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Explicit averaging cyclic algorithm for common fixed points of a finite family of asymptotically strictly pseudocontractive mappings in q-uniformly smooth Banach spaces

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Abstract
Let E be a real q-uniformly smooth Banach space which is also uniformly convex and K be a nonempty, closed and convex subset of E. We obtain a weak convergence theorem of the explicit averaging cyclic algorithm for a finite family of asymptotically strictly pseudocontractive mappings of K under suitable control conditions, and elicit a necessary and sufficient condition that guarantees strong convergence of an explicit averaging cyclic process to a common fixed point of a finite family of asymptotically strictly pseudocontractive mappings in q-uniformly smooth Banach spaces. The results of this paper are interesting extensions of those known results.
MSC: 47H09; 47H10

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1 Introduction

Let E and E^* be a real Banach space and the dual space of E, respectively. Let J_q $(q > 1)$ denote the generalized duality mapping from E into 2^{E^*} given by $J_q(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^q$ and $\|f\| = \|x\|^{q-1}\}$ for all $x \in E$, where $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing between E and E^*. In particular, J_2 is called the normalized duality mapping and it is usually denoted by J. If E is smooth or E^* is strictly convex, then J_q is single-valued. In the sequel, we will denote the single-valued generalized duality mapping by j_q.

Let K be a nonempty subset of E. A mapping $T : K \to K$ is called *asymptotically κ-strictly pseudocontractive* with sequence $\{\kappa_n\}_{n=1}^{\infty} \subseteq [1,\infty)$ such that $\lim_{n\to\infty} \kappa_n = 1$ (see, *e.g.*, [1–3]) if for all $x, y \in K$, there exist a constant $\kappa \in [0, 1)$ and $j_q(x - y) \in J_q(x - y)$ such that

$$\left\langle T^n x - T^n y, j_q(x - y)\right\rangle \le \kappa_n\|x - y\|^q - \kappa\left\|x - y - \left(T^n x - T^n y\right)\right\|^q, \quad \forall n \ge 1. \tag{1}$$

If I denotes the identity operator, then (1) can be written in the form

$$\begin{aligned} &\langle (I - T^n)x - (I - T^n)y, j_q(x - y)\rangle \\ &\quad \geq \kappa \| (I - T^n)x - (I - T^n)y\|^q - (\kappa_n - 1)\|x - y\|^q. \end{aligned} \tag{2}$$

The class of asymptotically κ-strictly pseudocontractive mappings was first introduced in Hilbert spaces by Qihou [3]. In Hilbert spaces, j_q is the identity, and it is shown by Osilike *et al.* [2] that (1) (and hence (2)) is equivalent to the inequality

$$\| T^n x - T^n y\|^2 \leq \lambda_n \|x - y\|^2 + \lambda \|x - y - (T^n x - T^n y)\|^2,$$

where $\lim_{n\to\infty} \lambda_n = \lim_{n\to\infty}[1 + 2(\kappa_n - 1)] = 1$, $\lambda = (1 - 2\kappa) \in [0,1)$.

A mapping T with domain $D(T)$ and range $R(T)$ in E is called *strictly pseudocontractive* of Browder-Petryshyn type [4] if for all $x, y \in D(T)$, there exist $\kappa \in [0,1)$ and $j_q(x - y) \in J_q(x - y)$ such that

$$\langle Tx - Ty, j_q(x - y)\rangle \leq \|x - y\|^q - \kappa \|x - y - (Tx - Ty)\|^q. \tag{3}$$

If I denotes the identity operator, then (3) can be written in the form

$$\langle (I - T)x - (I - T)y, j_q(x - y)\rangle \geq \kappa \|(I - T)x - (I - T)y\|^q. \tag{4}$$

In Hilbert spaces, (3) (and hence (4)) is equivalent to the inequality

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|x - y - (Tx - Ty)\|^2, \quad k = (1 - 2\kappa) < 1.$$

It is shown in [5] that the class of asymptotically κ-strictly pseudocontractive mappings and the class of κ-strictly pseudocontractive mappings are independent.

A mapping T is said to be *uniformly L-Lipschitzian* if there exists a constant $L > 0$ such that, for all $x, y \in K$,

$$\| T^n x - T^n y\| \leq L\|x - y\|, \quad n \geq 1.$$

Let $\{T_j\}_{j=0}^{N-1}$ be N asymptotically strictly pseudocontractive self-mappings of K, and denote the common fixed points set of $\{T_j\}_{j=0}^{N-1}$ by $F := \bigcap_{j=0}^{N-1} F(T_j)$, where $F(T_j) := \{x \in K : T_j x = x\}$. We consider the following explicit averaging cyclic algorithm.

