2}, & u \ge 1. \end{cases}$$

Obviously, the conditions (H1), (H2) are satisfied, furthermore $f_0 = \frac{1}{4}$, $f_\infty = 1$, $g_0 = 2$, $g_\infty = 3$, and $\lambda_1^0 \approx 0.000673$, $\lambda_1^\infty \approx 0.000072$. From Theorem 1.1, the problem (3.9) has at least one positive solution u on $\hat{I}$ if $0.000072 < \lambda < 0.000673$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
YL and RM completed the main study, YL carried out the results of this article and drafted the manuscript and RM checked the proofs and verified the calculation. All the authors read and approved the manuscript.

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References

1. Gaines, R: Difference equations associated with boundary value problems for second order nonlinear ordinary differential equations. SIAM J. Numer. Anal. **11**, 411-434 (1974)
2. Kelley, WG, Peterson, AC: Difference Equations: An Introduction with Applications, 2nd edn. Harcourt/Academic Press, San Diego (2001)
3. Agarwal, RP: Difference Equations and Inequalities: Theory, Methods, and Applications. 2nd edn. Monographs and Textbooks in Pure and Applied Mathematics, vol. 228. Dekker, New York (2000)
4. Bereanu, C, Mawhin, J: Existence and multiplicity results for periodic solutions of nonlinear difference equations. J. Differ. Equ. Appl. **12**, 677-695 (2006)
5. Agarwal, RP, O'Regan, D: Fixed-point approach for nonlinear discrete boundary value problems. Comput. Math. Appl. **36**(10-12), 115-121 (1998)
6. Atici, FM: Existence of positive solutions of nonlinear discrete Sturm-Liouville problems. Boundary value problems and related topics. Math. Comput. Model. **32**(5-6), 599-607 (2000)
7. Cabada, A, Otero-Espinar, V: Fixed sign solutions of second-order difference equations with Neumann boundary conditions. Comput. Math. Appl. **45**(6-9), 1125-1136 (2003)
8. Rodríguez, J: On the solvability of nonlinear discrete boundary value problems. J. Differ. Equ. Appl. **9**(9), 863-867 (2003)
9. Rodríguez, J: Nonlinear discrete Sturm-Liouville problems. J. Math. Anal. Appl. **308**(1), 380-391 (2005)
10. Rodríguez, J, Abernathy, Z: Nonlinear discrete Sturm-Liouville problems with global boundary conditions. J. Differ. Equ. Appl. **18**(3), 431-445 (2012)
11. Ma, R: Bifurcation from infinity and multiple solutions for some discrete Sturm-Liouville problems. Comput. Math. Appl. **54**, 535-543 (2007)
12. Henderson, J, Ntouyas, S, Purnaras, I: Positive solutions for systems of nonlinear discrete boundary value problems. J. Differ. Equ. Appl. **15**, 895-912 (2009)
13. Anderson, DR, Avery, RI, Henderson, J, Liu, X, Lyons, JW: Existence of a positive solution for a right focal discrete boundary value problem. J. Differ. Equ. Appl. **17**, 1635-1642 (2011)
14. Dancer, EN: Global solution branches for positive mappings. Arch. Ration. Mech. Anal. **52**, 181-192 (1973)
15. Deimling, K: Nonlinear Functional Analysis. Springer, Berlin (1988)

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