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A new iterative algorithm for solving common solutions of generalized mixed equilibrium problems, fixed point problems and variational inclusion problems with minimization problems

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Abstract

In this article, we introduce a new general iterative method for solving a common element of the set of solutions of fixed point for nonexpansive mappings, the set of solutions of generalized mixed equilibrium problems and the set of solutions of the variational inclusion for a β-inverse-strongly monotone mapping in a real Hilbert space. We prove that the sequence converges strongly to a common element of the above three sets under some mild conditions. Our results improve and extend the corresponding results of Marino and Xu (J. Math. Anal. Appl. 318:43-52, 2006), Su *et al.* (Nonlinear Anal. 69:2709-2719, 2008), Tan and Chang (Fixed Point Theory Appl. 2011:915629, 2011) and some authors.

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1 Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$, respectively. A mapping $S: C \to C$ is said to be *nonexpansive* if $\|Sx - Sy\| \le \|x - y\|$, $\forall x, y \in C$. If C is bounded closed convex and S is a nonexpansive mapping of C into itself, then $F(S) := \{x \in C : Sx = x\}$ is nonempty [1]. A mapping $S: C \to C$ is said to be a *k-strictly pseudo-contraction* [2] if there exists $0 \le k < 1$ such that $\|Sx - Sy\|^2 \le \|x - y\|^2 + k\|(I - S)x - (I - S)y\|^2$, $\forall x, y \in C$, where I denotes the identity operator on C. We denote weak convergence and strong convergence by notations $\rightharpoonup$ and $\to$, respectively. A mapping A of C into H is called *monotone* if $\langle Ax - Ay, x - y\rangle \ge 0$, $\forall x, y \in C$. A mapping A is called *α-inverse-strongly monotone* if there exists a positive real number α such that $\langle Ax - Ay, x - y\rangle \ge \alpha\|Ax - Ay\|^2$, $\forall x, y \in C$. A mapping A is called *α-strongly monotone* if there exists a positive real number α such that $\langle Ax - Ay, x - y\rangle \ge \alpha\|x - y\|^2$, $\forall x, y \in C$. It is obvious that any α-inverse-strongly monotone mappings A is a monotone and $\frac{1}{\alpha}$-Lipschitz continuous mapping. A linear bounded operator A is called *strongly positive* if there exists a constant $\bar{\gamma} > 0$ with the property $\langle Ax, x\rangle \ge \bar{\gamma}\|x\|^2$, $\forall x \in H$. A self mapping $f : C \to C$ is called *contraction* on C if there exists a constant $\alpha \in (0,1)$ such that $\|f(x) - f(y)\| \leq \alpha\|x - y\|$, $\forall x, y \in C$.

Let $B : H \to H$ be a single-valued nonlinear mapping and $M : H \to 2^H$ be a set-valued mapping. The *variational inclusion problem* is to find $x \in H$ such that

$$\theta \in B(x) + M(x), \tag{1.1}$$

where θ is the zero vector in H. The set of solutions of (1.1) is denoted by $I(B,M)$. The variational inclusion has been extensively studied in the literature. See, *e.g.* [3–10] and the reference therein.

A set-valued mapping $M : H \to 2^H$ is called *monotone* if $\forall x, y \in H, f \in M(x)$ and $g \in M(y)$ imply $\langle x - y, f - g\rangle \geq 0$. A monotone mapping M is *maximal* if its graph $G(M) := \{(f, x) \in H \times H : f \in M(x)\}$ of M is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping M is maximal if and only if for $(x,f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for all $(y,g) \in G(M)$ imply $f \in M(x)$.

Let B be an inverse-strongly monotone mapping of C into H and let $N_C v$ be normal cone to C at $v \in C$, *i.e.*, $N_C v = \{w \in H : \langle v - u, w\rangle \geq 0, \forall u \in C\}$, and define

$$Mv = \begin{cases} Bv + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases}$$

Then M is a maximal monotone and $\theta \in Mv$ if and only if $v \in \mathrm{VI}(C,B)$ (see [11]).

Let $M : H \to 2^H$ be a set-valued maximal monotone mapping, then the single-valued mapping $J_{M,\lambda} : H \to H$ defined by

$$J_{M,\lambda}(x) = (I + \lambda M)^{-1}(x), \quad x \in H \tag{1.2}$$

is called the *resolvent operator* associated with M, where λ is any positive number and I is the identity mapping. In the worth mentioning that the resolvent operator is nonexpansive, 1-inverse-strongly monotone and that a solution of problem (1.1) is a fixed point of the operator $J_{M,\lambda}(I - \lambda B)$ for all $\lambda > 0$ (see [12]).

Let F be a bifunction of $C \times C$ into $\mathcal{R}$, where $\mathcal{R}$ is the set of real numbers, $\Psi : C \to H$ be a mapping and $\psi : C \to \mathcal{R}$ be a real-valued function. The *generalized mixed equilibrium problem* for finding $x \in C$ such that

$$F(x,y) + \langle \Psi x, y - x\rangle + \psi(y) - \psi(x) \geq 0, \quad \forall y \in C. \tag{1.3}$$

The set of solutions of (1.3) is denoted by $\mathrm{GMEP}(F, \psi, \Psi)$, that is

$$\mathrm{GMEP}(F, \psi, \Psi) = \big\{x \in C : F(x,y) + \langle \Psi x, y - x\rangle + \psi(y) - \psi(x) \geq 0, \forall y \in C\big\}.$$

If $\Psi \equiv 0$ and $\psi \equiv 0$, the problem (1.3) is reduced into the *equilibrium problem* (see also [13]) for finding $x \in C$ such that

$$F(x,y) \geq 0, \quad \forall y \in C. \tag{1.4}$$

The set of solutions of (1.4) is denoted by EP(F), that is

$$\mathrm{EP}(F) = \{x \in C : F(x,y) \geq 0, \forall y \in C\}.$$

This problem contains fixed point problems, includes as special cases numerous problems in physics, optimization and economics. Some methods have been proposed to solve the equilibrium problem, please consult [14–16].

If $F \equiv 0$ and $\psi \equiv 0$, the problem (1.3) is reduced into the *Hartmann-Stampacchia variational inequality* [17] for finding $x \in C$ such that

$$\langle \Psi x, y - x \rangle \geq 0, \quad \forall y \in C. \tag{1.5}$$

The set of solutions of (1.5) is denoted by VI(C, Ψ). The variational inequality has been extensively studied in the literature [18].

If $F \equiv 0$ and $\Psi \equiv 0$, the problem (1.3) is reduced into the *minimize problem* for finding $x \in C$ such that

$$\psi(y) - \psi(x) \geq 0, \quad \forall y \in C. \tag{1.6}$$

The set of solutions of (1.6) is denoted by Argmin(ψ). Iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems. Convex minimization problems have a great impact and influence in the development of almost all branches of pure and applied sciences. A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\theta(x) = \frac{1}{2}\langle Ax, x\rangle - \langle x, y\rangle, \quad \forall x \in F(S), \tag{1.7}$$

where A is a linear bounded operator, $F(S)$ is the fixed point set of a nonexpansive mapping S and y is a given point in H [19].

In 2000, Moudafi [20] introduced the viscosity approximation method for nonexpansive mapping and prove that if H is a real Hilbert space, the sequence $\{x_n\}$ defined by the iterative method below, with the initial guess $x_0 \in C$ is chosen arbitrarily,

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Sx_n, \quad n \geq 0, \tag{1.8}$$

where $\{\alpha_n\} \subset (0,1)$ satisfies certain conditions, converge strongly to a fixed point of S (say $\bar{x} \in C$) which is the unique solution of the following variational inequality:

$$\langle (I - f)\bar{x}, x - \bar{x} \rangle \geq 0, \quad \forall x \in F(S). \tag{1.9}$$

In 2005, Iiduka and Takahashi [21] introduced following iterative process $x_0 \in C$,

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)SP_C(x_n - \lambda_n Ax_n), \quad \forall n \geq 0, \tag{1.10}$$

where $u \in C$, $\{\alpha_n\} \subset (0,1)$ and $\{\lambda_n\} \subset [a,b]$ for some a, b with $0 < a < b < 2\beta$. They proved that under certain appropriate conditions imposed on $\{\alpha_n\}$ and $\{\lambda_n\}$, the sequence $\{x_n\}$

generated by (1.10) converges strongly to a common element of the set of fixed points of nonexpansive mapping and the set of solutions of the variational inequality for an inverse-strongly monotone mapping (say $\bar{x} \in C$) which solve some variational inequality

$$\langle \bar{x} - u, x - \bar{x} \rangle \geq 0, \quad \forall x \in F(S). \tag{1.11}$$

In 2006, Marino and Xu [19] introduced a general iterative method for nonexpansive mapping. They defined the sequence $\{x_n\}$ generated by the algorithm $x_0 \in C$,

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Sx_n, \quad n \geq 0, \tag{1.12}$$

where $\{\alpha_n\} \subset (0,1)$ and A is a strongly positive linear bounded operator. They proved that if $C = H$ and the sequence $\{\alpha_n\}$ satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (1.12) converge strongly to a fixed point of S (say $\bar{x} \in H$) which is the unique solution of the following variational inequality:

$$\big\langle (A - \gamma f)\bar{x}, x - \bar{x} \big\rangle \geq 0, \quad \forall x \in F(S), \tag{1.13}$$

which is the optimality condition for the minimization problem

$$\min_{x \in F(S)} \frac{1}{2} \langle Ax, x \rangle - h(x), \tag{1.14}$$

where h is a potential function for γf (*i.e.*, $h'(x) = \gamma f(x)$ for $x \in H$).

In 2008, Su *et al.* [22] introduced the following iterative scheme by the viscosity approximation method in a real Hilbert space: $x_1, u_n \in H$

$$\begin{cases} F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) SP_C(u_n - \lambda_n A u_n), \end{cases} \tag{1.15}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [0,1)$ and $\{r_n\} \subset (0,\infty)$ satisfy some appropriate conditions. Furthermore, they proved $\{x_n\}$ and $\{u_n\}$ converge strongly to the same point $z \in F(S) \cap \mathrm{VI}(C,A) \cap \mathrm{EP}(F)$ where $z = P_{F(S)\cap \mathrm{VI}(C,A)\cap \mathrm{EP}(F)} f(z)$.

In 2011, Tan and Chang [10] introduced following iterative process for $\{T_n : C \to C\}$ is a sequence of nonexpansive mappings. Let $\{x_n\}$ be the sequence defined by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)\big(SP_C\big((1 - t_n)J_{M,\lambda}(I - \lambda A)T_n(I - \mu B)\big)x_n\big), \quad \forall n \geq 0, \tag{1.16}$$

where $\{\alpha_n\} \subset (0,1)$, $\lambda \in (0, 2\alpha]$ and $\mu \in (0, 2\beta]$. The sequence $\{x_n\}$ defined by (1.16) converges strongly to a common element of the set of fixed points of nonexpansive mappings, the set of solutions of the variational inequality and the generalized equilibrium problem.