For a given $x_0 \in K$, and a real sequence $\{\alpha_n\}_{n=0}^{\infty} \subseteq (0,1)$, the sequence $\{x_n\}_{n=0}^{\infty}$ is generated as follows:

$$\begin{aligned} &x_1 = \alpha_0 x_0 + (1 - \alpha_0)T_0 x_0, \\ &x_2 = \alpha_1 x_1 + (1 - \alpha_1)T_1 x_1, \\ &\vdots \\ &x_N = \alpha_{N-1} x_{N-1} + (1 - \alpha_{N-1})T_{N-1} x_{N-1}, \\ &x_{N+1} = \alpha_N x_N + (1 - \alpha_N)T_0^2 x_N, \end{aligned}$$

$$x_{N+2} = \alpha_{N+1}x_{N+1} + (1-\alpha_{N+1})T_1^2 x_{N+1},$$

$$\vdots$$

$$x_{2N} = \alpha_{2N-1}x_{2N-1} + (1-\alpha_{2N-1})T_{N-1}^2 x_{2N-1},$$

$$x_{2N+1} = \alpha_{2N}x_{2N} + (1-\alpha_{2N})T_0^3 x_{2N},$$

$$x_{2N+2} = \alpha_{2N+1}x_{2N+1} + (1-\alpha_{2N+1})T_1^3 x_{2N+1},$$

$$\vdots$$

The algorithm can be expressed in a compact form as

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)T_{i(n)}^{k(n)} x_n, \quad n \geq 0, \tag{5}$$

where $n = (k-1)N + i$ with $i = i(n) \in I = \{0, 1, 2, \ldots, N-1\}$, $k = k(n) \geq 1$ a positive integer and $\lim_{n\to\infty} k(n) = \infty$. The cyclic algorithm was first studied by Acedo and Xu [6] for the iterative approximation of common fixed points of a finite family of strictly pseudocontractive mappings in Hilbert spaces, and it is better than implicit iteration methods.

In [7] Xiaolong Qin *et al.* proved the following theorem in a Hilbert space.

Theorem QCKS *Let K be a closed and convex subset of a Hilbert space H and $N \geq 1$ be an integer. Let, for each $1 \leq i \leq N$, $T_i : K \to K$ be an asymptotically κ_i-strictly pseudocontractive mapping for some $0 \leq \kappa_i < 1$ and a sequence $\{k_{n,i}\}$ such that $\sum_{n=0}^{\infty}(k_{n,i} - 1) < \infty$. Let $\kappa = \max\{\kappa_i : 1 \leq i \leq N\}$ and $\kappa_n = \max\{\kappa_{n,i} : 1 \leq i \leq N\}$. Assume that $F \neq \emptyset$. For any $x_0 \in K$, let $\{x_n\}$ be the sequence generated by the cyclic algorithm* (5). *Assume that the control sequence $\{\alpha_n\}$ is chosen such that $\kappa + \epsilon \leq \alpha_n \leq 1 - \epsilon$ for all $n \geq 0$ and a small enough constant $\epsilon \in (0, 1)$. Then $\{x_n\}$ converges weakly to a common fixed point of the family $\{T_i\}_{i=1}^N$.*

Osilike and Shehu [8] extended the result of Theorem QCKS from a Hilbert space to 2-uniformly smooth Banach spaces which are also uniformly convex. They proved the following theorem.

Theorem OS *Let E be a real 2-uniformly smooth Banach space which is also uniformly convex, and K be a nonempty, closed and convex subset of E. Let $\{T_j\}_{j=0}^{N-1}$ be N asymptotically λ_j-strictly pseudocontractive self-mappings of K for some $0 \leq \lambda_j < 1$ with a sequence $\{\kappa_n^{(j)}\}_{n=0}^{\infty} \subset [1, \infty)$ such that $\sum_{n=0}^{\infty}(\kappa_n^{(j)} - 1) < \infty$, $\forall j \in J = \{0, 1, 2, \ldots, N-1\}$, and $F \neq \emptyset$. Let $\{\alpha_n\}$ satisfy the conditions*

(i*) $\quad 0 \leq \alpha_n < 1, \quad n \geq 0,$

(ii*) $\quad 0 < a \leq 1 - \alpha_n \leq b < \dfrac{2\lambda}{C_2},$

where $\lambda = \min_{j\in J}\{\lambda_j\}$ and C_2 is the constant appearing in the inequality (7) *with $q = 2$. Let $\{x_n\}$ be the sequence generated by the cyclic algorithm* (5). *Then $\{x_n\}$ converges weakly to a common fixed point of the family $\{T_j\}_{j=0}^{N-1}$.*

We would like to point out that the condition (ii*) in Theorem OS excludes the natural choice $1 - \frac{1}{n}$ for α_n. This is overcome by this paper. Moreover, we improve and extend the result of Theorem OS from 2-uniformly smooth Banach spaces to q-uniformly smooth Banach spaces which are also uniformly convex. We prove that if $\{\alpha_n\}$ satisfies the conditions

$$\begin{aligned} &\text{(i)} \quad \mu \le \alpha_n < 1, \quad n \ge 0, \\ &\text{(ii)} \quad \sum_{n=0}^{\infty}(1-\alpha_n)\big[q\lambda - C_q(1-\alpha_n)^{q-1}\big] = \infty, \end{aligned} \tag{6}$$

where $\mu = \max\{0, 1 - (\frac{q\lambda}{C_q})^{\frac{1}{q-1}}\}$, $\lambda = \min_{j\in J}\{\lambda_j\}$, then the iterative sequence (5) converges weakly to a common fixed point of the family $\{T_j\}_{j=0}^{N-1}$.

