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Global well-posedness of a class of stochastic equations with jumps

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Abstract

The existence and uniqueness of a mild solution to stochastic equations with jumps are established, a stochastic Fubini theorem and a type of Burkholder-Davis-Gundy inequality are proved, and the two formulas are used to study the regularity property of the mild solution of a general stochastic evolution equation perturbed by jump processes. As applications, the regularity of a stochastic heat equation with jump is given.

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1 Introduction

In recent years, there have been many monographs concerning stochastic partial differential equations with Lévy jump and their applications in physics, economics, statistical mechanics, fluid dynamics and finance etc. For this theory and its applications, one can see [1–3] and references therein. In this article, the existence, uniqueness, regularity for the mild solution of stochastic partial differential equations with Lévy jump are studied. There are a lot of works dealing with existence and uniqueness for stochastic partial differential equations with jump processes. In [4], the existence and uniqueness for solutions of stochastic reaction diffusion equations driven by Poisson random measures are obtained. In [5], Malliavin calculus is applied to study the absolute continuity of the law of the solutions of stochastic reaction diffusion equations driven by Poisson random measures. In [6], a minimal solution is obtained for the stochastic heat equation driven by non-negative Lévy noise with coefficients of polynomial growth. In [7], a weak solution is established for the stochastic heat equation driven by stable noise with coefficients of polynomial growth. In [8], the existence and uniqueness for solutions of stochastic generalized porous media equations with Lévy jump are obtained. In [9], the strong solutions to a large class of stochastic equations with Lévy noise are obtained in variational framework. And it is shown in [9] that the results can be applied to stochastic reaction-diffusion equations, Burgers-type equations, 2D Navier-Stokes equations, p -Laplace equations and porous media equations with locally monotone perturbations.

The main aim of this paper is to study the existence, uniqueness and regularity of the stochastic equation

$$dX(t) = [A(X(t)) + B(X(t))] dt + Q(X(t)) dW(t) + \int_Z F(X(t-), x) \tilde{N}(dt, dx). \quad (1.1)$$

In [10], the intensity measure λ of $\tilde{N}(dt, dx)$ is finite, while the intensity measure in this article is σ -finite, and also the classical Lipschitz condition (2.5) in [10] is relaxed to condition (A.5) in this article. The author of this article proves the existence and uniqueness of a mild solution of (1.1), the continuity of the solution with respect to initial data. And then a stochastic Fubini theorem is established for the compensated Poisson random measure whose intensity measure is σ -finite compared to a finite case in [10]. Furthermore, a new type of the Burkholder-Davis-Gundy inequality, which is more precise than Lemma 2.2 in [10], is gotten. The two formulas are basic in stochastic analysis. Using the formulas, the author gets the regularity property of a mild solution of (1.1) without conditions (2.15) and (2.16) in [10] which are critical there.

The article is organized as follows. In Section 2, we present the framework. Existence, uniqueness and regularity are proved in Section 3. In Section 4, two examples including stochastic heat equations with Lévy jump are given.

2 Some preliminaries

Let $(U, \|\cdot\|_U)$, $(H, \|\cdot\|)$ be separable Hilbert spaces and Z be a Banach space. Let $L_2(U, H)$ denote the space of all Hilbert-Schmidt operators from U to H , and set $\|\cdot\|_{L_2} := \|\cdot\|_{L_2(U, H)}$. Let $W(t)$, $t \geq 0$ be a cylindrical Wiener process in U on a probability space (Ω, \mathcal{F}, P) with a normal filtration \mathcal{F}_t , $t \geq 0$ and let $N(dt, dz)$ be a Poisson random measure associated with the compensated (\mathcal{F}_t) -martingale measure defined as $\tilde{N}(dt, dz) := N(dt, dz) - \lambda(dz) dt$, where $\lambda(dz)$ defined on a measurable space Z is the intensity measure of $N(dt, dz)$. In this article we study the following equation:

$$dX(t) = [AX(t) + B(X(t))] dt + Q(X(t)) dW(t) + \int_Z F(X(t-), z) \tilde{N}(dt, dz), \quad t \in [0, T] \tag{2.1}$$

with initial condition $X(0) = \eta$, and the coefficients in equation (2.1) satisfy the following conditions:

- (A.1) $A : D(A) \rightarrow H$ is the infinitesimal generator of a C_0 -semigroup $S(t)$, $t \geq 0$.
- (A.2) $B : H \rightarrow H$ is $\mathcal{B}(H)/\mathcal{B}(H)$ -measurable, and there exists a positive constant C satisfying

$$\|B(x) - B(y)\| \leq C\|x - y\| \quad \text{for all } x, y \in H.$$

- (A.3) $Q : H \rightarrow L(U, H)$ is strongly continuous, *i.e.* the mapping

$$x \rightarrow Q(x)u$$

is continuous from H to H for each $u \in U$.

- (A.4) For all $t \in]0, T]$ and $x \in H$, we have

$$S(t)Q(x) \in L_2(U, H).$$

- (A.5) There is a square integrable mapping $M : [0, +\infty[\rightarrow [0, \infty[$ such that

$$\|S(t)(Q(x) - Q(y))\|_{L_2} \leq M(t)\|x - y\|$$

and

$$\|S(t)Q(x)\|_{L_2} \leq M(t)(1 + \|x\|)$$

for all $t \in]0, T]$ and $x, y \in H$.

(A.6) There is a positive constant C like (A.2) such that

$$\int_Z \|F(x, z) - F(y, z)\|^2 \lambda(dz) \leq C\|x - y\|^2$$

and

$$\int_Z \|F(x, z)\|^2 \lambda(dz) \leq C(1 + \|x\|^2),$$

where $F : H \times Z \rightarrow H$ is $\mathcal{B}(H) \times \mathcal{B}(Z)/\mathcal{B}(H)$ measurable.

For fixed $T > 0$, we define:

$$\mathcal{H}^2(T, H) := \left\{ H \text{ valued predictable process } X \text{ with} \right. \\ \left. \|X\|_{H^2} := \sup_{t \in [0, T]} (E(\|X(t)\|^2))^{1/2} < \infty \right\}.$$

Obviously, $(\mathcal{H}^2(T, H), \|\cdot\|_{H^2})$ is a Banach space. For technical reasons we define $\|\cdot\|_{2, \lambda, T}$, $\lambda \geq 0$ on $\mathcal{H}^2(T, H)$ as

$$\|X\|_{2, \lambda, T} := \sup_{t \in [0, T]} e^{-\lambda t} (E(\|X(t)\|^2))^{1/2}.$$

Then, for $X \in \mathcal{H}^2(T, H)$ and $\lambda \geq 0$, we have

$$\|X\|_{2, \lambda, T} \leq \|X\|_{H^2} \leq e^{\lambda T} \|X\|_{2, \lambda, T}.$$

For the reader's convenience, before giving our main results, we cite some theorems here which will be needed later.

Contraction theorem (i) *Let $(E, \|\cdot\|_E)$ and $(\Lambda, \|\cdot\|_\Lambda)$ be two Banach spaces. Let the mapping $G : \Lambda \times E \rightarrow E$ satisfy*

$$\|G(\lambda, x) - G(\lambda, y)\|_E \leq \alpha \|x - y\|_E, \quad \alpha \in [0, 1[$$

for all $\lambda \in \Lambda$ and $x, y \in E$. Then there exists exactly one mapping $\varphi : \Lambda \rightarrow E$ such that

$$\varphi(\lambda) = G(\lambda, \varphi(\lambda))$$

for all $\lambda \in \Lambda$.

(ii) *If we assume in addition that the mapping $\lambda \rightarrow G(\lambda, x)$ is continuous from Λ to E for all $x \in E$, we get that $\varphi : \Lambda \rightarrow E$ is continuous.*

(iii) If the mapping $\lambda \rightarrow G(\lambda, x)$ is not only continuous from Λ to E for all $x \in E$, but there even exists an $L \geq 0$ such that

$$\|G(\lambda, x) - G(\tilde{\lambda}, x)\|_E \leq L\|\lambda - \tilde{\lambda}\|_\Lambda$$

for all $x \in E$, then the mapping $\varphi : \Lambda \rightarrow E$ is Lipschitz continuous.

Theorem [11] If a process Ψ is adapted to \mathcal{F}_t , $t \in [0, T]$, and stochastically continuous with values in a Banach space E , then there exists a predictable version of Ψ .

Definition 2.1 An H -valued predictable process $X(t)$, $t \in [0, T]$ is called a mild solution of (2.1) if

$$\begin{aligned} X(t) = & S(t)\eta + \int_0^t S(t-s)B(X(s)) ds + \int_0^t S(t-s)Q(X(s)) dW(s) \\ & + \int_0^t \int_Z S(t-s)F(X(s-), z)\tilde{N}(ds, dz) \end{aligned}$$

for each $t \in [0, T]$.