In this article, we mixed and modified the iterative methods (1.12), (1.15) and (1.16) by purposing the following new general viscosity iterative method: $x_0, u_n \in C$ and

$$\begin{cases} u_n = T_{r_n}^{(F_1,\psi_1)}(x_n - r_n B_1 x_n), \\ v_n = T_{s_n}^{(F_2,\psi_2)}(x_n - s_n B_2 x_n), \\ x_{n+1} = \xi_n P_C[\alpha_n \gamma f(x_n) + (I - \alpha_n A)SJ_{M,\lambda}(I - \lambda B)u_n] + (1 - \xi_n)v_n, \quad n \geq 0, \end{cases}$$

where $\{\alpha_n\}, \{\xi_n\} \subset (0,1)$, $\lambda \in (0, 2\beta)$ such that $0 < a \leq \lambda \leq b < 2\beta$, $\{r_n\} \in (0, 2\eta)$ with $0 < c \leq d \leq 1 - \eta$ and $\{s_n\} \in (0, 2\rho)$ with $0 < e \leq f \leq 1 - \rho$ satisfy some appropriate conditions. The purpose of this article, we show that under some control conditions the sequence $\{x_n\}$ converges strongly to a common element of the set of fixed points of nonexpansive mappings, the common solutions of the generalized mixed equilibrium problem and the set of solutions of the variational inclusion in a real Hilbert space.

2 Preliminaries

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$, respectively. Let C be a nonempty closed convex subset of H. Recall that the metric (nearest point) projection P_C from H onto C assigns to each $x \in H$, the unique point in $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|.$$

The following characterizes the projection P_C. We recall some lemmas which will be needed in the rest of this article.

Lemma 2.1 *The function $u \in C$ is a solution of the variational inequality (1.5) if and only if $u \in C$ satisfies the relation $u = P_C(u - \lambda \Psi u)$ for all $\lambda > 0$.*

Lemma 2.2 *For a given $z \in H$, $u \in C$, $u = P_C z \Leftrightarrow \langle u - z, v - u \rangle \geq 0$, $\forall v \in C$.*

It is well known that P_C is a firmly nonexpansive mapping of H onto C and satisfies

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle, \quad \forall x, y \in H. \tag{2.1}$$

Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and for all $x \in H$, $y \in C$,

$$\langle x - P_C x, y - P_C x \rangle \leq 0. \tag{2.2}$$

Lemma 2.3 ([23]) *Let $M : H \to 2^H$ be a maximal monotone mapping and let $B : H \to H$ be a monotone and Lipshitz continuous mapping. Then the mapping $L = M + B : H \to 2^H$ is a maximal monotone mapping.*

Lemma 2.4 ([24]) *Each Hilbert space H satisfies Opial's condition, that is, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality $\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$, hold for each $y \in H$ with $y \neq x$.*

Lemma 2.5 ([25]) *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n) a_n + \delta_n, \quad \forall n \geq 0,$$

where $\{\gamma_n\} \subset (0,1)$ and $\{\delta_n\}$ is a sequence in $\mathcal{R}$ such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(ii) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.6 ([26]) *Let C be a closed convex subset of a real Hilbert space H and let $T : C \to C$ be a nonexpansive mapping. Then $I - T$ is demiclosed at zero, that is,*

$$x_n \rightharpoonup x, \qquad x_n - Tx_n \to 0$$

implies $x = Tx$.

For solving the generalized mixed equilibrium problem, let us assume that the bifunction $F : C \times C \to \mathcal{R}$, the nonlinear mapping $\Psi : C \to H$ is continuous monotone and $\psi : C \to \mathcal{R}$ satisfies the following conditions:

(A1) $F(x,x) = 0$ for all $x \in C$;

(A2) F is monotone, *i.e.*, $F(x,y) + F(y,x) \leq 0$ for any $x,y \in C$;

(A3) for each fixed $y \in C$, $x \mapsto F(x,y)$ is weakly upper semicontinuous;

(A4) for each fixed $x \in C$, $y \mapsto F(x,y)$ is convex and lower semicontinuous;

(B1) for each $x \in C$ and $r > 0$, there exist a bounded subset $D_x \subseteq C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,

$$F(z,y_x) + \psi(y_x) - \psi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0, \tag{2.3}$$

(B2) C is a bounded set.

Lemma 2.7 ([27]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C \times C \to \mathcal{R}$ be a bifunction mapping satisfies (A1)-(A4) and let $\psi : C \to \mathcal{R}$ is convex and lower semicontinuous such that $C \cap \operatorname{dom}\psi \neq \emptyset$. Assume that either (B1) or (B2) holds. For $r > 0$ and $x \in H$, then there exists $u \in C$ such that*

$$F(u,y) + \psi(y) - \psi(u) + \frac{1}{r}\langle y - u, u - x\rangle \geq 0.$$

Define a mapping $T_r^{(F,\psi)} : H \to C$ as follows:

$$T_r^{(F,\psi)}(x) = \left\{u \in C : F(u,y) + \psi(y) - \psi(u) + \frac{1}{r}\langle y - u, u - x\rangle \geq 0, \forall y \in C\right\} \tag{2.4}$$

for all $x \in H$. Then, the following hold:

(i) $T_r^{(F,\psi)}$ is single-valued;

(ii) $T_r^{(F,\psi)}$ is firmly nonexpansive, i.e., for any $x,y \in H$,

$$\left\|T_r^{(F,\psi)}x - T_r^{(F,\psi)}y\right\|^2 \leq \left\langle T_r^{(F,\psi)}x - T_r^{(F,\psi)}y, x - y\right\rangle;$$

(iii) $F(T_r^{(F,\psi)}) = \mathrm{MEP}(F,\psi)$;

(iv) $\mathrm{MEP}(F,\psi)$ is closed and convex.

Lemma 2.8 ([19]) *Assume A is a strongly positive linear bounded operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$, then $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$.*

Lemma 2.9 ([28]) *Let H be a real Hilbert space and $A : H \to H$ a mapping.*

(i) If A is δ-strongly monotone and μ-strictly pseudo-contraction with $\delta + \mu > 1$, then $I - A$ is contraction with constant $\sqrt{(1-\delta)/\mu}$.

(ii) If A is δ-strongly monotone and μ-strictly pseudo-contraction with $\delta + \mu > 1$, then for any fixed number $\tau \in (0,1)$, $I - \tau A$ is contraction with constant $1 - \tau(1 - \sqrt{(1-\delta)/\mu})$.

3 Strong convergence theorems

In this section, we show a strong convergence theorem which solves the problem of finding a common element of $F(S)$, $\mathrm{GMEP}(F_1, \psi_1, B_1)$, $\mathrm{GMEP}(F_2, \psi_2, B_2)$ and $I(B,M)$ of an inverse-strongly monotone mapping in a real Hilbert space.

Theorem 3.1 *Let H be a real Hilbert space, C be a closed convex subset of H. Let F_1, F_2 be two bifunctions of $C \times C$ into $\mathcal{R}$ satisfying (A1)-(A4) and $B, B_1, B_2 : C \to H$ be β, η, ρ-inverse-strongly monotone mappings, $\psi_1, \psi_2 : C \to \mathcal{R}$ be convex and lower semicontinuous function, $f : C \to C$ be a contraction with coefficient α ($0 < \alpha < 1$), $M : H \to 2^H$ be a maximal monotone mapping and A be a δ-strongly monotone and μ-strictly pseudo-contraction mapping with $\delta + \mu > 1$, γ is a positive real number such that $\gamma < \frac{1}{\alpha}(1 - \sqrt{\frac{1-\delta}{\mu}})$. Assume that either (B1) or (B2) holds. Let S be a nonexpansive mapping of H into itself such that*

$$\Theta := F(S) \cap \mathrm{GMEP}(F_1, \psi_1, B_1) \cap \mathrm{GMEP}(F_2, \psi_2, B_2) \cap I(B, M) \neq \emptyset.$$

Suppose $\{x_n\}$ is a sequence generated by the following algorithm $x_0 \in C$ arbitrarily:

$$\begin{cases} u_n = T_{r_n}^{(F_1,\psi_1)}(x_n - r_n B_1 x_n), \\ v_n = T_{s_n}^{(F_2,\psi_2)}(x_n - s_n B_2 x_n), \\ x_{n+1} = \xi_n P_C[\alpha_n \gamma f(x_n) + (I - \alpha_n A) S J_{M,\lambda}(I - \lambda B) u_n] + (1 - \xi_n) v_n, \quad n \geq 0, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}, \{\xi_n\} \subset (0,1)$, $\lambda \in (0, 2\beta)$ such that $0 < a \leq \lambda \leq b < 2\beta$, $\{r_n\} \in (0, 2\eta)$ with $0 < c \leq d \leq 1 - \eta$ and $\{s_n\} \in (0, 2\rho)$ with $0 < e \leq f \leq 1 - \rho$ satisfy the following conditions:

(C1): $\lim_{n\to\infty} \alpha_n = 0$, $\Sigma_{n=0}^{\infty} \alpha_n = \infty$, $\Sigma_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,

(C2): $0 < \liminf_{n\to\infty} \xi_n < \limsup_{n\to\infty} \xi_n < 1$, $\Sigma_{n=1}^{\infty} |\xi_{n+1} - \xi_n| < \infty$,

(C3): $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$,

(C4): $\liminf_{n\to\infty} s_n > 0$ *and* $\lim_{n\to\infty} |s_{n+1} - s_n| = 0$.

Then $\{x_n\}$ converges strongly to $q \in \Theta$, where $q = P_\Theta(\gamma f + I - A)(q)$ which solves the following variational inequality:

$$\langle (\gamma f - A) q, p - q \rangle \leq 0, \quad \forall p \in \Theta$$

which is the optimality condition for the minimization problem

$$\min_{q \in \Theta} \frac{1}{2} \langle Aq, q \rangle - h(q), \tag{3.2}$$

where h is a potential function for γf (i.e., $h'(q) = \gamma f(q)$ for $q \in H$).

Proof Since B is β-inverse-strongly monotone mappings, we have

$$\begin{aligned} \|(I - \lambda B)x - (I - \lambda B)y\|^2 &= \|(x - y) - \lambda(Bx - By)\|^2 \\ &= \|x - y\|^2 - 2\lambda \langle x - y, Bx - By \rangle + \lambda^2 \|Bx - By\|^2 \end{aligned}$$

$$\begin{aligned}&\leq \|x-y\|^2+\lambda(\lambda-2\beta)\|Bx-By\|^2\\&\leq \|x-y\|^2. \qquad (3.3)\end{aligned}$$

And B_1, B_2 are η, ρ-inverse-strongly monotone mappings, we have

$$\begin{aligned}\big\|(I-r_nB_1)x-(I-r_nB_1)y\big\|^2 &= \big\|(x-y)-r_n(B_1x-B_1y)\big\|^2\\&= \|x-y\|^2-2r_n\langle x-y, B_1x-B_1y\rangle+r_n^2\|B_1x-B_1y\|^2\\&\leq \|x-y\|^2+r_n(r_n-2\eta)\|B_1x-B_1y\|^2\\&\leq \|x-y\|^2. \qquad (3.4)\end{aligned}$$

In similar way, we can obtain

$$\big\|(I-s_nB_2)x-(I-s_nB_2)y\big\|^2\leq \|x-y\|^2. \qquad (3.5)$$

It is clear that if $0<\lambda<2\beta$, $0<r_n<2\eta$, $0<s_n\leq 2\rho$ then $I-\lambda B$, $I-r_nB_1$, $I-s_nB_2$ are all nonexpansive. We will divide the proof into six steps.