Furthermore, we elicit a necessary and sufficient condition that guarantees strong convergence of the iterative sequence (5) to a common fixed point of the family $\{T_j\}_{j=0}^{N-1}$ in q-uniformly smooth Banach spaces.

We will use the notation:

1. $\rightharpoonup$ for weak convergence.
2. $\omega_{\mathcal{W}}(x_n) = \{x : \exists x_{n_j} \rightharpoonup x\}$ denotes the weak ω-limit set of $\{x_n\}$.

2 Preliminaries

Let E be a real Banach space. The *modulus of smoothness* of E is the function $\rho_E : [0,\infty) \to [0,\infty)$ defined by

$$\rho_E(\tau) = \sup\left\{\frac{1}{2}\big(\|x+y\| + \|x-y\|\big) - 1 : \|x\| \le 1, \|y\| \le \tau\right\}.$$

E is *uniformly smooth* if and only if $\lim_{\tau\to 0}[\rho_E(\tau)/\tau] = 0$.

Let $q > 1$. E is said to be *q-uniformly smooth* (or to have a modulus of smoothness of power type $q > 1$) if there exists a constant $c > 0$ such that $\rho_E(\tau) \le c\tau^q$. Hilbert spaces, L_p (or l_p) spaces ($1 < p < \infty$) and the Sobolev spaces W_m^p ($1 < p < \infty$) are q-uniformly smooth. Hilbert spaces are 2-uniformly smooth while

$$L_p \text{ (or } l_p) \quad \text{or} \quad W_m^p \quad \text{is} \begin{cases} p\text{-uniformly smooth} & \text{if } 1 < p \le 2, \\ 2\text{-uniformly smooth} & \text{if } p \ge 2. \end{cases}$$

Theorem HKX ([9, p.1130]) *Let $q > 1$ and let E be a real q-uniformly smooth Banach space. Then there exists a constant $C_q > 0$ such that, for all $x, y \in E$,*

$$\|x+y\|^q \le \|x\|^q + q\big\langle y, j_q(x)\big\rangle + C_q\|y\|^q. \tag{7}$$

E is said to have a *Fréchet differentiable norm* if, for all $x \in U = \{x \in E : \|x\| = 1\}$,

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists and is attained uniformly in $y \in U$. In this case, there exists an increasing function $b : [0,\infty) \to [0,\infty)$ with $\lim_{t\to 0^+}[b(t)/t] = 0$ such that, for all $x, h \in E$,

$$\frac{1}{2}\|x\|^2 + \big\langle h, j(x)\big\rangle \le \frac{1}{2}\|x+h\|^2 \le \frac{1}{2}\|x\|^2 + \big\langle h, j(x)\big\rangle + b\big(\|h\|\big). \tag{8}$$

It is well known (see, for example, [10, p.107]) that a q-uniformly smooth Banach space has a Fréchet differentiable norm.

Lemma 2.1 ([5, p.1338]) *Let E be a real q-uniformly smooth Banach space which is also uniformly convex. Let K be a nonempty, closed and convex subset of E and $T : K \to K$ be an asymptotically κ-strictly pseudocontractive mapping with a nonempty fixed point set. Then $(I - T)$ is demiclosed at zero, that is, if whenever $\{x_n\} \subset D(T)$ such that $\{x_n\}$ converges weakly to $x \in D(T)$ and $\{(I - T)x_n\}$ converges strongly to 0, then $Tx = x$.*

Lemma 2.2 ([2, p.80]) *Let $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$, $\{\delta_n\}_{n=0}^{\infty}$ be sequences of nonnegative real numbers satisfying the following inequality:*

$$a_{n+1} \le (1 + \delta_n)a_n + b_n, \quad \forall n \ge 0.$$

If $\sum_{n=0}^{\infty} \delta_n < \infty$ and $\sum_{n=0}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. If, in addition, $\{a_n\}_{n=0}^{\infty}$ has a subsequence which converges strongly to zero, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.3 ([2, p.78]) *Let E be a real Banach space, K be a nonempty subset of E and $T : K \to K$ be an asymptotically κ-strictly pseudocontractive mapping. Then T is uniformly L-Lipschitzian.*

Lemma 2.4 *Let E be a real q-uniformly smooth Banach space which is also uniformly convex, and let K be a nonempty, closed and convex subset of E. Let, for each $0 \le j \le N - 1$, $T_j : K \to K$ be an asymptotically λ_j-strictly pseudocontractive mapping with $F \ne \emptyset$. Let $\{x_n\}_{n=0}^{\infty}$ be the sequence satisfying the following conditions:*

(a) $\lim_{n\to\infty} \|x_n - p\|$ *exists for every $p \in F$;*
(b) $\lim_{n\to\infty} \|x_n - T_j x_n\| = 0$, *for each $0 \le j \le N - 1$;*
(c) $\lim_{n\to\infty} \|tx_n + (1 - t)p_1 - p_2\|$ *exists for all $t \in [0, 1]$ and for all $p_1, p_2 \in F$.*

Then the sequence $\{x_n\}$ converges weakly to a common fixed point of the family $\{T_j\}_{j=0}^{N-1}$.