3 Existence, uniqueness and regularity

Theorem 3.1 Assume that conditions from (A.1) to (A.6) hold, then there exists a unique mild solution $X(\eta) \in \mathcal{H}^2(T, H)$ of (2.1) with the initial condition

$$\eta \in L^2(\Omega, \mathcal{F}_0; H) =: L_0^2.$$

In addition we get that the mapping

$$\begin{aligned} X : L_0^2 &\rightarrow \mathcal{H}^2(T, H), \\ \eta &\rightarrow X(\eta) \end{aligned}$$

is Lipschitz continuous.

Proof Fix $t \in [0, T]$, $\eta \in L_0^2$ and $X \in \mathcal{H}^2(T, H)$, and define

$$\begin{aligned} \mathcal{F}(\eta, X)(t) := & S(t)\eta + \int_0^t S(t-s)B(X(s)) ds + \int_0^t S(t-s)Q(X(s)) dW(s) \\ & + \int_0^t S(t-s)F(X(s-), z)\tilde{N}(ds, dz). \end{aligned}$$

In the following two steps, it comes to prove that

$$\mathcal{F} : L_0^2 \times \mathcal{H}^2(T, H) \rightarrow \mathcal{H}^2(T, H).$$

The first step: It is proved that the mapping \mathcal{F} is well defined.

1. Let $h \in H$, as

$$\langle 1_{[0,t]}(s)S(t-s)B(X(s)), h \rangle_H = \langle B(X(s)), 1_{[0,t]}(s)S^*(t-s)h \rangle_H.$$

By the strongly continuity of $S(t)$, we know $1_{[0,t]}(s)S(t-s)B(X(s))$ is predictable. And together with (A.2), we know the Bochner integral $\int_0^t S(t-s)B(X(s)) ds, t \in [0, T]$, is well defined.

2. Let $\{e_i\}$ and $\{h_i\}$ be an orthonormal basis respectively of U and H . Because

$$\langle h_i, 1_{[0,t]}(s)S(t-s)Q(X(s))e_j \rangle_H$$

is predictable, and by (A.5), it is easy to see $1_{[0,t]}(s)S(t-s)Q(X(s)) \in \mathcal{N}_{\tilde{W}}^2(0, T)$. So, the stochastic integral $\int_0^t S(t-s)Q(X(s)) dW(s)$ is well defined.

3. Similarly to 1, it is easy to see that $1_{[0,t]}(s)S(t-s)F(X(s), z)$ is predictable, and by (A.6) we have

$$E \left\| \int_0^t \int_Z S(t-s)F(X(s-), z) \tilde{N}(ds, dz) \right\|^2 < +\infty.$$

Therefore the stochastic integral $\int_0^t \int_Z S(t-s)F(X(s-), z) \tilde{N}(ds, dz)$ is well defined.

The second step: We prove that $\mathcal{F}(\eta, X) \in \mathcal{H}^2(T, H)$ for all $\eta \in L^2_0$ and $X \in \mathcal{H}^2(T, H)$.

1. Obviously $S(t)\eta, t \in [0, T]$, is an element of $\mathcal{H}^2(T, H)$.

2. There is a version of the second summand $\int_0^t S(t-s)B(X(s)) ds, t \in [0, T]$, which is an element of $\mathcal{H}^2(T, H)$.

Similarly to the argument of the first step, we know $\int_0^t S(t-s)B(X(s)) ds$ is adapted for $(\mathcal{F}_t)_{t \geq 0}$. Then we will show $\int_0^t S(t-s)B(X(s)) ds$ is continuous a.s. Let $0 \leq t < t_0, r \in]1, 2]$. By (A.2), we have

$$\begin{aligned} & \left\| \int_0^t S(t-s)B(X(s)) ds - \int_0^{t_0} S(t_0-s)B(X(s)) ds \right\| \\ & \leq \left\| \int_0^t (S(t-s) - S(t_0-s))B(X(s)) ds \right\| + \left\| \int_t^{t_0} S(t_0-s)B(X(s)) ds \right\| \\ & \leq \left\| \int_0^{t/r} (S(t-rs) - S(t_0-rs))S((r-1)s)B(X(s)) ds \right\| + \left\| \int_{t/r}^t S(t-s)B(X(s)) ds \right\| \\ & \quad + \left\| \int_{t/r}^t S(t_0-s)B(X(s)) ds \right\| + \left\| \int_t^{t_0} S(t_0-s)B(X(s)) ds \right\| \\ & \leq \left\| \int_0^{t/r} (S(t-rs) - S(t_0-rs))S((r-1)s)B(X(s)) ds \right\| + \bar{C}M(T) \int_{t/r}^t (1 + \|X(s)\|) ds \\ & \quad + \bar{C}M(T) \int_{t/r}^t (1 + \|X(s)\|) ds + \bar{C}M(T) \int_t^{t_0} (1 + \|X(s)\|) ds \\ & \leq \int_0^{t/r} \|(S(t-rs) - S(t_0-rs))S((r-1)s)B(X(s))\| ds \\ & \quad + 4\bar{C}M(T)(T)^{\frac{1}{2}} \left(\int_{t/r}^t (1 + \|X(s)\|^2) ds \right)^{1/2} \\ & \quad + 2\bar{C}M(T)(T)^{\frac{1}{2}} \left(\int_t^{t_0} (1 + \|X(s)\|^2) ds \right)^{1/2}. \end{aligned}$$

In the above inequalities, we set $M_T := \sup_{t \in [0, T]} \|S(t)\|_{L(H)}$ and $\bar{C} := \max\{C, \|B(0)\|\}$, where C is as in (A.2). As we can derive that

$$\begin{aligned} \int_0^{t/r} \|S((r-1)s)B(X(s))\| ds &\leq \bar{C}M(T) \int_0^{t/r} (1 + \|X(s)\|) ds \\ &\leq \bar{C}M(T) \left(\int_0^t 2(1 + \|X(s)\|^2) ds \right)^{1/2} < +\infty. \end{aligned}$$

Letting $t \rightarrow t_0$, and then $r \rightarrow 1$, by (A.2), strong continuity of $S(t)$ and dominated convergence theorem, we know, for all most $w \in \Omega$, we have

$$\int_0^t S(t-s)B(X(s)) ds \rightarrow \int_0^{t_0} S(t_0-s)B(X(s)) ds \quad \text{for all } 0 \leq t < t_0 \leq T.$$

Similarly we can get the same conclusion for $0 \leq t_0 < t \leq T$ when $t \downarrow t_0$, so we have proved $\int_0^t S(t-s)B(X(s)) ds$ is continuous a.s. for $t \in [0, T]$.

As we have

$$\begin{aligned} E \left(\left\| \int_0^t S(t-s)B(X(s)) ds \right\|^2 \right) &\leq E \int_0^t (\|S(t-s)B(X(s))\|^2) ds T \\ &\leq (2T)(\bar{C}M(T))^2 E \int_0^t (1 + \|X(s)\|^2) ds \\ &\leq 2(T\bar{C}M(T))^2 (1 + \|X\|_{H^2}^2). \end{aligned}$$

Therefore we have proved $\int_0^t S(t-s)B(X(s)) ds \in \mathcal{H}^2(T, H)$.

3. There is a version of $\int_0^t S(t-s)Q(X(s)) dW(s)$, $t \in [0, T]$, which is in $\mathcal{H}^2(T, H)$.

First we fix $r \in]1, 2]$, then

$$\int_0^{t/r} S(t-s)Q(X(s)) dW(s) = \int_0^{t/r} S(t-rs)S((r-1)s)Q(X(s)) dW(s).$$

Let us define

$$\Phi^r := 1_{[0, T]}(s)S((r-1)s)Q(X(s)).$$

Then it is clear that $\Phi^r \in \mathcal{N}_{\bar{W}}^2(0, T)$. By (A.2), we have

$$\begin{aligned} E \int_0^T \|\Phi^r(s)\|_{L_2}^2 ds &= E \int_0^T \|1_{[0, T]}(s)S((r-1)s)Q(X(s))\|_{L_2}^2 ds \\ &\leq E \int_0^T 2(M((r-1)s))^2 (1 + \|X(s)\|^2) ds \\ &\leq \int_0^T 2(M((r-1)s))^2 ds \left(1 + \sup_{t \in [0, T]} E(\|X(t)\|^2) \right) \\ &\leq \frac{1}{r-1} \int_0^{(r-1)T} 2(M(s))^2 ds \left(1 + \sup_{t \in [0, T]} E(\|X(t)\|^2) \right). \end{aligned}$$