Step 1. We will show $\{x_n\}$ is bounded. Put $y_n=J_{M,\lambda}(u_n-\lambda Bu_n)$, $n\geq 0$. It follows that

$$\begin{aligned}\|y_n-q\| &= \big\|J_{M,\lambda}(u_n-\lambda Bu_n)-J_{M,\lambda}(q-\lambda Bq)\big\|\\&\leq \|u_n-q\|. \qquad (3.6)\end{aligned}$$

By Lemma 2.7, we have $u_n=T_{r_n}^{(F_1,\psi_1)}(x_n-r_nB_1x_n)$ for all $n\geq 0$. Then, we note that

$$\begin{aligned}\|u_n-q\|^2 &= \big\|T_{r_n}^{(F_1,\psi_1)}(x_n-r_nB_1x_n)-T_{r_n}^{(F_1,\psi_1)}(q-r_nB_1q)\big\|^2\\&\leq \big\|(x_n-r_nB_1x_n)-(q-r_nB_1q)\big\|^2\\&\leq \|x_n-q\|^2+r_n(r_n-2\eta)\|B_1x_n-B_1q\|^2\\&\leq \|x_n-q\|^2. \qquad (3.7)\end{aligned}$$

In similar way, we can obtain

$$\begin{aligned}\|v_n-q\|^2 &= \big\|T_{s_n}^{(F_2,\psi_2)}(x_n-s_nB_2x_n)-T_{s_n}^{(F_2,\psi_2)}(q-s_nB_2q)\big\|^2\\&\leq \big\|(x_n-s_nB_2x_n)-(q-s_nB_2q)\big\|^2\\&\leq \|x_n-q\|^2+s_n(s_n-2\rho)\|B_2x_n-B_2q\|^2\\&\leq \|x_n-q\|^2. \qquad (3.8)\end{aligned}$$

Put $z_n=P_C[\alpha_n\gamma f(x_n)+(I-\alpha_nA)Sy_n]$ for all $n\geq 0$. From (3.1) and Lemma 2.9(ii), we deduce that

$$\begin{aligned}&\|x_{n+1}-q\|\\&\quad= \big\|\xi_n(z_n-q)+(1-\xi_n)(v_n-q)\big\|\\&\quad\leq \xi_n\big\|P_C\big[\alpha_n\gamma f(x_n)+(I-\alpha_nA)Sy_n\big]-P_Cq\big\|+(1-\xi_n)\|v_n-q\|\end{aligned}$$

$$
\begin{aligned}
&\leq \xi_n \|\alpha_n \gamma f(x_n) + (I - \alpha_n A) S y_n - q\| + (1 - \xi_n)\|v_n - q\| \\
&= \xi_n \|\alpha_n(\gamma f(x_n) - Aq) + (I - \alpha_n A)(S y_n - q)\| + (1 - \xi_n)\|v_n - q\| \\
&\leq \xi_n \alpha_n \|\gamma f(x_n) - Aq\| + \xi_n \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right)\|y_n - q\| + (1 - \xi_n)\|v_n - q\| \\
&\leq \xi_n \alpha_n \gamma \alpha \|x_n - q\| + \xi_n \alpha_n \|\gamma f(q) - Aq\| + \xi_n \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right)\|x_n - q\| \\
&\quad + (1 - \xi_n)\|x_n - q\| \\
&= \left(1 - \left(1 - \sqrt{\frac{1-\delta}{\mu}} - \gamma\alpha\right)\xi_n \alpha_n\right)\|x_n - q\| + \xi_n \alpha_n \|\gamma f(q) - Aq\| \\
&\leq \left(1 - \left(1 - \sqrt{\frac{1-\delta}{\mu}} - \gamma\alpha\right)\xi_n \alpha_n\right)\|x_n - q\| \\
&\quad + \left(1 - \sqrt{\frac{1-\delta}{\mu}} - \gamma\alpha\right)\xi_n \alpha_n \frac{\|\gamma f(q) - Aq\|}{(1 - \sqrt{\frac{1-\delta}{\mu}} - \gamma\alpha)} \\
&\leq \max\left\{\|x_n - q\|, \frac{\|\gamma f(q) - Aq\|}{1 - \sqrt{\frac{1-\delta}{\mu}} - \gamma\alpha}\right\}.
\end{aligned}
\tag{3.9}
$$

It follows from induction that

$$
\|x_n - q\| \leq \max\left\{\|x_0 - q\|, \frac{\|\gamma f(q) - Aq\|}{1 - \sqrt{\frac{1-\delta}{\mu}} - \gamma\alpha}\right\}, \quad n \geq 0.
$$

Therefore $\{x_n\}$ is bounded, so are $\{v_n\}$, $\{y_n\}$, $\{z_n\}$, $\{Sy_n\}$, $\{f(x_n)\}$ and $\{ASy_n\}$.

Step 2. We claim that $\lim_{n\to\infty} \|x_{n+2} - x_{n+1}\| = 0$. From (3.1), we have

$$
\begin{aligned}
\|x_{n+2} - x_{n+1}\| &= \|\xi_{n+1} z_{n+1} + (1 - \xi_{n+1}) v_{n+1} - \xi_n z_n - (1 - \xi_n) v_n\| \\
&= \|\xi_{n+1}(z_{n+1} - z_n) + (\xi_{n+1} - \xi_n) z_n \\
&\quad + (1 - \xi_{n+1})(v_{n+1} - v_n) + (\xi_{n+1} - \xi_n) v_n\| \\
&\leq \xi_{n+1}\|z_{n+1} - z_n\| + (1 - \xi_{n+1})\|v_{n+1} - v_n\| \\
&\quad + |\xi_{n+1} - \xi_n|(\|z_n\| + \|v_n\|).
\end{aligned}
\tag{3.10}
$$

We will estimate $\|v_{n+1} - v_n\|$. On the other hand, from $v_{n-1} = T_{s_{n-1}}^{(F_2,\psi_2)}(x_{n-1} - s_{n-1}B_2 x_{n-1})$ and $v_n = T_{s_n}^{(F_2,\psi_2)}(x_n - s_n B_2 x_n)$, it follows that

$$
\begin{aligned}
&F_2(v_{n-1}, y) + \langle B_2 x_{n-1}, y - v_{n-1}\rangle + \psi_2(y) - \psi_2(v_{n-1}) \\
&\quad + \frac{1}{s_{n-1}}\langle y - v_{n-1}, v_{n-1} - x_{n-1}\rangle \geq 0, \quad \forall y \in C
\end{aligned}
\tag{3.11}
$$

and

$$
F_2(v_n, y) + \langle B_2 x_n, y - v_n\rangle + \psi_2(y) - \psi_2(v_n) + \frac{1}{s_n}\langle y - v_n, v_n - x_n\rangle \geq 0, \quad \forall y \in C. \tag{3.12}
$$

Substituting $y = v_n$ in (3.11) and $y = v_{n-1}$ in (3.12), we get

$$F_2(v_{n-1}, v_n) + \langle B_2 x_{n-1}, v_n - v_{n-1}\rangle + \psi_2(v_n) - \psi_2(v_{n-1}) + \frac{1}{s_{n-1}}\langle v_n - v_{n-1}, v_{n-1} - x_{n-1}\rangle \geq 0$$

and

$$F_2(v_n, v_{n-1}) + \langle B_2 x_n, v_{n-1} - v_n\rangle + \psi_2(v_{n-1}) - \psi_2(v_n) + \frac{1}{s_n}\langle v_{n-1} - v_n, v_n - x_n\rangle \geq 0.$$

From (A2), we obtain

$$\left\langle v_n - v_{n-1}, B_2 x_{n-1} - B_2 x_n + \frac{v_{n-1} - x_{n-1}}{s_{n-1}} - \frac{v_n - x_n}{s_n}\right\rangle \geq 0,$$

and then

$$\left\langle v_n - v_{n-1}, s_{n-1}(B_2 x_{n-1} - B_2 x_n) + v_{n-1} - x_{n-1} - \frac{s_{n-1}}{s_n}(v_n - x_n)\right\rangle \geq 0,$$

so

$$\left\langle v_n - v_{n-1}, s_{n-1} B_2 x_{n-1} - s_{n-1} B_2 x_n + v_{n-1} - v_n + v_n - x_{n-1} - \frac{s_{n-1}}{s_n}(v_n - x_n)\right\rangle \geq 0.$$

It follows that

$$\left\langle v_n - v_{n-1}, (I - s_{n-1}B_2)x_n - (I - s_{n-1}B_2)x_{n-1} + v_{n-1} - v_n + v_n - x_n - \frac{s_{n-1}}{s_n}(v_n - x_n)\right\rangle \geq 0,$$

$$\langle v_n - v_{n-1}, v_{n-1} - v_n\rangle + \left\langle v_n - v_{n-1}, x_n - x_{n-1} + \left(1 - \frac{s_{n-1}}{s_n}\right)(v_n - x_n)\right\rangle \geq 0.$$

Without loss of generality, let us assume that there exists a real number e such that $s_{n-1} > e > 0$, for all $n \in \mathbb{N}$. Then, we have

$$\begin{aligned}\|v_n - v_{n-1}\|^2 &\leq \left\langle v_n - v_{n-1}, x_n - x_{n-1} + \left(1 - \frac{s_{n-1}}{s_n}\right)(v_n - x_n)\right\rangle \\ &\leq \|v_n - v_{n-1}\|\left\{\|x_n - x_{n-1}\| + \left|1 - \frac{s_{n-1}}{s_n}\right|\|v_n - x_n\|\right\}\end{aligned}$$

and hence

$$\begin{aligned}\|v_n - v_{n-1}\| &\leq \|x_n - x_{n-1}\| + \frac{1}{s_n}|s_n - s_{n-1}|\|v_n - x_n\| \\ &\leq \|x_n - x_{n-1}\| + \frac{M_1}{e}|s_n - s_{n-1}|, \qquad (3.13)\end{aligned}$$

where $M_1 = \sup\{\|v_n - x_n\| : n \in \mathbb{N}\}$. Substituting (3.13) into (3.10) that

$$\begin{aligned}\|x_{n+2} - x_{n+1}\| &\leq \xi_{n+1}\|z_{n+1} - z_n\| + (1 - \xi_{n+1})\left\{\|x_{n+1} - x_n\| + \frac{M_1}{e}|s_n - s_{n-1}|\right\} \\ &\quad + |\xi_{n+1} - \xi_n|\big(\|z_n\| + \|v_n\|\big). \qquad (3.14)\end{aligned}$$

We note that

$$\begin{aligned}
\|z_{n+1}-z_n\| &= \big\|P_C\big[\alpha_{n+1}\gamma f(x_{n+1})+(I-\alpha_{n+1}A)Sy_{n+1}\big]-P_C\big[\alpha_n\gamma f(x_n)-(I-\alpha_nA)Sy_n\big]\big\| \\
&\le \big\|\alpha_{n+1}\gamma f(x_{n+1})+(I-\alpha_{n+1}A)Sy_{n+1}-\big(\alpha_n\gamma f(x_n)-(I-\alpha_nA)Sy_n\big)\big\| \\
&\le \big\|\alpha_{n+1}\gamma\big(f(x_{n+1})-f(x_n)\big)+(\alpha_{n+1}-\alpha_n)\gamma f(x_n)+(I-\alpha_{n+1}A)(Sy_{n+1}-Sy_n) \\
&\quad +(\alpha_n-\alpha_{n+1})ASy_n\big\| \\
&\le \alpha_{n+1}\gamma\alpha\|x_{n+1}-x_n\|+|\alpha_{n+1}-\alpha_n|\big\|\gamma f(x_n)\big\| \\
&\quad +\left(1-\alpha_{n+1}\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)\|y_{n+1}-y_n\| \\
&\quad +|\alpha_{n+1}-\alpha_n|\|ASy_n\| \\
&\le \alpha_{n+1}\gamma\alpha\|x_{n+1}-x_n\|+|\alpha_{n+1}-\alpha_n|\big(\big\|\gamma f(x_n)\big\|+\|ASy_n\|\big) \\
&\quad +\left(1-\alpha_{n+1}\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)\|y_{n+1}-y_n\|. \qquad (3.15)
\end{aligned}$$