Proof Since $\lim_{n\to\infty} \|x_n - p\|$ exists, then $\{x_n\}$ is bounded. By (b) and Lemma 2.1, we have $\omega_{\mathcal{W}}(x_n) \subset F$. Assume that $p_1, p_2 \in \omega_{\mathcal{W}}(x_n)$ and that $\{x_{n_i}\}$ and $\{x_{m_j}\}$ are subsequences of $\{x_n\}$ such that $x_{n_i} \rightharpoonup p_1$ and $x_{m_j} \rightharpoonup p_2$, respectively. Since E is a real q-uniformly smooth Banach space, which is also uniformly convex, then E has a Fréchet differentiable norm. Set $x = p_1 - p_2$, $h = t(x_n - p_1)$ in (8), we obtain

$$\begin{aligned}\frac{1}{2}\|p_1 - p_2\|^2 + t\langle x_n - p_1, j(p_1 - p_2)\rangle &\le \frac{1}{2}\big\|tx_n + (1 - t)p_1 - p_2\big\|^2 \\ &\le \frac{1}{2}\|p_1 - p_2\|^2 + b\big(t\|x_n - p_1\|\big) \\ &\quad + t\langle x_n - p_1, j(p_1 - p_2)\rangle,\end{aligned}$$

where b is an increasing function. Since $\|x_n - p_1\| \le M$, $\forall n \ge 0$, for some $M > 0$, then

$$\begin{aligned}\frac{1}{2}\|p_1 - p_2\|^2 + t\langle x_n - p_1, j(p_1 - p_2)\rangle &\le \frac{1}{2}\big\|tx_n + (1 - t)p_1 - p_2\big\|^2 \\ &\le \frac{1}{2}\|p_1 - p_2\|^2 + b(tM) + t\langle x_n - p_1, j(p_1 - p_2)\rangle.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{1}{2}\|p_1-p_2\|^2+t\limsup_{n\to\infty}\langle x_n-p_1,j(p_1-p_2)\rangle &\le \frac{1}{2}\lim_{n\to\infty}\|tx_n+(1-t)p_1-p_2\|^2\\ &\le \frac{1}{2}\|p_1-p_2\|^2+b(tM)\\ &\quad+t\liminf_{n\to\infty}\langle x_n-p_1,j(p_1-p_2)\rangle.\end{aligned}$$

Hence, $\limsup_{n\to\infty}\langle x_n-p_1,j(p_1-p_2)\rangle \le \liminf_{n\to\infty}\langle x_n-p_1,j(p_1-p_2)\rangle + b(tM)/t$. Since $\lim_{t\to 0^+}[b(tM)/t]=0$, then $\lim_{n\to\infty}\langle x_n-p_1,j(p_1-p_2)\rangle$ exists. Since $\lim_{n\to\infty}\langle x_n-p_1,j(p_1-p_2)\rangle=\langle p-p_1,j(p_1-p_2)\rangle$, for all $p\in\omega_{\mathcal{W}}(x_n)$. Set $p=p_2$. We have $\langle p_2-p_1,j(p_1-p_2)\rangle=0$, that is, $p_2=p_1$. Hence, $\omega_{\mathcal{W}}(x_n)$ is a singleton, so that $\{x_n\}$ converges weakly to a common fixed point of the family $\{T_j\}_{j=0}^{N-1}$. □

3 Main results

Theorem 3.1 *Let E be a real q-uniformly smooth Banach space which is also uniformly convex and K be a nonempty, closed and convex subset of E. Let $N\ge 1$ be an integer and $J=\{0,1,2,\dots,N-1\}$. Let, for each $j\in J$, $T_j:K\to K$ be an asymptotically λ_j-strictly pseudocontractive mapping for some $0\le\lambda_j<1$ with sequences $\{\kappa_{n,j}\}_{n=0}^{\infty}\subset[1,\infty)$ such that $\sum_{n=0}^{\infty}(\kappa_n-1)<\infty$, where $\kappa_n=\max_{j\in J}\{\kappa_{n,j}\}$, and $F:=\bigcap_{j=0}^{N-1}F(T_j)\neq\emptyset$. Let $\lambda=\min_{j\in J}\{\lambda_j\}$. Let $\{\alpha_n\}$ satisfy the conditions* (6) *and $\{x_n\}$ be the sequence generated by the cyclic algorithm* (5). *Then $\{x_n\}$ converges weakly to a common fixed point of the family $\{T_j\}_{j=0}^{N-1}$.*

Proof Pick a $p\in F$. We firstly show that $\lim_{n\to\infty}\|x_n-p\|$ exists. To see this, using (2) and (7), we obtain