Similarly we have $1_{[0,t]}(s)S(t-s)Q(X(s)) \in \mathcal{N}_W^2(0, T)$ and $\int_0^t S(t-s)Q(X(s)) dW(s)$ is \mathcal{F}_t -measurable. Next we are going to show

$$t \rightarrow \int_0^t S(t-s)Q(X(s)) dW(s)$$

is continuous in the mean square and therefore stochastically continuous. Indeed, let $0 \leq t < t_0 \leq T$, we get

$$\begin{aligned} & E\left(\left\|\int_0^t S(t-s)Q(X(s)) dW(s) - \int_0^{t_0} S(t_0-s)Q(X(s)) dW(s)\right\|^2\right) \\ & \leq 2E\left(\left\|\int_0^t (S(t_0-s) - S(t-s))Q(X(s)) dW(s)\right\|^2\right) \\ & \quad + 2E\left(\left\|\int_t^{t_0} S(t_0-s)Q(X(s)) dW(s)\right\|^2\right) \\ & = 2E \int_0^t \|(S(t_0-s) - S(t-s))Q(X(s))\|_{L_2}^2 ds + 2E \int_t^{t_0} \|S(t_0-s)Q(X(s))\|_{L_2}^2 ds \\ & = 2E \int_0^{t/r} \|(S(t_0-rs) - S(t-rs))\Phi^r(s)\|_{L_2}^2 ds \\ & \quad + 2E \int_{t/r}^t \|(S(t_0-s) - S(t-s))Q(X(s))\|_{L_2}^2 ds \\ & \quad + 2E \int_t^{t_0} \|S(t_0-s)Q(X(s))\|_{L_2}^2 ds \\ & \leq 2E \int_0^{t/r} \|(S(t_0-rs) - S(t-rs))\Phi^r(s)\|_{L_2}^2 ds + 4E \int_{t/r}^t \|(S(t_0-s))Q(X(s))\|_{L_2}^2 ds \\ & \quad + 4E \int_{t/r}^t \|(S(t-s))Q(X(s))\|_{L_2}^2 ds + 2E \int_t^{t_0} \|S(t_0-s)Q(X(s))\|_{L_2}^2 ds \\ & \leq 2E\left(\int_0^{t/r} \sum_i \|(S(t_0-rs) - S(t-rs))\Phi^r(s)e_i\|^2 ds\right) \\ & \quad + 8E \int_{t/r}^t M^2(t_0-s)(1 + \|X(s)\|^2) ds \\ & \quad + 8E \int_{t/r}^t M^2(t-s)(1 + \|X(s)\|^2) ds + 4E \int_t^{t_0} M^2(t_0-s)(1 + \|X(s)\|^2) ds \\ & = 2E\left(\int_0^{t/r} \sum_i \|(S(t_0-rs) - S(t-rs))\Phi^r(s)e_i\|^2 ds\right) \\ & \quad + 8[1 + \|X\|_{H^2}^2]\left(\int_{t_0-t}^{t_0-t/r} M^2(s) ds + \int_0^{t(r-1)/r} M^2(s) ds\right) \\ & \quad + 4E \int_0^{t_0-t} M^2(s) ds [1 + \|X\|_{H^2}^2] \\ & \rightarrow 0, \quad \text{let } t \rightarrow t_0 \text{ and then } r \rightarrow 1. \end{aligned}$$

The last step follows by a dominated convergence theorem since $\|(S(t_0-rs) - S(t-rs))\Phi^r(s)e_i\|^2 \leq 4M_T^2 \|\Phi^r(s)e_i\|^2$ and $\Phi^r(t) \in \mathcal{N}_W^2(0, T)$, $t \in [0, T]$.

For $0 \leq t_0 < t \leq T$, we can get the same conclusion. Therefore by Theorem [11] we know

$$\int_0^t S(t-s)Q(X(s)) dW(s)$$

has a predictable version for all $t \in [0, T]$. Furthermore, by (A.5), we have

$$\begin{aligned} & \left(E \left(\left\| \int_0^t S(t-s)Q(X(s)) dW(s) \right\|^2 \right) \right)^{1/2} \\ &= \left(\int_0^t (E(\|S(t-s)Q(X(s))\|_{L_2}^2)) ds \right)^{1/2} \\ &\leq \left(\int_0^T M^2(s) ds \right)^{1/2} (1 + \|X\|_{H^2}). \end{aligned}$$

So far we have proved $\int_0^t S(t-s)Q(X(s)) dW(s)$ has a predictable version which is an element in $\mathcal{H}^2(T, H)$.

4. There is a version of $\int_0^t \int_Z S(t-s)F(X(s-), z) \tilde{N}(ds, dz)$, $t \in [0, T]$, which is in $\mathcal{H}^2(T, H)$. It is easy to show $1_{[0,t]}(s)S(t-s)F(X(s-), z)$ is predictable. By (A.6), we have

$$\begin{aligned} & \left(E \left\| \int_0^t \int_Z S(t-s)F(X(s-), z) \tilde{N}(ds, dz) \right\|^2 \right)^{1/2} \\ &= \left\{ E \left(\int_0^t \int_Z 1_{[0,t]}(s) \|S(t-s)F(X(s), z)\|^2 \lambda(dz) ds \right) \right\}^{1/2} \\ &\leq C^{\frac{1}{2}} M(T) \left\{ E \left(\int_0^t (1 + \|X(s)\|^2) ds \right) \right\}^{1/2} \\ &\leq C^{\frac{1}{2}} TM(T)(1 + \|X\|_{H^2}). \end{aligned}$$

Letting $t < t_0$, we have

$$\begin{aligned} & \left\{ E \left\| \int_0^t \int_Z S(t-s)F(X(s-), z) \tilde{N}(ds, dz) - \int_0^{t_0} \int_Z S(t_0-s)F(X(s-), z) \tilde{N}(ds, dz) \right\|^2 \right\}^{1/2} \\ &\leq \left\{ E \left(\left\| \int_0^t \int_Z (S(t-s) - S(t_0-s))F(X(s-), z) \tilde{N}(ds, dz) \right\|^2 \right) \right\}^{1/2} \\ &\quad + \left\{ E \left(\left\| \int_t^{t_0} \int_Z S(t_0-s)F(X(s-), z) \tilde{N}(ds, dz) \right\|^2 \right) \right\}^{1/2} \\ &= \left\{ E \int_0^t \int_Z \|(S(t-s) - S(t_0-s))F(X(s), z)\|^2 \lambda(dz) ds \right\}^{1/2} \\ &\quad + \left\{ E \int_t^{t_0} \int_Z \|S(t_0-s)F(X(s), z)\|^2 \lambda(dz) ds \right\}^{1/2} \\ &\leq \left\{ E \int_0^{t/r} \int_Z \|(S(t-s) - S(t_0-s))F(X(s), z)\|^2 \lambda(dz) ds \right\}^{1/2} \\ &\quad + \left\{ E \int_{t/r}^t \int_Z \|(S(t-s) - S(t_0-s))F(X(s), z)\|^2 \lambda(dz) ds \right\}^{1/2} \end{aligned}$$

$$\begin{aligned}
 & + C^{\frac{1}{2}}M(T) \left\{ \int_t^{t_0} (1 + E(\|X(s)\|^2)) ds \right\}^{1/2} \\
 \leq & \left\{ E \int_0^{t/r} \int_Z \| (S(t-s) - S(t_0-s))F(X(s), z) \|^2 \lambda(dz) ds \right\}^{1/2} \\
 & + 2C^{\frac{1}{2}}M(T) \left\{ E \int_{t/r}^t (1 + \|X(s)\|^2) ds \right\}^{1/2} + C^{\frac{1}{2}}M(T)(t_0 - t)^{\frac{1}{2}} (1 + \|X\|_{H^2}) \\
 \leq & \left\{ E \int_0^{t/r} \int_Z \| (S(t-rs) - S(t_0-rs))S((r-1)s)F(X(s), z) \|^2 \lambda(dz) ds \right\}^{1/2} \\
 & + 2C^{\frac{1}{2}}M(T) \left\{ E \int_{t/r}^t (1 + \|X(s)\|^2) ds \right\}^{1/2} + C^{\frac{1}{2}}M(T)(t_0 - t)^{\frac{1}{2}} (1 + \|X\|_{H^2}).
 \end{aligned}$$

Since

$$E \int_0^{t/r} \int_Z \| S((r-1)s)F(X(s), z) \|^2 \lambda(dz) ds < +\infty,$$

letting $t \rightarrow t_0$ and $r \downarrow 1$, by a dominated convergence theorem, we have the following result:

$$\begin{aligned}
 & \int_0^t \int_Z S(t-s)F(X(s-), z) \tilde{N}(ds, dz) \\
 & \rightarrow \int_0^{t_0} \int_Z S(t_0-s)F(X(s-), z) \tilde{N}(ds, dz) \quad \text{in } L^2(\Omega), \text{ as } t \uparrow t_0.
 \end{aligned}$$

If $t > t_0$, similarly we can get

$$\begin{aligned}
 & \int_0^t \int_Z S(t-s)F(X(s-), z) \tilde{N}(ds, dz) \\
 & \rightarrow \int_0^{t_0} \int_Z S(t_0-s)F(X(s-), z) \tilde{N}(ds, dz) \quad \text{in } L^2(\Omega), \text{ as } t \downarrow t_0.
 \end{aligned}$$

As $\int_0^t \int_Z S(t-s)F(X(s-), z) \tilde{N}(ds, dz)$ is \mathcal{F}_t adapted, thus by Theorem [11] we know

$$\int_0^t \int_Z S(t-s)F(X(s-), z) \tilde{N}(ds, dz)$$

has a predictable version which is an element in $\mathcal{H}^2(T, H)$.