Since $I-\lambda B$ be nonexpansive, we have

$$\begin{aligned}
\|y_{n+1}-y_n\| &= \big\|J_{M,\lambda}(u_{n+1}-\lambda Bu_{n+1})-J_{M,\lambda}(u_n-\lambda Bu_n)\big\| \\
&\le \big\|(u_{n+1}-\lambda Bu_{n+1})-(u_n-\lambda Bu_n)\big\| \\
&\le \|u_{n+1}-u_n\|. \qquad (3.16)
\end{aligned}$$

On the other hand, from $u_{n-1}=T_{r_{n-1}}^{(F_1,\psi_1)}(x_{n-1}-r_{n-1}B_1x_{n-1})$ and $u_n=T_{r_n}^{(F_1,\psi_1)}(x_n-r_nB_1x_n)$, it follows that

$$\begin{aligned}
&F_1(u_{n-1},y)+\langle B_1x_{n-1},y-u_{n-1}\rangle+\psi_1(y)-\psi_1(u_{n-1}) \\
&\quad +\frac{1}{r_{n-1}}\langle y-u_{n-1},u_{n-1}-x_{n-1}\rangle\ge 0,\quad \forall y\in C \qquad (3.17)
\end{aligned}$$

and

$$F_1(u_n,y)+\langle B_1x_n,y-u_n\rangle+\psi_1(y)-\psi_1(u_n)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\ge 0,\quad \forall y\in C. \qquad (3.18)$$

Substituting $y=u_n$ in (3.17) and $y=u_{n-1}$ in (3.18), we get

$$F_1(u_{n-1},u_n)+\langle B_1x_{n-1},u_n-u_{n-1}\rangle+\psi_1(u_n)-\psi_1(u_{n-1})+\frac{1}{r_{n-1}}\langle u_n-u_{n-1},u_{n-1}-x_{n-1}\rangle\ge 0$$

and

$$F_1(u_n,u_{n-1})+\langle B_1x_n,u_{n-1}-u_n\rangle+\psi_1(u_{n-1})-\psi_1(u_n)+\frac{1}{r_n}\langle u_{n-1}-u_n,u_n-x_n\rangle\ge 0.$$

From (A2), we obtain

$$\left\langle u_n-u_{n-1},B_1x_{n-1}-B_1x_n+\frac{u_{n-1}-x_{n-1}}{r_{n-1}}-\frac{u_n-x_n}{r_n}\right\rangle\ge 0,$$

and then

$$\left\langle u_n - u_{n-1}, r_{n-1}(B_1x_{n-1} - B_1x_n) + u_{n-1} - x_{n-1} - \frac{r_{n-1}}{r_n}(u_n - x_n)\right\rangle \geq 0,$$

so

$$\left\langle u_n - u_{n-1}, r_{n-1}B_1x_{n-1} - r_{n-1}B_1x_n + u_{n-1} - u_n + u_n - x_{n-1} - \frac{r_{n-1}}{r_n}(u_n - x_n)\right\rangle \geq 0.$$

It follows that

$$\left\langle u_n - u_{n-1}, (I - r_{n-1}B_1)x_n - (I - r_{n-1}B_1)x_{n-1} + u_{n-1} - u_n + u_n - x_n - \frac{r_{n-1}}{r_n}(u_n - x_n)\right\rangle \geq 0,$$

$$\langle u_n - u_{n-1}, u_{n-1} - u_n\rangle + \left\langle u_n - u_{n-1}, x_n - x_{n-1} + \left(1 - \frac{r_{n-1}}{r_n}\right)(u_n - x_n)\right\rangle \geq 0.$$

Without loss of generality, let us assume that there exists a real number c such that $r_{n-1} > c > 0$, for all $n \in \mathbb{N}$. Then, we have

$$\begin{aligned}\|u_n - u_{n-1}\|^2 &\leq \left\langle u_n - u_{n-1}, x_n - x_{n-1} + \left(1 - \frac{r_{n-1}}{r_n}\right)(u_n - x_n)\right\rangle \\ &\leq \|u_n - u_{n-1}\| \left\{ \|x_n - x_{n-1}\| + \left|1 - \frac{r_{n-1}}{r_n}\right| \|u_n - x_n\| \right\}\end{aligned}$$

and hence

$$\begin{aligned}\|u_n - u_{n-1}\| &\leq \|x_n - x_{n-1}\| + \frac{1}{r_n}|r_n - r_{n-1}|\|u_n - x_n\| \\ &\leq \|x_n - x_{n-1}\| + \frac{M_2}{c}|r_n - r_{n-1}|, \end{aligned} \tag{3.19}$$

where $M_2 = \sup\{\|u_n - x_n\| : n \in \mathbb{N}\}$. Substituting (3.19) into (3.16), we have

$$\|y_n - y_{n-1}\| \leq \|x_n - x_{n-1}\| + \frac{M_2}{c}|r_n - r_{n-1}|. \tag{3.20}$$

Substituting (3.20) into (3.15), we obtain that

$$\begin{aligned}\|z_{n+1} - z_n\| &\leq \alpha_{n+1}\gamma\alpha\|x_{n+1} - x_n\| + |\alpha_{n+1} - \alpha_n|\big(\|\gamma f(x_n)\| + \|ASy_n\|\big) \\ &\quad + \left(1 - \alpha_{n+1}\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right)\left\{\|x_n - x_{n-1}\| + \frac{M_2}{c}|r_n - r_{n-1}|\right\}. \end{aligned} \tag{3.21}$$

And substituting (3.13), (3.21) into (3.10), we get

$$\begin{aligned}\|x_{n+2} - x_{n+1}\| &\leq \xi_{n+1}\left\{\alpha_{n+1}\gamma\alpha\|x_{n+1} - x_n\| + |\alpha_{n+1} - \alpha_n|\big(\|\gamma f(x_n)\| + \|ASy_n\|\big)\right. \\ &\quad \left. + \left(1 - \alpha_{n+1}\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right)\|x_n - x_{n-1}\| + \frac{M_2}{c}|r_n - r_{n-1}|\right\} \\ &\quad + (1 - \xi_{n+1})\left\{\|x_n - x_{n-1}\| + \frac{M_1}{e}|s_n - s_{n-1}|\right\} + |\xi_{n+1} - \xi_n|\big(\|z_n\| + \|v_n\|\big)\end{aligned}$$

$$\le \left(1-\left(\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)-\gamma\alpha\right)\xi_{n+1}\alpha_{n+1}\right)\|x_{n+1}-x_n\| + \big(|\alpha_{n+1}-\alpha_n| + |\xi_{n+1}-\xi_n|\big)M_3 + \frac{M_1}{e}|s_n-s_{n-1}| + \frac{M_2}{c}|r_n-r_{n-1}|, \tag{3.22}$$

where $M_3 > 0$ is a constant satisfying

$$\sup_n\big\{\big\|\gamma f(x_n)\big\| + \|ASy_n\|, \|z_n\| + \|v_n\|\big\} \le M_3.$$

This together with (C1)-(C4) and Lemma 2.5, imply that

$$\lim_{n\to\infty}\|x_{n+2}-x_{n+1}\| = 0. \tag{3.23}$$

From (3.20), we also have $\|y_{n+1}-y_n\| \to 0$ as $n\to\infty$.

Step 3. We show the followings:

(i) $\lim_{n\to\infty}\|Bu_n - Bq\| = 0$;
(ii) $\lim_{n\to\infty}\|B_1x_n - B_1q\| = 0$;
(iii) $\lim_{n\to\infty}\|B_2x_n - B_2q\| = 0$.

For $q\in\Theta$ and $q = J_{M,\lambda}(q-\lambda Bq)$, then we get

$$\begin{aligned}\|y_n-q\|^2 &= \big\|J_{M,\lambda}(u_n-\lambda Bu_n) - J_{M,\lambda}(q-\lambda Bq)\big\|^2\\ &\le \big\|(u_n-\lambda Bu_n)-(q-\lambda Bq)\big\|^2\\ &\le \|u_n-q\|^2 + \lambda(\lambda-2\beta)\|Bu_n-Bq\|^2\\ &\le \|x_n-q\|^2 + \lambda(\lambda-2\beta)\|Bu_n-Bq\|^2.\end{aligned} \tag{3.24}$$

It follows that

$$\begin{aligned}\|z_n-q\|^2 &= \big\|P_C\big(\alpha_n\gamma f(x_n) + (I-\alpha_nA)Sy_n\big) - P_C(q)\big\|^2\\ &\le \big\|\alpha_n\big(\gamma f(x_n)-Aq\big) + (I-\alpha_nA)(Sy_n-q)\big\|^2\\ &\le \alpha_n\big\|\gamma f(x_n)-Aq\big\|^2 + \left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)\|y_n-q\|^2\\ &\quad + 2\alpha_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)\big\|\gamma f(x_n)-Aq\big\|\|y_n-q\|\\ &\le \alpha_n\big\|\gamma f(x_n)-Aq\big\|^2 + 2\alpha_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)\big\|\gamma f(x_n)-Aq\big\|\|y_n-q\|\\ &\quad + \left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)\big\{\|x_n-q\|^2 + \lambda(\lambda-2\beta)\|Bu_n-Bq\|^2\big\}\\ &\le \alpha_n\big\|\gamma f(x_n)-Aq\big\|^2 + 2\alpha_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)\big\|\gamma f(x_n)-Aq\big\|\|y_n-q\|\\ &\quad + \|x_n-q\|^2 + \left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)\lambda(\lambda-2\beta)\|Bu_n-Bq\|^2.\end{aligned} \tag{3.25}$$

By the convexity of the norm $\|\cdot\|$, we have

$$\begin{aligned}\|x_{n+1}-q\|^2 &= \left\|\xi_n z_n + (1-\xi_n)v_n - q\right\|^2 \\ &\le \left\|\xi_n(z_n-q) + (1-\xi_n)(v_n-q)\right\|^2 \\ &\le \xi_n\|z_n-q\|^2 + (1-\xi_n)\|v_n-q\|^2. \end{aligned} \tag{3.26}$$

Substituting (3.8), (3.25) into (3.26), we obtain

$$\begin{aligned}&\|x_{n+1}-q\|^2 \\ &\quad\le \xi_n\Bigg\{\alpha_n\left\|\gamma f(x_n)-Aq\right\|^2 + 2\alpha_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)\left\|\gamma f(x_n)-Aq\right\|\|y_n-q\| \\ &\qquad + \|x_n-q\|^2 + \left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)\lambda(\lambda-2\beta)\|Bu_n-Bq\|^2\Bigg\} + (1-\xi_n)\|x_n-q\|^2 \\ &\quad\le \xi_n\alpha_n\left\|\gamma f(x_n)-Aq\right\|^2 + 2\xi_n\alpha_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)\left\|\gamma f(x_n)-Aq\right\|\|y_n-q\| \\ &\qquad + \xi_n\|x_n-q\|^2 + \xi_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)\lambda(\lambda-2\beta)\|Bu_n-Bq\|^2 \\ &\qquad + (1-\xi_n)\|x_n-q\|^2.\end{aligned}$$