$$\begin{aligned}\|x_{n+1}-p\|^q &= \|x_n-p-(1-\alpha_n)[x_n-p-(T_{i(n)}^{k(n)}x_n-p)]\|^q\\ &\le \|x_n-p\|^q+C_q(1-\alpha_n)^q\|x_n-p-(T_{i(n)}^{k(n)}x_n-p)\|^q\\ &\quad -q(1-\alpha_n)\langle x_n-p-(T_{i(n)}^{k(n)}x_n-p),j_q(x_n-p)\rangle\\ &\le \|x_n-p\|^q+C_q(1-\alpha_n)^q\|x_n-p-(T_{i(n)}^{k(n)}x_n-p)\|^q\\ &\quad -q(1-\alpha_n)\{\lambda_{i(n)}\|x_n-p-(T_{i(n)}^{k(n)}x_n-p)\|^q\\ &\quad -(\kappa_{k(n),i(n)}-1)\|x_n-p\|^q\}\\ &= [1+q(1-\alpha_n)(\kappa_{k(n),i(n)}-1)]\|x_n-p\|^q\\ &\quad -(1-\alpha_n)[q\lambda_{i(n)}-C_q(1-\alpha_n)^{q-1}]\|x_n-T_{i(n)}^{k(n)}x_n\|^q\\ &\le [1+q(1-\alpha_n)(\kappa_{k(n)}-1)]\|x_n-p\|^q\\ &\quad -(1-\alpha_n)[q\lambda-C_q(1-\alpha_n)^{q-1}]\|x_n-T_{i(n)}^{k(n)}x_n\|^q,\end{aligned}\tag{9}$$

where $\kappa_{k(n)}=\max_{i\in J}\{\kappa_{k(n),i(n)}\}$. Since $\mu\le\alpha_n<1$ for all n, where $\mu=\max\{0,1-(\frac{q\lambda}{C_q})^{\frac{1}{q-1}}\}$, we get $(1-\alpha_n)[q\lambda-C_q(1-\alpha_n)^{q-1}]\ge 0$. Therefore, (9) implies

$$\|x_{n+1}-p\|^q\le[1+q(1-\alpha_n)(\kappa_{k(n)}-1)]\|x_n-p\|^q.\tag{10}$$

Let $\delta_n = 1 + q(1-\alpha_n)(\kappa_{k(n)} - 1)$. Since $\sum_{n=0}^{\infty}(\kappa_n - 1) < \infty$, we have

$$\sum_{n=0}^{\infty}(\delta_n - 1) = q\sum_{n=0}^{\infty}(1-\alpha_n)(\kappa_{k(n)} - 1) \le qN\sum_{n=1}^{\infty}(\kappa_n - 1) < \infty,$$

then (10) implies $\lim_{n\to\infty}\|x_n - p\|$ exists by Lemma 2.2 (and hence the sequence $\{\|x_n - p\|\}$ is bounded, that is, there exists a constant $M > 0$ such that $\|x_n - p\| < M$).

Then we prove $\lim_{n\to\infty}\|x_n - T_j x_n\| = 0$, $\forall j \in J$. In fact, it follows from (9) that

$$\begin{aligned}(1-\alpha_n)\big[q\lambda - C_q(1-\alpha_n)^{q-1}\big]\big\|x_n - T_{i(n)}^{k(n)}x_n\big\|^q &\le \|x_n - p\|^q - \|x_{n+1} - p\|^q \\ &\quad + q(1-\alpha_n)(\kappa_{k(n)} - 1)\|x_n - p\|^q.\end{aligned}$$

Then

$$\sum_{n=0}^{\infty}(1-\alpha_n)\big[q\lambda - C_q(1-\alpha_n)^{q-1}\big]\big\|x_n - T_{i(n)}^{k(n)}x_n\big\|^q < \|x_0 - p\|^q + M^q\sum_{n=0}^{\infty}(\delta_{k(n)} - 1) < \infty. \quad (11)$$

Since $\sum_{n=0}^{\infty}(1-\alpha_n)[q\lambda - C_q(1-\alpha_n)^{q-1}] = \infty$, then (11) implies that $\liminf_{n\to\infty}\|x_n - T_{i(n)}^{k(n)}x_n\| = 0$. Thus $\lim_{n\to\infty}\|x_n - T_{i(n)}^{k(n)}x_n\| = 0$.

For all $n > N$, we have $k(n) - 1 = k(n-N)$ and $i(n) = i(n-N)$. By Lemma 2.3, we know that T_j is uniformly L_j-Lipschitzian, then there exists a constant $L = \max_{j\in J}\{L_j\}$, such that

$$\big\|T_j^n x - T_j^n y\big\| \le L\|x - y\|, \quad \forall n \ge 0, \forall x, y \in K \text{ and } \forall j \in J.$$

Thus

$$\begin{aligned}\|x_n - T_{i(n)}x_n\| &\le \big\|x_n - T_{i(n)}^{k(n)}x_n\big\| + \big\|T_{i(n)}^{k(n)}x_n - T_{i(n)}x_n\big\| \\ &\le \big\|x_n - T_{i(n)}^{k(n)}x_n\big\| + L\big\|T_{i(n)}^{k(n)-1}x_n - x_n\big\| \\ &\le \big\|x_n - T_{i(n)}^{k(n)}x_n\big\| + L\big\|T_{i(n)}^{k(n)-1}x_n - T_{i(n-N)}^{k(n)-1}x_{n-N}\big\| \\ &\quad + L\big\|T_{i(n-N)}^{k(n)-1}x_{n-N} - x_{n-N-1}\big\| + L\|x_{n-N-1} - x_n\| \\ &\le \big\|x_n - T_{i(n)}^{k(n)}x_n\big\| + L^2\|x_n - x_{n-N}\| \\ &\quad + L\big\|T_{i(n-N)}^{k(n-N)}x_{n-N} - x_{n-N-1}\big\| + L\|x_{n-N-1} - x_n\|.\end{aligned}$$