The third step: We are going to show

$$\mathcal{F}(\eta, \cdot) : \mathcal{H}^{2,\lambda}(T, H) \rightarrow \mathcal{H}^{2,\lambda}(T, H)$$

is a contraction mapping for all $\eta \in L_0^2$.

Let $X, \tilde{X} \in \mathcal{H}^2(T, H)$, $\eta \in L_0^2$ and $t \in [0, T]$. Then we have

$$\begin{aligned}
 & \| \mathcal{F}(\eta, X) - \mathcal{F}(\eta, \tilde{X}) \|_{L^2} \\
 & \leq \left\| \int_0^t S(t-s) [B(X(s)) - B(\tilde{X}(s))] ds \right\|_{L^2}
 \end{aligned}$$

$$\begin{aligned}
 & + \left\| \int_0^t S(t-s)[Q(X(s)) - Q(\tilde{X}(s))] dW(s) \right\|_{L^2} \\
 & + \left\| \int_0^t \int_Z S(t-s)[F(X(s-), z) - F(\tilde{X}(s-), z)] \tilde{N}(ds, dz) \right\|_{L^2}, \tag{3.1}
 \end{aligned}$$

where $\| \cdot \|_{L^2} := (E(\| \cdot \|^2))^{1/2}$. For the first term of the right-hand side of (3.1), we have

$$\begin{aligned}
 & \left\| \int_0^t S(t-s)[B(X(s)) - B(\tilde{X}(s))] ds \right\|_{L^2} \\
 & = \left\{ E \left(\left\| \int_0^t S(t-s)[B(X(s)) - B(\tilde{X}(s))] ds \right\|^2 \right) \right\}^{1/2} \\
 & \leq \left\{ E \left(\int_0^t \|S(t-s)[B(X(s)) - B(\tilde{X}(s))]\|^2 ds \right) t \right\}^{1/2} \\
 & \leq M(T) \left\{ E \left(\int_0^t e^{2\lambda s} e^{-2\lambda s} \|X(s) - \tilde{X}(s)\|^2 ds \right) \right\}^{1/2} t^{1/2} \\
 & \leq M(T) \left(\int_0^t e^{2\lambda s} ds \right)^{1/2} \sup_{t \in [0, T]} e^{-\lambda t} (E(\|X(t) - \tilde{X}(t)\|^2))^{1/2} T^{1/2} \\
 & = T^{1/2} M(T) e^{\lambda t} \left(\int_0^t e^{-2\lambda s} ds \right)^{1/2} \|X - \tilde{X}\|_{2, \lambda, T}. \tag{3.2}
 \end{aligned}$$

Dividing by $e^{\lambda t}$ both sides of (3.2), we have

$$\left\| \int_0^t S(t-s)[B(X(s)) - B(\tilde{X}(s))] ds \right\|_{2, \lambda, T} \leq T^{1/2} M(T) \left(\int_0^t e^{-2\lambda s} ds \right)^{1/2} \|X - \tilde{X}\|_{2, \lambda, T}.$$

Obviously $T^{1/2} M(T) (\int_0^t e^{-2\lambda s} ds)^{1/2} \rightarrow 0$ as $\lambda \rightarrow \infty$.

For the second term of (3.1), by the Burkholder-Davis-Gundy inequality and (A.5), we have

$$\begin{aligned}
 & \left\| \int_0^t S(t-s)[Q(X(s)) - Q(\tilde{X}(s))] dW(s) \right\|_{L^2} \\
 & = \left(E \left(\left\| \int_0^t S(t-s)[Q(X(s)) - Q(\tilde{X}(s))] dW(s) \right\|^2 \right) \right)^{1/2} \\
 & \leq \left(\int_0^t (E(\|S(t-s)[Q(X(s)) - Q(\tilde{X}(s))]\|_{L^2}^2)) ds \right)^{1/2} \\
 & \leq \left(\int_0^t M^2(t-s) e^{2\lambda s} e^{-2\lambda s} E\|X(s) - \tilde{X}(s)\|_{L^2}^2 ds \right)^{1/2} \\
 & \leq e^{\lambda t} \left(\int_0^T M^2(s) e^{-2\lambda s} ds \right)^{1/2} \|X - \tilde{X}\|_{2, \lambda, T}. \tag{3.3}
 \end{aligned}$$

Dividing by $e^{\lambda t}$ both sides of (3.3), we have

$$\left\| \int_0^t S(t-s)[Q(X(s)) - Q(\tilde{X}(s))] dW(s) \right\|_{2, \lambda, T} \leq \left(\int_0^T M^2(s) e^{-2\lambda s} ds \right)^{1/2} \|X - \tilde{X}\|_{2, \lambda, T}.$$

Obviously,

$$\int_0^T M^2(s)e^{-2\lambda s} ds \rightarrow 0, \quad \text{as } \lambda \rightarrow +\infty.$$

By the Burkholder-Davis-Gundy inequality and (A.6), we have the following estimate for the third term of the right-hand side of (3.1):

$$\begin{aligned} & \left\{ E \left(\left\| \int_0^t \int_Z S(t-s)(F(X(s-), z) - F(\tilde{X}(s-), z)) \tilde{N}(ds, dz) \right\|^2 \right) \right\}^{1/2} \\ & \leq \left\{ E \left(\int_0^t \int_Z \|S(t-s)(F(X(s), z) - F(\tilde{X}(s), z))\|^2 \lambda(dz) ds \right) \right\}^{1/2} \\ & \leq C^{\frac{1}{2}} M(T) \left\{ E \left(\int_0^t \|X(s) - \tilde{X}(s)\|^2 ds \right) \right\}^{1/2} \\ & \leq C^{\frac{1}{2}} M(T) \left\{ \int_0^t e^{2\lambda s} e^{-2\lambda s} E(\|X(s) - \tilde{X}(s)\|^2) ds \right\}^{1/2} \\ & \leq C^{\frac{1}{2}} M(T) \left\{ \int_0^t e^{2\lambda s} ds \right\}^{1/2} \left(\sup_{t \in [0, T]} e^{-\lambda t} (E(\|X(t) - \tilde{X}(t)\|^2))^{1/2} \right) \\ & = C^{\frac{1}{2}} M(T) e^{\lambda t} \left\{ \int_0^t e^{-2\lambda s} ds \right\}^{1/2} \|X - \tilde{X}\|_{2, \lambda, T}. \end{aligned}$$

Thus we have

$$\begin{aligned} & \left\| \int_0^t \int_Z S(t-s)(F(X(s-), z) - F(\tilde{X}(s-), z)) \tilde{N}(ds, dz) \right\|_{2, \lambda, T} \\ & \leq C^{\frac{1}{2}} M(T) \left\{ \int_0^t e^{-2\lambda s} ds \right\}^{1/2} \|X - \tilde{X}(s)\|_{2, \lambda, T}. \end{aligned}$$

Obviously,

$$\int_0^t e^{-2\lambda s} ds \rightarrow 0, \quad \text{as } \lambda \rightarrow +\infty.$$

Therefore we have finally proved that there exists an $a(\lambda) < 1$ with

$$\|\mathcal{F}(\eta, X) - \mathcal{F}(\eta, \tilde{X}(s))\|_{2, \lambda, T} \leq a(\lambda) \|X - \tilde{X}\|_{2, \lambda, T}.$$

So, there exists a unique

$$X : L_0^2 \rightarrow \mathcal{H}^2(T, H)$$

satisfying

$$X(\eta) = \mathcal{F}(\eta, X(\eta)),$$

which is the unique solution of (2.1).

The fourth step : We will show the Lipschitz continuity of $X : L_0^2 \rightarrow \mathcal{H}^2(T, H)$. By contraction theorem (iii), we only have to prove that the mapping

$$\mathcal{F}(\cdot, Y) : L_0^2 \rightarrow \mathcal{H}^2(T, H)$$

is Lipschitz continuous for all $Y \in \mathcal{H}^2(T, H)$, where the Lipschitz constant does not depend on Y .