So, we obtain

$$\begin{aligned}&\xi_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)\lambda(2\beta-\lambda)\|Bu_n-Bq\|^2 \\ &\quad\le \xi_n\alpha_n\left\|\gamma f(x_n)-Aq\right\|^2 + \epsilon_n \\ &\qquad + \|x_n-x_{n+1}\|\big(\|x_n-q\| + \|x_{n+1}-q\|\big),\end{aligned}$$

where $\epsilon_n = 2\xi_n\alpha_n(1-\alpha_n(1-\sqrt{\frac{1-\delta}{\mu}}))\|\gamma f(x_n)-Aq\|\|y_n-q\|$. Since conditions (C1)-(C3) and $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$, then we obtain that $\|Bu_n-Bq\|\to 0$ as $n\to\infty$. We consider this inequality in (3.25) that

$$\begin{aligned}\|z_n-q\|^2 &\le \alpha_n\left\|\gamma f(x_n)-Aq\right\|^2 + \left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)\|y_n-q\|^2 \\ &\quad + 2\alpha_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)\left\|\gamma f(x_n)-Aq\right\|\|y_n-q\|.\end{aligned} \tag{3.27}$$

Substituting (3.6) and (3.8) into (3.27), we have

$$\begin{aligned}\|z_n-q\|^2 &\le \alpha_n\left\|\gamma f(x_n)-Aq\right\|^2 + \left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right) \\ &\quad \times\left\{\|x_n-q\|^2 + r_n(r_n-2\eta)\|B_1x_n-B_1q\|^2\right\}\end{aligned}$$

$$
\begin{aligned}
&+ 2\alpha_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)\left\|\gamma f(x_n)-Aq\right\|\|y_n-q\| \\
&= \alpha_n\left\|\gamma f(x_n)-Aq\right\|^2 + \left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)\|x_n-q\|^2 \\
&\quad + \left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)r_n(r_n-2\eta)\|B_1x_n-B_1q\|^2 \\
&\quad + 2\alpha_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)\left\|\gamma f(x_n)-Aq\right\|\|y_n-q\| \\
&\le \alpha_n\left\|\gamma f(x_n)-Aq\right\|^2 + \|x_n-q\|^2 \\
&\quad + \left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)r_n(r_n-2\eta)\|B_1x_n-B_1q\|^2 \\
&\quad + 2\alpha_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)\left\|\gamma f(x_n)-Aq\right\|\|y_n-q\|.
\end{aligned}
\tag{3.28}
$$

Substituting (3.7) and (3.28) into (3.26), we obtain

$$
\begin{aligned}
\|x_{n+1}-q\|^2 \le{}& \xi_n\Bigg\{\alpha_n\left\|\gamma f(x_n)-Aq\right\|^2 + \|x_n-q\|^2 \\
&+ \left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)r_n(r_n-2\eta)\|B_1x_n-B_1q\|^2 \\
&+ 2\alpha_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)\left\|\gamma f(x_n)-Aq\right\|\|y_n-q\|\Bigg\} \\
&+ (1-\xi_n)\|x_n-q\|^2 \\
={}& \xi_n\alpha_n\left\|\gamma f(x_n)-Aq\right\|^2 \\
&+\xi_n\|x_n-q\|^2 + \xi_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)r_n(r_n-2\eta)\|B_1x_n-B_1q\|^2 \\
&+ 2\xi_n\alpha_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)\left\|\gamma f(x_n)-Aq\right\|\|y_n-q\| \\
&+ (1-\xi_n)\|x_n-q\|^2.
\end{aligned}
\tag{3.29}
$$

So, we also have

$$
\begin{aligned}
&\xi_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)r_n(2\eta-r_n)\|B_1x_n-B_1q\|^2 \\
&\quad\le \xi_n\alpha_n\left\|\gamma f(x_n)-Aq\right\|^2 + \epsilon_n \\
&\qquad + \|x_n-x_{n+1}\|\big(\|x_n-q\| + \|x_{n+1}-q\|\big),
\end{aligned}
$$

where $\epsilon_n = 2\xi_n\alpha_n(1-\alpha_n(1-\sqrt{\frac{1-\delta}{\mu}}))\|\gamma f(x_n)-Aq\|\|y_n-q\|$. Since conditions (C1)-(C3), $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$, then we obtain that $\|B_1x_n-B_1q\| \to 0$ as $n\to\infty$. Substituting

(3.24) into (3.27), we have

$$\begin{aligned}
&\|z_n - q\|^2 \\
&\quad \le \alpha_n \big\| \gamma f(x_n) - Aq \big\|^2 \\
&\qquad + \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right)\left\{\|x_n - q\|^2 + \lambda(\lambda - 2\beta)\|Bu_n - Bq\|^2\right\} \\
&\qquad + 2\alpha_n \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right)\big\| \gamma f(x_n) - Aq \big\| \|y_n - q\| \\
&\quad \le \alpha_n \big\| \gamma f(x_n) - Aq \big\|^2 + \|x_n - q\|^2 \\
&\qquad + \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right)\lambda(\lambda - 2\beta)\|Bu_n - Bq\|^2 \\
&\qquad + 2\alpha_n \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right)\big\| \gamma f(x_n) - Aq \big\| \|y_n - q\|.
\end{aligned} \tag{3.30}$$

Substituting (3.8) and (3.30) into (3.26), we obtain

$$\begin{aligned}
&\|x_{n+1} - q\|^2 \\
&\quad \le \xi_n \Bigg\{ \alpha_n \big\| \gamma f(x_n) - Aq \big\|^2 + \|x_n - q\|^2 \\
&\qquad + \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right)\lambda(\lambda - 2\beta)\|Bu_n - Bq\|^2 \\
&\qquad + 2\alpha_n \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right)\big\| \gamma f(x_n) - Aq \big\| \|y_n - q\| \Bigg\} \\
&\qquad + (1 - \xi_n)\left\{\|x_n - q\|^2 + s_n(s_n - 2\rho)\|B_2 x_n - B_2 q\|^2\right\} \\
&\quad = \xi_n \alpha_n \big\| \gamma f(x_n) - Aq \big\|^2 \\
&\qquad + \xi_n \|x_n - q\|^2 + \xi_n \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right)\lambda(\lambda - 2\beta)\|Bu_n - Bq\|^2 \\
&\qquad + 2\xi_n \alpha_n \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right)\big\| \gamma f(x_n) - Aq \big\| \|y_n - q\| \\
&\qquad + (1 - \xi_n)\|x_n - q\|^2 + (1 - \xi_n)s_n(s_n - 2\rho)\|B_2 x_n - B_2 q\|^2 \\
&\quad = \xi_n \alpha_n \big\| \gamma f(x_n) - Aq \big\|^2 \\
&\qquad + \|x_n - q\|^2 + \xi_n \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right)\lambda(\lambda - 2\beta)\|Bu_n - Bq\|^2 \\
&\qquad + 2\xi_n \alpha_n \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right)\big\| \gamma f(x_n) - Aq \big\| \|y_n - q\| \\
&\qquad + (1 - \xi_n)s_n(s_n - 2\rho)\|B_2 x_n - B_2 q\|^2.
\end{aligned} \tag{3.31}$$

So, we also have

$$\begin{aligned}
&(1-\xi_n)s_n(2\rho-s_n)\|B_2x_n-B_2q\|^2\\
&\quad\le \xi_n\alpha_n\big\|\gamma f(x_n)-Aq\big\|^2+\epsilon_n+\|x_n-x_{n+1}\|\big(\|x_n-q\|+\|x_{n+1}-q\|\big)\\
&\qquad+\xi_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)\lambda(\lambda-2\beta)\|Bu_n-Bq\|^2,
\end{aligned}$$

where $\epsilon_n = 2\xi_n\alpha_n(1-\alpha_n(1-\sqrt{\frac{1-\delta}{\mu}}))\|\gamma f(x_n)-Aq\|\|y_n-q\|$. Since conditions (C1), (C2), (C4), $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$ and $\lim_{n\to\infty}\|Bu_n-Bq\|=0$, then we obtain that $\|B_2x_n-B_2q\|\to 0$ as $n\to\infty$.

Step 4. We show the followings:

(i) $\lim_{n\to\infty}\|x_n-u_n\|=0$;

(ii) $\lim_{n\to\infty}\|u_n-y_n\|=0$;

(iii) $\lim_{n\to\infty}\|y_n-Sy_n\|=0$.

Since $T_{r_n}^{(F_1,\psi_1)}$ is firmly nonexpansive, we observe that

$$\begin{aligned}
\|u_n-q\|^2 &= \big\|T_{r_n}^{(F_1,\psi_1)}(x_n-r_nB_1x_n)-T_{r_n}^{(F_1,\psi_1)}(q-r_nB_1q)\big\|^2\\
&\le \big\langle (x_n-r_nB_1x_n)-(q-r_nB_1q),u_n-q\big\rangle\\
&= \frac{1}{2}\big(\big\|(x_n-r_nB_1x_n)-(q-r_nB_1q)\big\|^2+\|u_n-q\|^2\\
&\quad-\big\|(x_n-r_nB_1x_n)-(q-r_nB_1q)-(u_n-q)\big\|^2\big)\\
&\le \frac{1}{2}\big(\|x_n-q\|^2+\|u_n-q\|^2-\big\|(x_n-u_n)-r_n(B_1x_n-B_1q)\big\|^2\big)\\
&= \frac{1}{2}\big(\|x_n-q\|^2+\|u_n-q\|^2-\|x_n-u_n\|^2\\
&\quad+2r_n\langle B_1x_n-B_1q,x_n-u_n\rangle-r_n^2\|B_1x_n-B_1q\|^2\big).
\end{aligned}$$

Hence, we have

$$\|u_n-q\|^2\le\|x_n-q\|^2-\|x_n-u_n\|^2+2r_n\|B_1x_n-B_1q\|\|x_n-u_n\|. \tag{3.32}$$

Since $J_{M,\lambda}$ is 1-inverse-strongly monotone, we compute

$$\begin{aligned}
\|y_n-q\|^2 &= \big\|J_{M,\lambda}(u_n-\lambda Bu_n)-J_{M,\lambda}(q-\lambda Bq)\big\|^2\\
&\le \big\langle (u_n-\lambda Bu_n)-(q-\lambda Bq),y_n-q\big\rangle\\
&= \frac{1}{2}\big(\big\|(u_n-\lambda Bu_n)-(q-\lambda Bq)\big\|^2+\|y_n-q\|^2\\
&\quad-\big\|(u_n-\lambda Bu_n)-(q-\lambda Bq)-(y_n-q)\big\|^2\big)\\
&= \frac{1}{2}\big(\|u_n-q\|^2+\|y_n-q\|^2-\big\|(u_n-y_n)-\lambda(Bu_n-Bq)\big\|^2\big)\\
&\le \frac{1}{2}\big(\|u_n-q\|^2+\|y_n-q\|^2-\|u_n-y_n\|^2\\
&\quad+2\lambda\langle u_n-y_n,Bu_n-Bq\rangle-\lambda^2\|Bu_n-Bq\|^2\big),
\end{aligned} \tag{3.33}$$

which implies that

$$\|y_n - q\|^2 \leq \|u_n - q\|^2 - \|u_n - y_n\|^2 + 2\lambda\|u_n - y_n\|\|Bu_n - Bq\|. \tag{3.34}$$

Substitute (3.32) into (3.34), we have

$$\begin{aligned}\|y_n - q\|^2 \leq{}& \{\|x_n - q\|^2 - \|x_n - u_n\|^2 + 2r_n\|B_1x_n - B_1q\|\|x_n - u_n\|\} \\ &- \|u_n - y_n\|^2 + 2\lambda\|u_n - y_n\|\|Bu_n - Bq\|. \end{aligned}\tag{3.35}$$