Observe that

$$\|x_n - x_{n+1}\| = (1-\alpha_n)\big\|x_n - T_{i(n)}^{k(n)}x_n\big\| \to 0 \quad \text{as } n \to \infty.$$

Consequently,

$$\|x_n - x_{n+l}\| \to 0 \quad \text{as } n \to \infty, \text{ for all integer } l.$$

Observe also that

$$\big\|x_{n-1} - T_{i(n)}^{k(n)}x_n\big\| \le \|x_n - x_{n-1}\| + \big\|x_n - T_{i(n)}^{k(n)}x_n\big\| \to 0 \quad \text{as } n \to \infty.$$

Hence,

$$\lim_{n\to\infty} \|x_n - T_{i(n)}x_n\| = 0.$$

Consequently, for all $j \in J$, we have

$$\|x_n - T_{n+j}x_n\| \le \|x_n - x_{n+j}\| + \|x_{n+j} - T_{n+j}x_{n+j}\| + L\|x_n - x_{n+j}\| \to 0 \quad \text{as } n \to \infty.$$

Thus,

$$\lim_{n\to\infty} \|x_n - T_jx_n\| = 0, \quad \forall j \in J.$$

Now we prove that for all $p_1, p_2 \in F$, $\lim_{n\to\infty} \|tx_n + (1-t)p_1 - p_2\|$ exists for all $t \in [0,1]$. Let $a_n(t) = \|tx_n + (1-t)p_1 - p_2\|$. It is obvious that $\lim_{n\to\infty} a_n(0) = \|p_1 - p_2\|$ and $\lim_{n\to\infty} a_n(1) = \lim_{n\to\infty} \|x_n - p_2\|$ exist. So, we only need to consider the case of $t \in (0,1)$. Define $A_n : K \to K$ by

$$A_nx = \alpha_nx + (1-\alpha_n)T_{i(n)}^{k(n)}x, \quad x \in K.$$

Then for all $x, y \in K$

$$\begin{aligned}
\|A_nx - A_ny\|^q &\le \|x-y\|^q - q(1-\alpha_n)\big\langle \big(I - T_{i(n)}^{k(n)}\big)x - \big(I - T_{i(n)}^{k(n)}\big)y, j_q(x-y)\big\rangle \\
&\quad + C_q(1-\alpha_n)^q \big\| x - y - \big(T_{i(n)}^{k(n)}x - T_{i(n)}^{k(n)}y\big)\big\|^q \\
&\le \big[1 + q(1-\alpha_n)(\kappa_{k(n)} - 1)\big]\|x-y\|^q \\
&\quad - (1-\alpha_n)\big[q\lambda - C_q(1-\alpha_n)^{q-1}\big]\big\| x - y - \big(T_{i(n)}^{k(n)}x - T_{i(n)}^{k(n)}y\big)\big\|^q.
\end{aligned}$$

By the choice of α_n, we have $(1-\alpha_n)[q\lambda - C_q(1-\alpha_n)^{q-1}] \ge 0$, so it follows that $\|A_nx - A_ny\|^q \le [1 + q(1-\alpha_n)(\kappa_{k(n)} - 1)]\|x-y\|^q = \delta_n\|x-y\|^q$. For the convenience of the following discussion, set $\eta_n = (\delta_n)^{\frac{1}{q}}$, then $\|A_nx - A_ny\| \le \eta_n\|x-y\|$.

Set $S_{n,m} = A_{n+m-1}A_{n+m-2}\cdots A_n$, $m \ge 1$. We have

$$\|S_{n,m}x - S_{n,m}y\| \le \left(\prod_{j=n}^{n+m-1} \eta_j\right)\|x-y\| \quad \text{for all } x, y \in K,$$

and

$$S_{n,m}x_n = x_{n+m}, \qquad S_{n,m}p = p \quad \text{for all } p \in F.$$

Set $b_{n,m} = \|S_{n,m}(tx_n + (1-t)p_1) - tS_{n,m}x_n - (1-t)S_{n,m}p_1\|$. If $\|x_n - p_1\| = 0$ for some n_0, then $x_n = p_1$ for any $n \ge n_0$ so that $\lim_{n\to\infty} \|x_n - p_1\| = 0$, in fact, $\{x_n\}$ converges strongly to $p_1 \in F$. Thus we may assume $\|x_n - p_1\| > 0$ for any $n \ge 0$. Let δ denote the modulus of convexity of E. It is well known (see, for example, [11, p.108]) that

$$\begin{aligned}
\big\|tx + (1-t)y\big\| &\le 1 - 2\min\big\{t, (1-t)\big\}\delta\big(\|x-y\|\big) \\
&\le 1 - 2t(1-t)\delta\big(\|x-y\|\big) \qquad (12)
\end{aligned}$$

for all $t \in [0,1]$ and for all $x, y \in E$ such that $\|x\| \leq 1$, $\|y\| \leq 1$. Set

$$w_{n,m} = \frac{S_{n,m}p_1 - S_{n,m}(tx_n + (1-t)p_1)}{t(\prod_{j=n}^{n+m-1}\eta_j)\|x_n - p_1\|}, \qquad z_{n,m} = \frac{S_{n,m}(tx_n + (1-t)p_1) - S_{n,m}x_n}{(1-t)(\prod_{j=n}^{n+m-1}\eta_j)\|x_n - p_1\|}.$$