But this is obvious for all $\xi_1, \xi_2 \in L_0^2$ and $Y \in \mathcal{H}^2(T, H)$, as

$$\|\mathcal{F}(\xi_1, Y) - \mathcal{F}(\xi_2, Y)\|_{\mathcal{H}^2} = \|S(\cdot)(\xi_1 - \xi_2)\|_{\mathcal{H}^2} \leq M_T \|\xi_1 - \xi_2\|_{L^2}. \quad \square$$

Before giving the regularity of the mild solution of (2.1), we first need a stochastic Fubini theorem with respect to compensated Poisson measure. Set $\Omega_T \times Z = [0, T] \times \Omega \times Z$, let \mathcal{Z} be σ -algebra generated by open subsets in Z , \mathcal{P}_T be the predictable σ -algebra of $[0, T] \times \Omega$, $ds \otimes P \otimes \lambda(dz)$ be the product of the Lebesgue measure, P and $\lambda(dz)$ on $[0, T] \times \Omega \times Z$. $L^2(\Omega_T \times Z) := L^2([0, T] \times \Omega \times Z, ds \otimes P \otimes \lambda(dz); H)$.

Proposition 3.1 *Let (Y, \mathcal{Y}, μ) be a finite measure space and let $\psi : \Omega_T \times Y \times Z \rightarrow H$ be a $\mathcal{P}_T \otimes \mathcal{Y} \otimes \mathcal{Z}$ -measurable mapping such that*

$$\int_Y \|\psi(y)\|_{L^2(\Omega_T \times Z)} d\mu < \infty.$$

Then

- (1) *the process indexed by $t \in [0, T]$*

$$\int_Y \psi(y) d\mu$$

is progressively measurable and belongs to $L^2(\Omega_T \times Z)$,

- (2) *the process indexed by $y \in Y$*

$$\int_0^T \int_Z \psi(s-, z) \tilde{N}(ds, dz)$$

has an $\mathcal{F}_T \otimes \mathcal{Y}$ -measurable version $m : \Omega \times Y \rightarrow H$ such that

$$P \left[m(y) = \int_0^T \int_Z \psi(y) \tilde{N}(ds, dz) \right] = 1,$$

- (3) *we have*

$$P \left[\int_Y m(y) d\mu = \int_0^T \int_Z \left(\int_Y \psi(y) d\mu \right) \tilde{N}(ds, dz) \right] = 1.$$

Proof (1) follows from the following inequality: Let f be a nonnegative $\mathcal{P}_T \otimes \mathcal{Y} \otimes \mathcal{Z}$ -measurable function. Then

$$\sqrt{\int_{\Omega_T \times Z} \left(\int_Y f d\mu \right)^2 ds \times P \times \lambda(dz)} \leq \int_Y \|f\|_{L^2(\Omega_T \times Z, R)} d\mu \tag{3.4}$$

because

$$\left(\int_Y f(a, y) d\mu \right)^2 = \int_{Y \times Y} f(a, y_1) f(a, y_2) d\mu d\mu$$

and we get (3.4) by the Schwarz inequality. Now suppose that m in (2) exists. Then, by taking Ω instead of $\Omega_T \times Z$ in (3.4), we get

$$\sqrt{E \left(\int_Y \|m(y)\| d\mu \right)^2} \leq \int_Y \sqrt{E \|m(y)\|^2} d\mu \leq (\mu(Y))^{1/2} \|\psi(y)\|_{L^2(\Omega_T \times Y \times Z)} \tag{3.5}$$

by the Burkholder inequality. Hence

$$\int_Y m(y) d\mu$$

is defined P -almost everywhere. Now take ψ_n satisfying the assumption of the proposition such that the sequence of the integrals

$$\int_Y \|\psi_n(y) - \psi(y)\|_{L^2(\Omega_T \times Z)} d\mu$$

converges to zero. Then there exists a subsequence $(n_k : k \in \mathbb{N})$ such that:

- (a) $\int_0^T \int_Z \psi_{n_k}(y) \tilde{N}(ds, dz) \rightarrow \int_0^T \int_Z \psi(y) \tilde{N}(ds, dz)$ in $L^1(\Omega; H)$ for μ -almost all $y \in Y$.
- (b) $\int_0^T \int_Z (\int_Y \psi_{n_k}(y) d\mu) \tilde{N}(ds, dz) \rightarrow \int_0^T \int_Z (\int_Y \psi(y) d\mu) \tilde{N}(ds, dz)$ in $L^2(\Omega; H)$ for μ -almost all $y \in Y$.

Let us introduce the set \mathcal{D} of all $\mathcal{P}_T \otimes \mathcal{Y} \otimes \mathcal{Z}$ -measurable processes ψ with

$$\int_Y \|\psi(y)\|_{L^2(\Omega_T \times Z)} d\mu < \infty$$

such that there exists an m satisfying (2) and (3). It is easy to see that \mathcal{D} is a linear space and if we can find $\psi_n \in \mathcal{D}$ such that

$$\int_Y \|\psi_n(y) - \psi(y)\|_{L^2(\Omega_T \times Z)} d\mu \rightarrow 0,$$

then we would finish the proof. Indeed, take the corresponding functions m_n . Then the sequence $(m_n : n \in \mathbb{N})$ is Cauchy in $L^1(\Omega \times Y; H)$ due to (3.5) and ψ belongs to \mathcal{D} due to (a) and (b). Now we will show how to construct the approximating sequence ψ_n . We assume that $\lambda(dz)$ is finite. By Lemma A.1.4 in [11] we can find mappings F_n on H . The simple functions $F_n \psi$ take values in the finite dimensional subspace of H . Moreover,

$$\int_Y \|F_n \psi(y) - \psi(y)\|_{L^2(\Omega_T \times Z)} d\mu \downarrow 0,$$

and if $F_n \psi \in \mathcal{D}$, $n \in \mathbb{N}$, then $\psi \in \mathcal{D}$. Now to show that $F_n \psi \in \mathcal{D}$, we will take advantage of the fact that each $F_n \psi$ is bounded in H and

$$\int_Y \|\phi_n(y) - \phi(y)\|_{L^2(\Omega_T \times Z)} d\mu \rightarrow 0$$

if and only if

$$\int_Y \|\phi_n(y) - \phi(y)\|_{L^1(\Omega_T \times Y \times Z)} d\mu \rightarrow 0$$

for ϕ_n uniformly bounded in H . So as F_n is of the form

$$\sum_{k=1}^m I_{C_k} B_k,$$

where $(C_k : k \leq m)$ is a $\mathcal{P}_T \otimes \mathcal{Y} \otimes \mathcal{Z}$ -decomposition of $\Omega_T \times Y \times Z$ and $B_k, k \leq m$ are elements in the finite dimensional subspace of H . We conclude that $F_n \psi \in \mathcal{D}$ provided $I_{C_k} B_k \in \mathcal{D}$ due to linearity of \mathcal{D} . Another reduction shows that this is true if

$$I_{C_k^1 \times C_k^2 \times C_k^3} B_k \in \mathcal{D}$$

for every $C_k^1 \in \mathcal{P}_T, C_k^2 \in \mathcal{Y}, C_k^3 \in \mathcal{Z}$ as I_{C_k} can be approximated by $I_{C_k^0}$ in $L^1(\Omega_T \times Y \times Z)$ where C_k^0 is a disjoint union of sets of the type $C_k^1 \times C_k^2 \times C_k^3$. Finally, as $I_{C_k^1}$ is a predictable process, it can be approximated by simple bounded real processes in $L^1(\Omega_T)$. So, we will finish the proof by showing that

$$I_{[s,t] \times C_s \times C_k^2 \times C_k^3} B_k \in \mathcal{D}$$

for $s < t, C_s \in \mathcal{F}_s$, but this is obvious. When $\lambda(dz)$ is σ -finite, there exists a sequence A_n satisfying $\lambda(A_n) < +\infty$ and $A_n \uparrow Z$. Obviously,

$$\psi(y) I_{A_n} \in \mathcal{D} \quad \text{and} \quad \int_Z \|\psi I_{A_n}(z) - \psi(z)\|_{L^2(\Omega_T \times Z)} d\mu \rightarrow 0,$$

therefore $\psi \in \mathcal{D}$. □

In the following we give a type of the Burkholder-Davis-Gundy inequality which will play an important role in proving the regularity property of the mild solution of (2.1).