Substitute (3.35) into (3.27), we have

$$\begin{aligned}\|z_n - q\|^2 \leq{}& \alpha_n\|\gamma f(x_n) - Aq\|^2 + \left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right)\{\|x_n - q\|^2 - \|x_n - u_n\|^2 \\ &+ 2r_n\|B_1x_n - B_1q\|\|x_n - u_n\| - \|u_n - y_n\|^2 + 2\lambda\|u_n - y_n\|\|Bu_n - Bq\|\} \\ &+ 2\alpha_n\left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right)\|\gamma f(x_n) - Aq\|\|y_n - q\| \\ \leq{}& \alpha_n\|\gamma f(x_n) - Aq\|^2 + \|x_n - q\|^2 - \|x_n - u_n\|^2 \\ &+ 2\left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right)r_n\|B_1x_n - B_1q\|\|x_n - u_n\| - \|u_n - y_n\|^2 \\ &+ 2\left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right)\lambda\|u_n - y_n\|\|Bu_n - Bq\| \\ &+ 2\alpha_n\left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right)\|\gamma f(x_n) - Aq\|\|y_n - q\|. \end{aligned}\tag{3.36}$$

Since $T_{s_n}^{(F_2,\psi_2)}$ is firmly nonexpansive, we observe that

$$\begin{aligned}\|v_n - q\|^2 ={}& \|T_{s_n}^{(F_2,\psi_2)}(x_n - s_nB_2x_n) - T_{s_n}^{(F_2,\psi_2)}(q - s_nB_2q)\|^2 \\ \leq{}& \langle (x_n - s_nB_2x_n) - (q - s_nB_2q), v_n - q\rangle \\ ={}& \frac{1}{2}\big(\|(x_n - s_nB_2x_n) - (q - s_nB_2q)\|^2 + \|v_n - q\|^2 \\ &- \|(x_n - s_nB_2x_n) - (q - s_nB_2q) - (v_n - q)\|^2\big) \\ \leq{}& \frac{1}{2}\big(\|x_n - q\|^2 + \|v_n - q\|^2 - \|(x_n - v_n) - s_n(B_2x_n - B_2q)\|^2\big) \\ ={}& \frac{1}{2}\big(\|x_n - q\|^2 + \|v_n - q\|^2 - \|x_n - v_n\|^2 \\ &+ 2s_n\langle B_2x_n - B_2q, x_n - v_n\rangle - s_n^2\|B_2x_n - B_2q\|^2\big). \end{aligned}$$

Hence, we have

$$\|v_n - q\|^2 \leq \|x_n - q\|^2 - \|x_n - v_n\|^2 + 2s_n\|B_2x_n - B_2q\|\|x_n - v_n\|. \tag{3.37}$$

Substitute (3.36) and (3.37) into (3.26), we obtain

$$
\begin{aligned}
&\|x_{n+1}-q\|^2\\
&\quad\le \xi_n\|z_n-q\|^2+(1-\xi_n)\|v_n-q\|^2\\
&\quad\le \xi_n\Bigg\{\alpha_n\big\|\gamma f(x_n)-Aq\big\|^2+\|x_n-q\|^2-\|x_n-u_n\|^2-\|u_n-y_n\|^2\\
&\qquad+2\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)r_n\|B_1x_n-B_1q\|\|x_n-u_n\|\\
&\qquad+2\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)\lambda\|u_n-y_n\|\|Bu_n-Bq\|\\
&\qquad+2\alpha_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)\big\|\gamma f(x_n)-Aq\big\|\|y_n-q\|\Bigg\}\\
&\qquad+(1-\xi_n)\big\{\|x_n-q\|^2-\|x_n-v_n\|^2+2s_n\|B_2x_n-B_2q\|\|x_n-v_n\|\big\}\\
&\quad\le \xi_n\alpha_n\big\|\gamma f(x_n)-Aq\big\|^2+\xi_n\|x_n-q\|^2-\|x_n-u_n\|^2-\|u_n-y_n\|^2\\
&\qquad+2\xi_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)r_n\|B_1x_n-B_1q\|\|x_n-u_n\|\\
&\qquad+2\xi_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)\lambda\|u_n-y_n\|\|Bu_n-Bq\|\\
&\qquad+2\xi_n\alpha_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)\big\|\gamma f(x_n)-Aq\big\|\|y_n-q\|\\
&\qquad+(1-\xi_n)\|x_n-q\|^2-\|x_n-v_n\|^2+2(1-\xi_n)s_n\|B_2x_n-B_2q\|\|x_n-v_n\|.
\end{aligned}
\tag{3.38}
$$

Then, we derive

$$
\begin{aligned}
&\|x_n-u_n\|^2+\|u_n-y_n\|^2+\|x_n-v_n\|^2\\
&\quad\le \xi_n\alpha_n\big\|\gamma f(x_n)-Aq\big\|^2+\|x_n-q\|^2-\|x_{n+1}-q\|^2\\
&\qquad+2\xi_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)r_n\|B_1x_n-B_1q\|\|x_n-u_n\|\\
&\qquad+2\xi_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)\lambda\|u_n-y_n\|\|Bu_n-Bq\|\\
&\qquad+2\xi_n\alpha_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)\big\|\gamma f(x_n)-Aq\big\|\|y_n-q\|\\
&\qquad+2(1-\xi_n)s_n\|B_2x_n-B_2q\|\|x_n-v_n\|\\
&\quad\le \xi_n\alpha_n\big\|\gamma f(x_n)-Aq\big\|^2+\|x_{n+1}-x_n\|\big(\|x_n-q\|+\|x_{n+1}-q\|\big)\\
&\qquad+2\xi_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)r_n\|B_1x_n-B_1q\|\|x_n-u_n\|
\end{aligned}
$$

$$+ 2\xi_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)\lambda\|u_n-y_n\|\|Bu_n-Bq\|$$
$$+ 2\xi_n\alpha_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)\left\|\gamma f(x_n)-Aq\right\|\|y_n-q\|$$
$$+ 2(1-\xi_n)s_n\|B_2x_n-B_2q\|\|x_n-v_n\|. \tag{3.39}$$

By conditions (C1)-(C4), $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$, $\lim_{n\to\infty}\|Bu_n-Bq\| = 0$, $\lim_{n\to\infty}\|B_1x_n - B_1q\| = 0$ and $\lim_{n\to\infty}\|B_2x_n - B_2q\| = 0$. So, we have $\|x_n-u_n\| \to 0$, $\|u_n-y_n\| \to 0$, $\|x_n-v_n\| \to 0$ as $n\to\infty$. We note that $x_{n+1}-x_n = \xi_n(z_n-x_n) + (1-\xi_n)(v_n-x_n)$. From $\lim_{n\to\infty}\|x_n-v_n\| = 0$, $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$, and hence

$$\lim_{n\to\infty}\|z_n-x_n\| = 0. \tag{3.40}$$

It follows that

$$\|x_n-y_n\| \le \|x_n-u_n\| + \|u_n-y_n\| \to 0, \quad \text{as } n\to\infty. \tag{3.41}$$

Since

$$\|z_n-y_n\| \le \|z_n-x_n\| + \|x_n-y_n\|.$$

So, by (3.40) and $\lim_{n\to\infty}\|x_n-y_n\| = 0$, we obtain

$$\lim_{n\to\infty}\|z_n-y_n\| = 0. \tag{3.42}$$

Therefore, we observe that

$$\begin{aligned}\|Sy_n-z_n\| &= \left\|P_CSy_n - P_C\big(\alpha_n\gamma f(x_n) + (I-\alpha_nA)Sy_n\big)\right\| \\ &\le \left\|Sy_n-\alpha_n\gamma f(x_n) - (I-\alpha_nA)Sy_n\right\| \\ &= \alpha_n\left\|\gamma f(x_n) - ASy_n\right\|.\end{aligned} \tag{3.43}$$

By condition (C1), we have $\|Sy_n-z_n\| \to 0$ as $n\to\infty$. Next, we observe that

$$\|Sy_n-y_n\| \le \|Sy_n-z_n\| + \|z_n-y_n\|.$$

By (3.42) and (3.43), we have $\|Sy_n-y_n\| \to 0$ as $n\to\infty$.

Step 5. We show that $q\in\Theta := F(S)\cap \text{GMEP}(F_1,\psi_1,B_1)\cap\text{GMEP}(F_2,\psi_2,B_2)\cap I(B,M)$ and $\limsup_{n\to\infty}\langle(\gamma f-A)q, Sy_n-q\rangle \le 0$. It is easy to see that $P_\Theta(\gamma f + (I-A))$ is a contraction of H into itself. In fact, from Lemma 2.9, we have

$$\begin{aligned}&\left\|P_\Theta\big(\gamma f + (I-A)\big)x - P_\Theta\big(\gamma f + (I-A)\big)y\right\| \\ &\quad\le \left\|\big(\gamma f + (I-A)\big)x - \big(\gamma f + (I-A)\big)y\right\| \\ &\quad\le \gamma\left\|f(x)-f(y)\right\| + (I-A)\|x-y\|\end{aligned}$$

$$\le \gamma\alpha\|x-y\| + \left(1-\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)\|x-y\|$$
$$= \left(\sqrt{\frac{1-\delta}{\mu}} + \gamma\alpha\right)\|x-y\|.$$

Hence H is complete, there exists a unique fixed point $q \in H$ such that $q = P_\Theta(\gamma f + (I - A))(q)$. By Lemma 2.2 we obtain that $\langle(\gamma f - A)q, w - q\rangle \le 0$ for all $w \in \Theta$.

Next, we show that $\limsup_{n\to\infty}\langle(\gamma f - A)q, Sy_n - q\rangle \le 0$, where $q = P_\Theta(\gamma f + I - A)(q)$ is the unique solution of the variational inequality $\langle(\gamma f - A)q, p - qr\rangle \ge 0$, $\forall p \in \Theta$. We can choose a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that

$$\limsup_{n\to\infty}\big\langle(\gamma f - A)q, Sy_n - q\big\rangle = \lim_{i\to\infty}\big\langle(\gamma f - A)q, Sy_{n_i} - q\big\rangle.$$

As $\{y_{n_i}\}$ is bounded, there exists a subsequence $\{y_{n_{i_j}}\}$ of $\{y_{n_i}\}$ which converges weakly to w. We may assume without loss of generality that $y_{n_i} \rightharpoonup w$. We claim that $w \in \Theta$. Since $\|y_n - Sy_n\| \to 0$ and by Lemma 2.6, we have $w \in F(S)$.