Then $\|w_{n,m}\| \leq 1$ and $\|z_{n,m}\| \leq 1$ so that it follows from (12) that

$$2t(1-t)\delta\big(\|w_{n,m} - z_{n,m}\|\big) \leq 1 - \big\|tw_{n,m} + (1-t)z_{n,m}\big\|. \tag{13}$$

Observe that

$$\|w_{n,m} - z_{n,m}\| = \frac{b_{n,m}}{t(1-t)(\prod_{j=n}^{n+m-1}\eta_j)\|x_n - p_1\|}$$

and

$$\big\|tw_{n,m} + (1-t)z_{n,m}\big\| = \frac{\|S_{n,m}x_n - S_{n,m}p_1\|}{(\prod_{j=n}^{n+m-1}\eta_j)\|x_n - p_1\|},$$

it follows from (13) that

$$\begin{aligned}
&2t(1-t)\left(\prod_{j=n}^{n+m-1}\eta_j\right)\|x_n - p_1\|\delta\left(\frac{b_{n,m}}{t(1-t)(\prod_{j=n}^{n+m-1}\eta_j)\|x_n - p_1\|}\right)\\
&\quad\leq \left(\prod_{j=n}^{n+m-1}\eta_j\right)\|x_n - p_1\| - \|S_{n,m}x_n - S_{n,m}p_1\|\\
&\quad= \left(\prod_{j=n}^{n+m-1}\eta_j\right)\|x_n - p_1\| - \|x_{n+m} - p_1\|.
\end{aligned} \tag{14}$$

Since E is uniformly convex, then $\delta(s)/s$ is nondecreasing, and since $(\prod_{j=n}^{n+m-1}\eta_j)\|x_n - p_1\| \leq (\prod_{j=n}^{n+m-1}\eta_j)\eta_{n-1}\|x_{n-1} - p_1\| \leq \cdots \leq (\prod_{j=n}^{n+m-1}\eta_j)(\prod_{j=0}^{n-1}\eta_j)\|x_0 - p_1\| = (\prod_{j=0}^{n+m-1}\eta_j)\|x_0 - p_1\|$, hence it follows from (14) that

$$\begin{aligned}
&\frac{(\prod_{j=0}^{n+m-1}\eta_j)\|x_0 - p_1\|}{2}\delta\left(\frac{4}{(\prod_{j=0}^{n+m-1}\eta_j)\|x_0 - p_1\|}b_{n,m}\right)\\
&\quad\leq \left(\prod_{j=n}^{n+m-1}\eta_j\right)\|x_n - p_1\| - \|x_{n+m} - p_1\| \quad \left(\text{since } t(1-t) \leq \frac{1}{4} \text{ for all } t \in [0,1]\right).
\end{aligned}$$

Since $\lim_{n\to\infty}\prod_{j=0}^{n+m-1}\eta_j$ exits and $\lim_{n\to\infty}\prod_{j=0}^{n+m-1}\eta_j \neq 0$. Also since $\lim_{n\to\infty}\prod_{j=n}^{n+m-1}\eta_j = 1$ and $\lim_{n\to\infty}\|x_n - p_1\|$ exists, then the continuity of δ and $\delta(0) = 0$ yield $\lim_{n\to\infty} b_{n,m} = 0$ uniformly for all $m \geq 1$. Observe that

$$\begin{aligned}
a_{n+m}(t) \leq{}& \big\|tx_{n+m} + (1-t)p_1 - p_2 + \big(S_{n,m}\big(tx_n + (1-t)p_1\big)\\
&- tS_{n,m}x_n - (1-t)S_{n,m}p_1\big)\big\|\\
&+ \big\|S_{n,m}\big(tx_n + (1-t)p_1\big) - tS_{n,m}x_n - (1-t)S_{n,m}p_1\big\|
\end{aligned}$$

$$= \left\| S_{n,m}\bigl(tx_n + (1-t)p_1\bigr) - S_{n,m}p_2 \right\| + b_{n,m}$$
$$\leq \left(\prod_{j=n}^{n+m-1} \eta_j\right) \left\| tx_n + (1-t)p_1 - p_2 \right\| + b_{n,m} = \left(\prod_{j=n}^{n+m-1} \eta_j\right) a_n(t) + b_{n,m}.$$

Hence $\limsup_{n\to\infty} a_n(t) \leq \liminf_{n\to\infty} a_n(t)$, this ensures that $\lim_{n\to\infty} a_n(t)$ exists for all $t \in (0,1)$.