Lemma 3.1 *If A is the infinitesimal generator of pseudo-contraction C_0 -semigroup $S(t), t \leq T$, and $g(t, z)$ is an H -valued progressively measurable mapping with respect to $\{\mathcal{F}_t\}_{t \geq 0}$ such that*

$$\int_0^T \int_Z \|g(t, z)\|^2 \lambda(dz) dt < \infty, \quad a.s.,$$

then for any $t \in [0, T]$,

$$E \left(\sup_{0 \leq s \leq t} \left\| \int_0^s \int_Z S(s-u) g(u-, z) \tilde{N}(du, dz) \right\|^2 \right) \leq 8e^{4rt} E \int_0^t \int_Z \|g(s, z)\|^2 \lambda(dz) ds$$

for some number $r \geq 0$.

Proof Let us define

$$Y(t) = \int_0^t \int_Z S(t-s)g(s-, z)\tilde{N}(ds, dz).$$

Assume that $g(t, z)$ takes values in $\mathcal{D}(A)$ and

$$E \int_0^t \int_Z \|g(s, z)\|^2 \lambda(dz) ds + E \int_0^t \int_Z \|Ag(s, z)\|^2 \lambda(dz) ds < +\infty \quad \text{for all } t > 0. \quad (3.6)$$

Since A is the infinitesimal generator of a C_0 -semigroup, A is closed. And by (3.6) we can check that

$$\int_0^t \int_Z S(t-s)g(s-, z)\tilde{N}(ds, dz) \in \mathcal{D}(A)$$

for all $t \geq 0$ and

$$A \int_0^t \int_Z S(t-s)g(s-, z)\tilde{N}(ds, dz) = \int_0^t \int_Z AS(t-s)g(s-, z)\tilde{N}(ds, dz).$$

By (3.6) and Proposition 3.1 with Y replaced with $[0, T]$, we have

$$\begin{aligned} \int_0^t AY(s) ds &= \int_0^T \int_0^T \int_Z 1_{[0,t]}(s)1_{[0,s]}(u)AS(s-u)g(u-, z)\tilde{N}(du, dz) ds \\ &= \int_0^T \int_Z \int_0^T 1_{[u,t]}(s)1_{[0,t]}(u)AS(s-u)g(u-, z) ds \tilde{N}(du, dz) \\ &= \int_0^t \int_Z S(t-u)g(u-, z)\tilde{N}(du, dz) - \int_0^t \int_Z g(u-, z)\tilde{N}(du, dz) \\ &= Y(t) - \int_0^t \int_Z g(u-, z)\tilde{N}(du, dz). \end{aligned}$$

By Itô's formula, we get

$$\begin{aligned} \|Y(t)\|^2 &= 2 \int_0^t \langle AY(s), Y(s) \rangle ds + \int_0^t \int_Z \|g(s, z)\|^2 \lambda(dz) ds \\ &\quad + \int_0^t \int_Z (2\langle Y(s-), g(s-, z) \rangle + \|g(s-, z)\|^2) \tilde{N}(ds, dz). \end{aligned}$$

Set

$$\tilde{Y}(t) = \sup_{0 \leq s \leq t} \|Y(s)\| \quad \text{and} \quad \tilde{g}(t) = \int_0^t \int_Z \|g(s, z)\|^2 \lambda(dz) ds.$$

So, we have

$$\begin{aligned} (\tilde{Y}(t))^2 &\leq \sup_{0 \leq s \leq t} 2 \int_0^s \langle AY(u), Y(u) \rangle du + \int_0^t \int_Z \|g(u, z)\|^2 \lambda(dz) du \\ &\quad + \sup_{0 \leq s \leq t} \int_0^s \int_Z (2\langle Y(u-), g(u-, z) \rangle + \|g(u-, z)\|^2) \tilde{N}(du, dz). \end{aligned} \quad (3.7)$$

Since A is pseudo-contraction, there exists $r \geq 0$ such that

$$\langle Ax, x \rangle \leq r\|x\|^2 \quad \text{for all } x \in \mathcal{D}(A). \tag{3.8}$$

Define $T_n := \inf\{t \mid \|Y(t)\| > n\}$. Then by (3.7), (3.8) and B-D-G inequality, we can get the following estimate:

$$\begin{aligned} E(\tilde{Y}(t \wedge T_n))^2 &\leq 2rE \int_0^{t \wedge T_n} \|Y(s)\|^2 ds + E\tilde{g}(t \wedge T_n) \\ &\quad + 2E \sup_{0 \leq s \leq t \wedge T_n} \int_0^s \int_Z \langle Y(u-), g(u-, z) \rangle \tilde{N}(du, dz) \\ &\quad + E \sup_{0 \leq s \leq t \wedge T_n} \int_0^s \int_Z \|g(u-, z)\|^2 \tilde{N}(du, dz) \\ &\leq 2rE \int_0^{t \wedge T_n} \|Y(s)\|^2 ds + E\tilde{g}(t \wedge T_n) \\ &\quad + 2E \left(\int_0^{t \wedge T_n} \int_Z \langle Y(s-), g(s-, z) \rangle^2 N(ds, dz) \right)^{1/2} \\ &\quad + E \left(\int_0^{t \wedge T_n} \int_Z \|g(s-, z)\|^4 N(ds, dz) \right)^{1/2} \\ &\leq 2rE \int_0^{t \wedge T_n} \|Y(s)\|^2 ds + E\tilde{g}(t \wedge T_n) \\ &\quad + 2E \left(\int_0^{t \wedge T_n} \int_Z \langle Y(s-), g(s-, z) \rangle^2 N(ds, dz) \right)^{1/2} \\ &\quad + E \left(\int_0^{t \wedge T_n} \int_Z \|g(s-, z)\|^2 N(ds, dz) \right) \\ &= 2rE \int_0^{t \wedge T_n} \|Y(s)\|^2 ds + 2E\tilde{g}(t \wedge T_n) \\ &\quad + 2E \left(\int_0^{t \wedge T_n} \int_Z \langle Y(s-), g(s-, z) \rangle^2 N(dz, ds) \right)^{1/2} \\ &\leq 2rE \int_0^{t \wedge T_n} \|Y(s)\|^2 ds + 2E\tilde{g}(t \wedge T_n) \\ &\quad + 2E \left(\int_0^{t \wedge T_n} \int_Z \|Y(s-)\|^2 \|g(s-, z)\|^2 N(ds, dz) \right)^{1/2} \\ &\leq 2rE \int_0^{t \wedge T_n} \|Y(s)\|^2 ds + 2E\tilde{g}(t \wedge T_n) \\ &\quad + 2E \left[\tilde{Y}(t \wedge T_n) \left(\int_0^{t \wedge T_n} \int_Z \|g(s-, z)\|^2 N(ds, dz) \right)^{1/2} \right] \\ &\leq 2rE \int_0^{t \wedge T_n} \|Y(s)\|^2 ds + 2E\tilde{g}(t \wedge T_n) + \frac{1}{2}E(\tilde{Y}(t \wedge T_n))^2 + 2E\tilde{g}(t \wedge T_n). \end{aligned}$$

So, by Gronwall inequality, we have

$$E(\tilde{Y}(t \wedge T_n))^2 \leq 8e^{4rt} E \int_0^{t \wedge T_n} \int_Z \|g(s, z)\|^2 \lambda(dz) ds.$$

By the Fatou lemma, we have

$$E(\tilde{Y}(t))^2 \leq 8e^{4rt} E \int_0^t \int_Z \|g(s, z)\|^2 \lambda(dz) ds. \tag{3.9}$$

Without assuming $g(t, z)$ takes values in $\mathcal{D}(A)$, we define $g_n(t, z) = nR(n, A)g(t, z)$ where $R(n, A)$, $n \in \mathbb{N}$, is the resolvent of A , then we know $g_n(t, z)$ takes values in $\mathcal{D}(A)$ and satisfies (3.6) under the following condition:

$$E \int_0^t \int_Z \|g(s, z)\|^2 \lambda(dz) ds < +\infty. \tag{3.10}$$

Define

$$Y_n(t) = \int_0^t \int_Z S(t-s)g_n(s-, z)\tilde{N}(ds, dz), \quad \tilde{Y}_n(t) = \sup_{0 \leq s \leq t} \|Y_n(s)\|.$$

Then instead $\tilde{Y}(t)$, $\tilde{Y}_n(t)$ satisfies (3.9). By the Burkholder-Davis-Gundy inequality and dominated convergence theorem, we have

$$E \int_0^t \int_Z \|g_n(s, z) - g(s, z)\|^2 \lambda(dz) ds \rightarrow 0, \quad \text{as } n \rightarrow +\infty.$$

Then it is easy to check

$$E \left(\sup_{0 \leq s \leq t} \|Y_n(s) - Y(s)\|^2 \right) \rightarrow 0, \quad \text{as } n \rightarrow +\infty.$$