Next, we show that $w \in \mathrm{GMEP}(F_1, \psi_1, B_1)$. Since $u_n = T_{r_n}^{(F_1,\psi_1)}(x_n - r_nB_1x_n)$, we know that

$$F_1(u_n, y) + \psi_1(y) - \psi_1(u_n) + \langle B_1x_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C.$$

It follows by (A2) that

$$\psi_1(y) - \psi_1(u_n) + \langle B_1x_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge F_1(y, u_n), \quad \forall y \in C.$$

Hence,

$$\psi_1(y) - \psi_1(u_{n_i}) + \langle B_1x_{n_i}, y - u_{n_i}\rangle + \frac{1}{r_{n_i}}\langle y - u_{n_i}, u_{n_i} - x_{n_i}\rangle \ge F_1(y, u_{n_i}), \quad \forall y \in C. \tag{3.44}$$

For $t \in (0,1]$ and $y \in H$, let $y_t = ty + (1-t)w$. From (3.44), we have

$$\begin{aligned}
&\langle y_t - u_{n_i}, B_1y_t\rangle \\
&\quad\ge \langle y_t - u_{n_i}, B_1y_t\rangle - \psi_1(y_t) + \psi_1(u_{n_i}) - \langle B_1x_{n_i}, y_t - u_{n_i}\rangle \\
&\qquad - \frac{1}{r_{n_i}}\langle y_t - u_{n_i}, u_{n_i} - x_{n_i}\rangle + F_1(y_t, u_{n_i}) \\
&\quad= \langle y_t - u_{n_i}, B_1y_t - B_1u_{n_i}\rangle + \langle y_t - u_{n_i}, B_1u_{n_i} - B_1x_{n_i}\rangle - \psi_1(y_t) + \psi_1(u_{n_i}) \\
&\qquad - \frac{1}{r_{n_i}}\langle y_t - u_{n_i}, u_{n_i} - x_{n_i}\rangle + F_1(y_t, u_{n_i}).
\end{aligned}$$

From $\|u_{n_i} - x_{n_i}\| \to 0$, we have $\|B_1u_{n_i} - B_1x_{n_i}\| \to 0$. Further, from (A4) and the weakly lower semicontinuity of ψ_1, $\frac{u_{n_i} - x_{n_i}}{r_{n_i}} \to 0$ and $u_{n_i} \rightharpoonup w$, we have

$$\langle y_t - w, B_1y_t\rangle \ge -\psi_1(y_t) + \psi_1(w) + F_1(y_t, w). \tag{3.45}$$

From (A1), (A4) and (3.45), we have

$$\begin{aligned}
0 &= F_1(y_t, y_t) - \psi_1(y_t) + \psi_1(y_t) \\
&\leq tF_1(y_t, y) + (1-t)F_1(y_t, w) + t\psi_1(y) + (1-t)\psi_1(w) - \psi_1(y_t) \\
&= t\big[F_1(y_t, y) + \psi_1(y) - \psi_1(y_t)\big] + (1-t)\big[F_1(y_t, w) + \psi_1(w) - \psi_1(y_t)\big] \\
&\leq t\big[F_1(y_t, y) + \psi_1(y) - \psi_1(y_t)\big] + (1-t)\langle y_t - w, B_1 y_t\rangle \\
&= t\big[F_1(y_t, y) + \psi_1(y) - \psi_1(y_t)\big] + (1-t)t\langle y - w, B_1 y_t\rangle,
\end{aligned}$$

and hence

$$0 \leq F_1(y_t, y) + \psi_1(y) - \psi_1(y_t) + (1-t)\langle y - w, B_1 y_t\rangle.$$

Letting $t \to 0$, we have, for each $y \in C$,

$$F_1(w, y) + \psi_1(y) - \psi_1(w) + \langle y - w, B_1 w\rangle \geq 0.$$

This implies that $w \in \mathrm{GMEP}(F_1, \psi_1, B_1)$. By following the same arguments, we can show that $w \in \mathrm{GMEP}(F_2, \psi_2, B_2)$.

Lastly, we show that $w \in I(B, M)$. In fact, since B is a β-inverse-strongly monotone, B is monotone and Lipschitz continuous mapping. It follows from Lemma 2.3 that $M + B$ is a maximal monotone. Let $(v, g) \in G(M + B)$, since $g - Bv \in M(v)$. Again since $y_{n_i} = J_{M,\lambda}(u_{n_i} - \lambda Bu_{n_i})$, we have $u_{n_i} - \lambda Bu_{n_i} \in (I + \lambda M)(y_{n_i})$, that is, $\frac{1}{\lambda}(u_{n_i} - y_{n_i} - \lambda Bu_{n_i}) \in M(y_{n_i})$. By virtue of the maximal monotonicity of $M + B$, we have

$$\left\langle v - y_{n_i}, g - Bv - \frac{1}{\lambda}(u_{n_i} - y_{n_i} - \lambda Bu_{n_i})\right\rangle \geq 0,$$

and hence

$$\begin{aligned}
\langle v - y_{n_i}, g\rangle &\geq \left\langle v - y_{n_i}, Bv + \frac{1}{\lambda}(u_{n_i} - y_{n_i} - \lambda Bu_{n_i})\right\rangle \\
&= \langle v - y_{n_i}, Bv - By_{n_i}\rangle + \langle v - y_{n_i}, By_{n_i} - Bu_{n_i}\rangle \\
&\quad + \left\langle v - y_{n_i}, \frac{1}{\lambda}(u_{n_i} - y_{n_i})\right\rangle.
\end{aligned}$$

It follows from $\lim_{n\to\infty}\|u_n - y_n\| = 0$, we have $\lim_{n\to\infty}\|Bu_n - By_n\| = 0$ and $y_{n_i} \rightharpoonup w$ that

$$\limsup_{n\to\infty}\langle v - y_n, g\rangle = \langle v - w, g\rangle \geq 0.$$

It follows from the maximal monotonicity of $B + M$ that $\theta \in (M + B)(w)$, that is, $w \in I(B, M)$. Therefore, $w \in \Theta$. It follows that

$$\limsup_{n\to\infty}\big\langle(\gamma f - A)q, Sy_n - q\big\rangle = \lim_{i\to\infty}\big\langle(\gamma f - A)q, Sy_{n_i} - q\big\rangle = \big\langle(\gamma f - A)q, w - q\big\rangle \leq 0.$$

Step 6. We prove $x_n \to q$. By using (3.1) and together with Schwarz inequality, we have

$$\begin{aligned}
\|x_{n+1}-q\|^2 &= \big\|\xi_n P_C\big[\big(\alpha_n\gamma f(x_n)+(I-\alpha_n A)Sy_n\big)-q\big]+(1-\xi_n)(v_n-q)\big\|^2\\
&\le \xi_n\big\|P_C\big[\big(\alpha_n\gamma f(x_n)+(I-\alpha_n A)Sy_n\big)-P_C(q)\big]\big\|^2+(1-\xi_n)\|v_n-q\|^2\\
&\le \xi_n\big\|\alpha_n\big(\gamma f(x_n)-Aq\big)+(I-\alpha_n A)(Sy_n-q)\big\|^2+(1-\xi_n)\|x_n-q\|^2\\
&\le \xi_n(I-\alpha_n A)^2\|Sy_n-q\|^2+\xi_n\alpha_n^2\big\|\gamma f(x_n)-Aq\big\|^2\\
&\quad+2\xi_n\alpha_n\big\langle (I-\alpha_n A)(Sy_n-q),\gamma f(x_n)-Aq\big\rangle+(1-\xi_n)\|x_n-q\|^2\\
&\le \xi_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)^2\|y_n-q\|^2+\xi_n\alpha_n^2\big\|\gamma f(x_n)-Aq\big\|^2\\
&\quad+2\xi_n\alpha_n\big\langle Sy_n-q,\gamma f(x_n)-Aq\big\rangle-2\xi_n\alpha_n^2\big\langle A(Sy_n-q),\gamma f(x_n)-Aq\big\rangle\\
&\quad+(1-\xi_n)\|x_n-q\|^2\\
&\le \xi_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)^2\|x_n-q\|^2+\xi_n\alpha_n^2\big\|\gamma f(x_n)-Aq\big\|^2\\
&\quad+2\xi_n\alpha_n\big\langle Sy_n-q,\gamma f(x_n)-\gamma f(q)\big\rangle+2\xi_n\alpha_n\big\langle Sy_n-q,\gamma f(q)-Aq\big\rangle\\
&\quad-2\xi_n\alpha_n^2\big\langle A(Sy_n-q),\gamma f(x_n)-Aq\big\rangle+(1-\xi_n)\|x_n-q\|^2\\
&\le \xi_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)^2\|x_n-q\|^2+\xi_n\alpha_n^2\big\|\gamma f(x_n)-Aq\big\|^2\\
&\quad+2\xi_n\alpha_n\|Sy_n-q\|\big\|\gamma f(x_n)-\gamma f(q)\big\|+2\xi_n\alpha_n\big\langle Sy_n-q,\gamma f(q)-Aq\big\rangle\\
&\quad-2\xi_n\alpha_n^2\big\langle A(Sy_n-q),\gamma f(x_n)-Aq\big\rangle+(1-\xi_n)\|x_n-q\|^2\\
&\le \xi_n\left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)\right)^2\|x_n-q\|^2+\xi_n\alpha_n^2\big\|\gamma f(x_n)-Aq\big\|^2\\
&\quad+2\xi_n\gamma\alpha\alpha_n\|y_n-q\|\|x_n-q\|+2\xi_n\alpha_n\big\langle Sy_n-q,\gamma f(q)-Aq\big\rangle\\
&\quad-2\xi_n\alpha_n^2\big\langle A(Sy_n-q),\gamma f(x_n)-Aq\big\rangle+(1-\xi_n)\|x_n-q\|^2\\
&\le \left(\xi_n-2\xi_n\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)+\xi_n\alpha_n^2\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)^2\right)\|x_n-q\|^2\\
&\quad+\xi_n\alpha_n^2\big\|\gamma f(x_n)-Aq\big\|^2+2\xi_n\gamma\alpha\alpha_n\|x_n-q\|^2\\
&\quad+2\xi_n\alpha_n\big\langle Sy_n-q,\gamma f(q)-Aq\big\rangle-2\xi_n\alpha_n^2\big\langle A(Sy_n-q),\gamma f(x_n)-Aq\big\rangle\\
&\quad+(1-\xi_n)\|x_n-q\|^2\\
&\le \left(1-2\xi_n\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)+2\xi_n\gamma\alpha\alpha_n\right)\|x_n-q\|^2\\
&\quad+\alpha_n\Bigg\{\xi_n\alpha_n\big\|\gamma f(x_n)-Aq\big\|^2\\
&\quad+2\xi_n\big\langle Sy_n-q,\gamma f(q)-Aq\big\rangle-2\xi_n\alpha_n\big\|A(Sy_n-q)\big\|\big\|\gamma f(x_n)-Aq\big\|\\
&\quad+\xi_n\alpha_n\left(1-\sqrt{\frac{1-\delta}{\mu}}\right)^2\|x_n-q\|^2\Bigg\}
\end{aligned}$$

$$= \left(1 - 2\left(\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right) - \gamma\alpha\right)\xi_n\alpha_n\right)\|x_n - q\|^2 + \alpha_n\left\{\xi_n\alpha_n\left\|\gamma f(x_n) - Aq\right\|^2 + 2\xi_n\langle Sy_n - q, \gamma f(q) - Aq\rangle - 2\xi_n\alpha_n\left\|A(Sy_n - q)\right\|\left\|\gamma f(x_n) - Aq\right\| + \xi_n\alpha_n\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)^2\|x_n - q\|^2\right\}.$$

Since $\{x_n\}$ is bounded, where $\eta \geq \xi_n\|\gamma f(x_n) - Aq\|^2 - 2\xi_n\|A(Sy_n - q)\|\|\gamma f(x_n) - Aq\| + \xi_n(1 - \sqrt{\frac{1-\delta}{\mu}})^2\|x_n - q\|^2$ for all $n \geq 0$. It follows that

$$\|x_{n+1} - q\|^2 \leq \left(1 - 2\left(\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right) - \gamma\alpha\right)\xi_n\alpha_n\right)\|x_n - q\|^2 + \alpha_n\varsigma_n, \tag{3.46}$$

where $\varsigma_n = 2\xi_n\langle Sy_n - q, \gamma f(q) - Aq\rangle + \eta\alpha_n$. By $\limsup_{n\to\infty}\langle(\gamma f - A)q, Sy_n - q\rangle \leq 0$, we get $\limsup_{n\to\infty}\varsigma_n \leq 0$. Applying Lemma 2.5, we can conclude that $x_n \to q$. This completes the proof. □