Now apply Lemma 2.4 to conclude that $\{x_n\}$ converges weakly to a common fixed point of the family $\{T_j\}_{j=0}^{N-1}$. □

Theorem 3.2 *Let E be a real q-uniformly smooth Banach space, and let K be a nonempty, closed and convex subset of E. Let $N \geq 1$ be an integer and $J = \{0,1,2,\dots,N-1\}$. Let, for each $j \in J$, $T_j : K \to K$ be an asymptotically λ_j-strictly pseudocontractive mapping for some $0 \leq \lambda_j < 1$ with sequences $\{\kappa_{n,j}\}_{n=0}^{\infty} \subset [1,\infty)$ such that $\sum_{n=0}^{\infty}(\kappa_n - 1) < \infty$, where $\kappa_n = \max_{j\in J}\{\kappa_{n,j}\}$, and $F := \bigcap_{j=0}^{N-1} F(T_j) \neq \emptyset$. Let $\lambda = \min_{j\in J}\{\lambda_j\}$. Let $\{\alpha_n\}$ satisfy the conditions* (6) *and $\{x_n\}$ be the sequence generated by the cyclic algorithm* (5). *Then $\{x_n\}$ converges strongly to a common fixed point of the family $\{T_j\}_{j=0}^{N-1}$ if and only if*

$$\liminf_{n\to\infty} d(x_n, F) = 0,$$

where $d(x_n, F) = \inf_{p\in F} \|x_n - p\|$.

Proof It follows from (10) that

$$\|x_{n+1} - p\|^q \leq \delta_n \|x_n - p\|^q.$$

Thus $[d(x_{n+1} - p)]^q \leq \delta_n [d(x_n - p)]^q$, and it follows from Lemma 2.2 that $\lim_{n\to\infty} d(x_n, F)$ exists.

Now if $\{x_n\}$ converges strongly to a common fixed point p of the family $\{T_j\}_{j=0}^{N-1}$, then $\lim_{n\to\infty} \|x_n - p\| = 0$. Since

$$0 \leq d(x_n, F) \leq \|x_n - p\|,$$

we have $\liminf_{n\to\infty} d(x_n, F) = 0$.

Conversely, suppose $\liminf_{n\to\infty} d(x_n, F) = 0$, then the existence of $\lim_{n\to\infty} d(x_n, F)$ implies that $\lim_{n\to\infty} d(x_n, F) = 0$. Thus, for arbitrary $\epsilon > 0$, there exists a positive integer n_0 such that $d(x_n, F) < \frac{\epsilon}{2}$ for any $n \geq n_0$.

From (10), we have

$$\|x_{n+1} - p\|^q \leq \|x_n - p\|^q + M^q(\delta_n - 1), \quad n \geq 0,$$

and for some $M > 0$, $\|x_n - p\| < M$. Now, an induction yields

$$\begin{aligned} \|x_n - p\|^q &\leq \|x_{n-1} - p\|^q + M^q(\delta_{n-1} - 1) \\ &\leq \|x_{n-2} - p\|^q + M^q(\delta_{n-2} - 1) + M^q(\delta_{n-1} - 1) \\ &\leq \cdots \leq \|x_l - p\|^q + M^q \sum_{j=l}^{n-1} (\delta_j - 1), \quad n - 1 \geq l \geq 0. \end{aligned}$$

Since $\sum_{n=0}^{\infty}(\delta_n - 1) < \infty$, then there exists a positive integer n_1 such that $\sum_{j=n}^{\infty}(\delta_j - 1) < (\frac{\epsilon}{2M})^q$, $\forall n \geq n_1$. Choose $N = \max\{n_0, n_1\}$, then for all $n, m \geq N+1$ and for all $p \in F$, we have

$$\begin{aligned}
\|x_n - x_m\| &\leq \|x_n - p\| + \|x_m - p\| \\
&\leq \left[\|x_N - p\|^q + M^q \sum_{j=N}^{n-1}(\delta_j - 1)\right]^{\frac{1}{q}} + \left[\|x_N - p\|^q + M^q \sum_{j=N}^{m-1}(\delta_j - 1)\right]^{\frac{1}{q}} \\
&\leq \left[\|x_N - p\|^q + M^q \sum_{j=N}^{\infty}(\delta_j - 1)\right]^{\frac{1}{q}} + \left[\|x_N - p\|^q + M^q \sum_{j=N}^{\infty}(\delta_j - 1)\right]^{\frac{1}{q}} \\
&= 2\left[\|x_N - p\|^q + M^q \sum_{j=N}^{\infty}(\delta_j - 1)\right]^{\frac{1}{q}}.
\end{aligned}$$

Taking infimum over all $p \in F$, we obtain

$$\begin{aligned}
\|x_n - x_m\| &\leq 2\left\{\left[d(x_N, F)\right]^q + M^q \sum_{j=N}^{\infty}(\delta_j - 1)\right\}^{\frac{1}{q}} \\
&< 2\left[\left(\frac{\epsilon}{2}\right)^q + M^q\left(\frac{\epsilon}{2M}\right)^q\right]^{\frac{1}{q}} < 2\epsilon.
\end{aligned}$$

Thus $\{x_n\}_{n=0}^{\infty}$ is Cauchy. Suppose $\lim_{n\to} x_n = u$. Then for all $j \in J$ we have

$$0 \leq \|u - T_j u\| \leq \|u - x_n\| + \|x_n - T_j x_n\| + L\|x_n - u\| \to 0 \quad \text{as } n \to \infty.$$

Thus $u \in F(T_j)$, $\forall j \in J$, and hence $u \in F$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All the authors have read and approved the final manuscript.

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