So, under condition (3.10), (3.9) also follows without assumption that $g(t, z)$ takes values in $\mathcal{D}(A)$. In order to relax (3.10), we define stopping times

$$\tau_n := \inf \left\{ t \mid \int_0^t \int_Z \|g(s, z)\|^2 \lambda(dz) ds \geq n \right\}.$$

Then

$$\begin{aligned} & E \left(\sup_{\{0 \leq s \leq t \wedge \tau_n\}} \left\| \int_0^s \int_Z S(s-u)g(u-, z) \right\|^2 \tilde{N}(du, dz) \right) \\ & \leq 8e^{4rt} E \left(\int_0^{t \wedge \tau_n} \int_Z \|g(s, z)\|^2 \lambda(dz) ds \right) \\ & \leq 8e^{4rt} E \left(\int_0^t \int_Z \|g(s, z)\|^2 \lambda(dz) ds \right). \end{aligned}$$

Therefore by Fatou lemma, the assertion follows. □

In order to study the regularity property of the mild solution of (2.1), we introduce an approximation system of (2.1) in the following:

$$\begin{aligned} X(t) &= R(t)\eta + \int_0^t [AX(s) ds + R(t)B(X(s))] ds + \int_0^t R(t)Q(X(s)) dW(s) \\ &+ \int_0^t \int_Z R(t)F(X(s-), z)\tilde{N}(ds, dz), \end{aligned} \tag{3.11}$$

where $l \in \rho(A)$ which is the resolvent set of A , and $R(l) := lR(l, A)$, $R(l, A)$ is the resolvent of A . We say a stochastic process is *RCLL* if each of its sample path is right continuous with left limit. So a stochastic process $X(t)$ is called a strong solution of (2.1), if it is *RCLL*, \mathcal{F}_t -adapted, $X(t) \in \mathcal{D}(A)$ and satisfies (2.1).

Theorem 3.2 *Let $\eta \in L^2(\Omega, H)$ and \mathcal{F}_0 -measurable. In addition to assumptions in Theorem 3.1 we assume A is dissipative with $x \in H$. Then the mild solution of (2.1) is *RCLL*.*

Proof Let $l \in \rho(A)$, obviously $AR(l) = A l R(l, A) = l - l^2 R(l, A)$ are bounded operators. So (3.11) has unique strong solution. In fact by Theorem 3.1 we know (3.11) has a unique mild solution denoted by $X^l(t)$ and the following hold:

$$\begin{aligned} & \int_0^T \int_0^t \|AS(t-s)R(l)B(X^l(s))\| ds dt < \infty, \\ & E\left(\int_0^T \int_0^t \|AS(t-s)R(l)Q(X^l(s))\|_{L_2}^2 ds dt\right) < \infty, \\ & E\left(\int_0^T \int_0^t \int_Z \|AS(t-s)R(l)F(X^l(s), z)\|^2 \lambda(dz) ds dt\right) < \infty. \end{aligned}$$

Thus by Fubini theorem, we have

$$\begin{aligned} & \int_0^t \int_0^s AS(u-s)R(l)B(X^l(s)) ds du \\ &= \int_0^t \int_s^t AS(u-s)R(l)B(X^l(s)) du ds \\ &= \int_0^t S(t-s)R(l)B(X^l(s)) ds - \int_0^t R(l)B(X^l(s)) ds. \end{aligned}$$

On the other hand, by the stochastic Fubini theorem for Q -Wiener processes in [11], we have

$$\begin{aligned} & \int_0^t \int_0^u AS(u-s)R(l)Q(X^l(s)) dW(s) du \\ &= \int_0^t \int_s^t AS(u-s)R(l)Q(X^l(s)) du dW(s) \\ &= \int_0^t S(t-s)R(l)Q(X^l(s)) dW(s) - \int_0^t R(l)Q(X^l(s)) dW(s). \end{aligned}$$

And by Proposition 3.1

$$\begin{aligned} & \int_0^t \int_0^u \int_Z AS(u-s)R(l)F(X^l(s-), z) \tilde{N}(ds, dz) du \\ &= \int_0^t \int_Z \int_s^t AS(u-s)R(l)F(X^l(s-), z) du \tilde{N}(ds, dz) \\ &= \int_0^t \int_Z S(t-s)R(l)F(X^l(s-), z) \tilde{N}(ds, dz) - \int_0^t \int_Z R(l)F(X^l(s-), z) \tilde{N}(ds, dz). \end{aligned}$$

Hence, $AX^l(t)$ is integrable almost surely and

$$\begin{aligned} \int_0^t AX^l(s) ds &= S(t)R(l)\eta - R(l)\eta + \int_0^t S(t-s)R(l)B(X^l(s)) ds - \int_0^t R(l)B(X^l(s)) ds \\ &\quad + \int_0^t S(t-s)R(l)Q(X^l(s)) dW(s) - \int_0^t R(l)Q(X^l(s)) dW(s) \\ &\quad + \int_0^t \int_Z S(t-s)R(l)F(X^l(s-), z) \tilde{N}(ds, dz) \\ &\quad - \int_0^t \int_Z R(l)F(X^l(s-), z) \tilde{N}(ds, dz) \\ &= X^l(t) - R(l)\eta - \int_0^t R(l)B(X^l(s)) ds - \int_0^t R(l)Q(X^l(s)) dW(s) \\ &\quad - \int_0^t \int_Z R(l)F(X^l(s-), z) \tilde{N}(ds, dz). \end{aligned}$$

So far, we prove $X^l(t) \in \mathcal{D}(A)$, $t \in [0, T]$, is the unique strong solution of (3.11). Let $X(t)$ be the mild solution of (2.1). Then we consider

$$\begin{aligned} X(t) - X^l(t) &= S(t)(\eta - R(l)\eta) \\ &\quad + \int_0^t S(t-s)[B(X(s)) - R(l)B(X^l(s))] ds \\ &\quad + \int_0^t S(t-s)[Q(X(s)) - R(l)Q(X^l(s))] dW(s) \\ &\quad + \int_0^t \int_Z S(t-s)[F(X(s-), z) - R(l)F(X^l(s-), z)] \tilde{N}(ds, dz) \end{aligned}$$

for any $t \geq 0$. We have that for any $T \geq 0$,

$$\begin{aligned} E \sup_{0 \leq t \leq T} \|X(t) - X^l(t)\|^2 &\leq 16E \sup_{0 \leq t \leq T} \left\| \int_0^t S(t-s)R(l)[B(X(s)) - B(X^l(s))] ds \right\|^2 \\ &\quad + 16E \sup_{0 \leq t \leq T} \left\| \int_0^t S(t-s)R(l)[Q(X(s)) - Q(X^l(s))] dW(s) \right\|^2 \\ &\quad + 16E \sup_{0 \leq t \leq T} \left\| \int_0^t \int_Z S(t-s)R(l)[F(X(s-), z) - F(X^l(s-), z)] \tilde{N}(ds, dz) \right\|^2 \\ &\quad + 16 \left\{ E \sup_{0 \leq t \leq T} \left\| S(t)(\eta - R(l)\eta) + \int_0^t S(t-s)[I - R(l)]B(X(s)) ds \right\|^2 \right. \\ &\quad + \int_0^t S(t-s)[I - R(l)]Q(X(s)) dW(s) \\ &\quad \left. + \int_0^t \int_Z S(t-s)[I - R(l)]F(X(s-), z) \tilde{N}(ds, dz) \right\} \\ &:= 16[I_1 + I_2 + I_3 + I_4]. \end{aligned}$$

We consider

$$\begin{aligned}
 I_1 &= E \sup_{0 \leq t \leq T} \left\| \int_0^t S(t-s)R(l)[B(X(s)) - B(X^l(s))] ds \right\|^2 \\
 &\leq E \sup_{0 \leq t \leq T} \left(\int_0^t \|R(l)S(t-s)[B(X(s)) - B(X^l(s))]\| ds \right)^2.
 \end{aligned}$$

When l is big enough, there exists constant M such that $\|R(l)\| \leq M$, so we have

$$\begin{aligned}
 I_1 &\leq C^2 M^2 M_T^2 E \sup_{0 \leq t \leq T} \left(\int_0^t \|X(s) - X^l(s)\| ds \right)^2 \\
 &\leq C^2 M^2 M_T^2 \int_0^T E \sup_{0 \leq u \leq s} \|X(u) - X^l(u)\|^2 ds.
 \end{aligned}$$

Then we consider

$$\begin{aligned}
 I_2 &:= E \sup_{0 \leq t \leq T} \left\| \int_0^t S(t-s)R(l)[Q(X(s)) - Q(X^l(s))] dW(s) \right\|^2 \\
 &= E \sup_{0 \leq t \leq T} \left\| \int_0^t S(t-s)lR(l,A)[Q(X(s)) - Q(X^l(s))] dW(s) \right\|^2 \\
 &= E \sup_{0 \leq t \leq T} \left\| \int_0^t S(t-s) \int_0^{+\infty} le^{-lu} T(u)[Q(X(s)) - Q(X^l(s))] du dW(s) \right\|^2 \\
 &\leq E \int_0^T \left\| \int_0^{+\infty} le^{-lu} T(u)[Q(X(s)) - Q(X^l(s))] du \right\|_{L_2}^2 ds \\
 &\leq E \int_0^T \left(\int_0^{+\infty} le^{-lu} \|T(u)[Q(X(s)) - Q(X^l(s))]\|_{L_2} du \right)^2 ds \\
 &\leq C(l)E \int_0^T \|X(s) - X^l(s)\|^2 ds \leq C(l)E \int_0^T \sup_{0 \leq u \leq s} \|X(u) - X^l(u)\|^2 ds,
 \end{aligned}$$

where $C(l) \rightarrow 0$ as $l \rightarrow \infty$. By (A.6) and Lemma 3.1 we have

$$\begin{aligned}
 I_3 &= E \sup_{0 \leq t \leq T} \left\| \int_0^t \int_Z S(t-s)R(l)[F(X(s-), z) - F(X^l(s-), z)] \tilde{N}(ds, dz) \right\|^2 \\
 &\leq C(T)E \int_0^T \int_Z \|F(X(s), z) - F(X^l(s), z)\|^2 \lambda(dz) ds \\
 &\leq C(T) \int_0^T E \sup_{0 \leq u \leq s} \|X(u) - X^l(u)\|^2 ds.
 \end{aligned}$$

Finally we consider

$$\begin{aligned}
 I_4 &= E \sup_{0 \leq t \leq T} \left\| S(t)(\eta - R(l)\eta) + \int_0^t S(t-s)[I - R(l)]B(X(s)) ds \right. \\
 &\quad \left. + \int_0^t S(t-s)[I - R(l)]Q(X(s)) dW(s) \right. \\
 &\quad \left. + \int_0^t \int_Z S(t-s)[I - R(l)]F(X(s-), z) \tilde{N}(ds, dz) \right\|^2.
 \end{aligned}$$

It is easy to check

$$E \sup_{0 \leq t \leq T} \|S(t)(\eta - R(l)\eta)\|^2 \rightarrow 0, \quad \text{as } l \rightarrow \infty.$$

By dominated convergence theorem, we have

$$\begin{aligned} & E \sup_{0 \leq t \leq T} \left\| \int_0^t S(t-s)[I - R(l)]B(X(s)) ds \right\|^2 \\ & \leq E \sup_{0 \leq t \leq T} \left(\int_0^t \|[I - R(l)]S(t-s)B(X(s))\| ds \right)^2 \\ & \leq M_T^2 E \left(\int_0^T \|[I - R(l)]B(X(s))\| ds \right)^2 \\ & \leq TM_T^2 \int_0^T E \|[I - R(l)]B(X(s))\|^2 ds \rightarrow 0, \quad \text{as } l \rightarrow \infty. \end{aligned}$$

By Burkholder-Davis-Gundy inequality, we have

$$\begin{aligned} & E \sup_{0 \leq t \leq T} \left\| \int_0^t S(t-s)(I - R(l))Q(X(s)) dW(s) \right\|^2 \\ & \leq E \int_0^T \|(I - R(l))S(t-s)Q(X(s))\|_{L_2}^2 ds \\ & \leq \tilde{C} E \int_0^T \|S(t-s)Q(X(s))\|_{L_2}^2 ds < +\infty \end{aligned} \tag{3.12}$$

for $\tilde{C} > 0$. So, by dominated convergence theorem and the fact that

$$\|(I - R(l))h\| \rightarrow 0, \quad \text{as } l \rightarrow +\infty$$

for $h \in H$, we get

$$(3.12) = E \int_0^T \sum_i \|(I - R(l))S(t-s)Q(X(s))e_i\|^2 ds \rightarrow 0, \quad \text{as } l \rightarrow \infty.$$

By Lemma 3.1, we get

$$\begin{aligned} & E \sup_{0 \leq t \leq T} \left\| \int_0^t \int_Z S(t-s)[I - R(l)]F(X(s-), z) \tilde{N}(ds, dz) \right\|^2 \\ & \leq M_T^2 E \int_0^T \int_Z \|[I - R(l)]F(X(s), z)\|^2 \lambda(dz) ds \rightarrow 0, \quad \text{as } l \rightarrow \infty. \end{aligned}$$

After discussing about $I_i, i = 1, 2, 3, 4$, we can get the following estimate

$$\begin{aligned} & E \sup_{0 \leq t \leq T} \|X(t) - X^l(t)\|^2 \\ & \leq [C(T) + C(l) + C^2 M^2 M_T^2] \int_0^T E \sup_{0 \leq u \leq s} \|X(u) - X^l(u)\|^2 ds + g(l), \end{aligned}$$

where $g(l) \rightarrow 0$ as $l \rightarrow \infty$. By Gronwall theorem, we have

$$E \sup_{0 \leq t \leq T} \|X(t) - X^l(t)\|^2 \leq g(l)e^{[C(T)+C(l)+C^2M^2M_T^2]T} \rightarrow 0, \quad \text{as } l \rightarrow \infty.$$

Therefore we know $X(t)$ is *RCLL* on $[0, T]$ a.s. □

4 Example

Example 4.1 Consider the following stochastic heat equation

$$dy(x, t) = \frac{\partial^2 y(x, t)}{\partial x^2} dt + b(x)f(y(\cdot, t)) dB_t + \int_{|z| \leq 1} z^2 x \tilde{N}(dt, dz), \quad t \geq 0, 0 < x < 1 \quad (4.1)$$

and

$$y(0, t) = y(1, t) = 0, \quad t \geq 0; \quad y(x, 0) = y_0(x), \quad 0 \leq x \leq 1, \quad (4.2)$$

where $B_t, t \geq 0$, is a real standard Brownian motion, $\tilde{N}(dt, dz)$ is a Poisson compensated martingale measure with character measure $\lambda(dz)$ on \mathbb{R} , $b \in L^2(0, 1) := H$ and f is a real Lipschitz continuous function on H satisfying $|f(u)| \leq c\|u\|$ for some $c > 0$ and $u \in H$. Let $A = \frac{\partial^2}{\partial x^2}$ with $\mathcal{D}(A) = H_0^1(0, 1) \cap H^2(0, 1)$, then we have

$$\langle u, Au \rangle_H \leq -\pi \|u\|^2 \quad \text{for } u \in \mathcal{D}(A).$$

For the sake of simplicity, we assume $\|b\| = 1$. It is easy to check that (A.1)-(A.5) are satisfied. So Theorem 3.1 and Theorem 3.2 hold true for (4.1).

Example 4.2 Denote $X(t, x)$ a process satisfying $X(0, x) = x$. Then we consider the semi-linear stochastic partial differential equation:

$$\begin{aligned} dX(t, x) &= a \frac{\partial^2}{\partial x^2} X(t, x) dt + [\varphi(t)X(t, x)] dt + b(X(t, x)) dW(t) \\ &\quad + \int_{|z| \leq c} zX(t-, x) \tilde{N}(dz, dt), \end{aligned} \quad (4.3)$$

$$X(t, 0) = X(t, \pi) = 0 \quad \text{for } t \geq 0, a > 0, c > 0, x \in H := L^2(0, \pi)$$

for some negative constant $a < 0$, and some $M > 0$, $\varphi(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}^1$ is a bounded function with $|\varphi(t)| \leq M$ for all $t \geq 0$, $b(x) : H \rightarrow \mathbb{R}^1$ is nonlinear and Lipschitz continuous with $b(0) = 0$, $|b(x)| \leq M\|x\|$, $W(t), t \geq 0$, is an H -valued Q -wiener process with covariance operator Q , $\text{tr } Q < \infty$.

Let $C = \int_{|z| \leq c} z^2 \lambda(dz) < +\infty$ and $A = a \frac{\partial^2}{\partial x^2}$ with domain

$$\mathcal{D}(A) = \left\{ h \in H : \frac{\partial h}{\partial x}, \frac{\partial^2 h}{\partial x^2} \in H, h(0) = h(\pi) = 0 \right\}.$$

So, it is easy to deduce

$$\langle Ax, x \rangle_H \leq a\|x\|^2, \quad x \in \mathcal{D}(A).$$

It is easy to check that (A.1)-(A.5) are satisfied. So, Theorem 3.1 and Theorem 3.2 hold true for (4.3).

Competing interests

The author declares that he has no competing interests.

Author's contributions

The author read and approved the final manuscript.

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