Corollary 3.2 *Let H be a real Hilbert space, C be a closed convex subset of H. Let F_1, F_2 be two bifunctions of $C \times C$ into $\mathcal{R}$ satisfying (A1)-(A4) and $B, B_1, B_2 : C \to H$ be β, η, ρ-inverse-strongly monotone mappings, $\psi_1, \psi_2 : C \to \mathcal{R}$ be convex and lower semicontinuous function, $f : C \to C$ be a contraction with coefficient α $(0 < \alpha < 1)$, $M : H \to 2^H$ be a maximal monotone mapping. Assume that either (B1) or (B2) holds. Let S be a nonexpansive mapping of H into itself such that*

$$\Theta := F(S) \cap \mathrm{GMEP}(F_1, \psi_1, B_1) \cap \mathrm{GMEP}(F_2, \psi_2, B_2) \cap I(B, M) \neq \emptyset.$$

Suppose $\{x_n\}$ is a sequence generated by the following algorithm $x_0 \in C$ arbitrarily:

$$\begin{cases} u_n = T_{r_n}^{(F_1,\psi_1)}(x_n - r_nB_1x_n), \\ v_n = T_{s_n}^{(F_2,\psi_2)}(x_n - s_nB_2x_n), \\ x_{n+1} = \xi_nP_C[\alpha_nf(x_n) + (I - \alpha_n)SJ_{M,\lambda}(I - \lambda B)u_n] + (1 - \xi_n)v_n, \end{cases}$$

where $\{\alpha_n\}, \{\xi_n\} \subset (0,1)$, $\lambda \in (0, 2\beta)$ such that $0 < a \leq \lambda \leq b < 2\beta$, $\{r_n\} \in (0, 2\eta)$ with $0 < c \leq d \leq 1 - \eta$ and $\{s_n\} \in (0, 2\rho)$ with $0 < e \leq f \leq 1 - \rho$ satisfy the conditions (C1)-(C4).

Then $\{x_n\}$ converges strongly to $q \in \Theta$, where $q = P_\Theta(f + I)(q)$ which solves the following variational inequality:

$$\langle(f - I)q, p - q\rangle \leq 0, \quad \forall p \in \Theta.$$

Proof Putting $A \equiv I$ and $\gamma \equiv 1$ in Theorem 3.1, we can obtain desired conclusion immediately. □

Corollary 3.3 *Let H be a real Hilbert space, C be a closed convex subset of H. Let F_1, F_2 be two bifunctions of $C \times C$ into $\mathcal{R}$ satisfying (A1)-(A4) and $B, B_1, B_2 : C \to H$ be β, η, ρ-inverse-strongly monotone mappings, $\psi_1, \psi_2 : C \to \mathcal{R}$ be convex and lower semicontinuous* *function. Assume that either (B1) or (B2) holds. Let S be a nonexpansive mapping of H into itself such that*

$$\Theta := F(S) \cap \mathrm{GMEP}(F_1, \psi_1, B_1) \cap \mathrm{GMEP}(F_2, \psi_2, B_2) \cap I(B, M) \neq \emptyset.$$

Suppose $\{x_n\}$ *is a sequence generated by the following algorithm* $x_0 \in C$ *arbitrarily:*

$$\begin{cases} u_n = T_{r_n}^{(F_1,\psi_1)}(x_n - r_n B_1 x_n), \\ v_n = T_{s_n}^{(F_2,\psi_2)}(x_n - s_n B_2 x_n), \\ x_{n+1} = \xi_n P_C[\alpha_n u + (I - \alpha_n) S J_{M,\lambda}(I - \lambda B) u_n] + (1 - \xi_n) v_n, \end{cases}$$

where $\{\alpha_n\}, \{\xi_n\} \subset (0,1)$, $\lambda \in (0, 2\beta)$ *such that* $0 < a \leq \lambda \leq b < 2\beta$, $\{r_n\} \in (0, 2\eta)$ *with* $0 < c \leq d \leq 1 - \eta$ *and* $\{s_n\} \in (0, 2\rho)$ *with* $0 < e \leq f \leq 1 - \rho$ *satisfy the conditions (C1)-(C4).*

Then $\{x_n\}$ *converges strongly to* $q \in \Theta$, *where* $q = P_\Theta(q)$ *which solves the following variational inequality:*

$$\langle u - q, p - q \rangle \leq 0, \quad \forall p \in \Theta.$$

Proof Putting $f \equiv u \in C$ is a constant in Corollary 3.2, we can obtain desired conclusion immediately. □

Corollary 3.4 *Let H be a real Hilbert space, C be a closed convex subset of H. Let* F_1, F_2 *be two bifunctions of* $C \times C$ *into* $\mathcal{R}$ *satisfying (A1)-(A4) and* $B, B_1, B_2 : C \to H$ *be* β, η, ρ*-inverse-strongly monotone mappings,* $\psi_1, \psi_2 : C \to \mathcal{R}$ *be convex and lower semicontinuous function,* $f : C \to C$ *be a contraction with coefficient* α $(0 < \alpha < 1)$ *and A is* δ*-strongly monotone and* μ*-strictly pseudo-contraction with* $\delta + \mu > 1$, γ *is a positive real number such that* $\gamma < \frac{1}{\alpha}(1 - \sqrt{\frac{1-\delta}{\mu}})$. *Assume that either (B1) or (B2) holds. Let S be a nonexpansive mapping of C into itself such that*

$$\Theta := F(S) \cap \mathrm{GMEP}(F_1, \psi_1, B_1) \cap \mathrm{GMEP}(F_2, \psi_2, B_2) \cap \mathrm{VI}(C, B) \neq \emptyset.$$

Suppose $\{x_n\}$ *is a sequence generated by the following algorithm* $x_0 \in C$ *arbitrarily:*

$$\begin{cases} u_n = T_{r_n}^{(F_1,\psi_1)}(x_n - r_n B_1 x_n), \\ v_n = T_{s_n}^{(F_2,\psi_2)}(x_n - s_n B_2 x_n), \\ x_{n+1} = \xi_n P_C[\alpha_n \gamma f(x_n) + (I - \alpha_n A) S P_C(I - \lambda B) u_n] + (1 - \xi_n) v_n, \end{cases}$$

where $\{\alpha_n\}, \{\xi_n\} \subset (0,1)$, $\lambda \in (0, 2\beta)$ *such that* $0 < a \leq \lambda \leq b < 2\beta$, $\{r_n\} \in (0, 2\eta)$ *with* $0 < c \leq d \leq 1 - \eta$ *and* $\{s_n\} \in (0, 2\rho)$ *with* $0 < e \leq f \leq 1 - \rho$ *satisfy the conditions (C1)-(C4).*

Then $\{x_n\}$ *converges strongly to* $q \in \Theta$, *where* $q = P_\Theta(\gamma f + I - A)(q)$ *which solves the following variational inequality:*

$$\langle (\gamma f - A) q, p - q \rangle \leq 0, \quad \forall p \in \Theta.$$

Proof Taking $J_{M,\lambda} = P_C$ in Theorem 3.1, we can obtain desired conclusion immediately. □

Corollary 3.5 *Let H be a real Hilbert space, C be a closed convex subset of H. Let $f : C \to C$ be a contraction with coefficient α $(0 < \alpha < 1)$, A is δ-strongly monotone and μ-strictly pseudo-contraction with $\delta + \mu > 1$, γ is a positive real number such that $\gamma < \frac{1}{\alpha}(1 - \sqrt{\frac{1-\delta}{\mu}})$. Let S be a nonexpansive mapping of C into itself such that*

$$\Theta := F(S) \neq \emptyset.$$

Suppose $\{x_n\}$ is a sequence generated by the following algorithm $x_0 \in C$ arbitrarily:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Sx_n,$$

where $\{\alpha_n\} \subset (0,1)$ and satisfy the condition $\lim_{n\to\infty} \alpha_n = 0$. Then $\{x_n\}$ converges strongly to $q \in \Theta$, where $q = P_\Theta(\gamma f + I - A)(q)$ which solves the following variational inequality:

$$\langle (\gamma f - A)q, p - q \rangle \leq 0, \quad \forall p \in \Theta.$$

Proof Taking $\xi_n \equiv 1$, $P_C \equiv I$ and $B, B_1, B_2 \equiv 0$ in Corollary 3.4, we can obtain desired conclusion immediately. □

Remark 3.6 Corollary 3.5 generalizes and improves the result of Marino and Xu [19].

Corollary 3.7 *Let H be a real Hilbert space, C be a closed convex subset of H. Let F_1, F_2 be two bifunctions of $C \times C$ into $\mathcal{R}$ satisfying (A1)-(A4) and $B_1, B_2 : C \to H$ be η, ρ-inverse-strongly monotone mappings, $\psi_1, \psi_2 : C \to \mathcal{R}$ be convex and lower semicontinuous function, $f : C \to C$ be a contraction with coefficient α $(0 < \alpha < 1)$. Assume that either (B1) or (B2) holds. Let S be a nonexpansive mapping of C into itself such that*

$$\Theta := F(S) \cap \mathrm{GMEP}(F_1, \psi_1, B_1) \cap \mathrm{GMEP}(F_2, \psi_2, B_2) \neq \emptyset.$$

Suppose $\{x_n\}$ is a sequence generated by the following algorithm $x_0 \in C$ arbitrarily:

$$\begin{cases} u_n = T_{r_n}^{(F_1,\psi_1)}(x_n - r_n B_1 x_n), \\ v_n = T_{s_n}^{(F_2,\psi_2)}(x_n - s_n B_2 x_n), \\ x_{n+1} = \xi_n P_C[\alpha_n f(x_n) + (I - \alpha_n)Su_n] + (1 - \xi_n)v_n, \end{cases}$$

where $\{\alpha_n\}, \{\xi_n\} \subset (0,1)$, $\{r_n\} \in (0, 2\eta)$ with $0 < c \leq d \leq 1 - \eta$ and $\{s_n\} \in (0, 2\rho)$ with $0 < e \leq f \leq 1 - \rho$ satisfy the conditions (C1)-(C4).

Then $\{x_n\}$ converges strongly to $q \in \Theta$, where $q = P_\Theta(f + I)(q)$ which solves the following variational inequality:

$$\langle (f - I)q, p - q \rangle \leq 0, \quad \forall p \in \Theta.$$

Proof Taking $\gamma \equiv 1$, $A \equiv I$, $J_{M,\lambda} \equiv I$ and $B \equiv 0$ in Theorem 3.1, we can obtain desired conclusion immediately. □

Remark 3.8 Corollary 3.7 generalizes and improves the result of Yao and Liou [29].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contribute equally and significantly in this research work. All authors read and approved the final manuscript.

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References
1. Kirk, WA: Fixed point theorem for mappings which do not increase distance. Am. Math. Mon. **72**, 1004-1006 (1965)
2. Browder, FE, Petryshym, WV: Construction of fixed points of nonlinear mappings in Hilbert spaces. J. Math. Anal. Appl. **20**, 197-228 (1963)
3. Chamnarnpan, T, Kumam, P: Iterative algorithms for solving the system of mixed equilibrium problems, fixed-point problems, and variational inclusions with application to minimization problem. J. Appl. Math. **2012**, Article ID 538912 (2012)
4. Hao, Y: Some results of variational inclusion problems and fixed point problems with applications. Appl. Math. Mech. **30**(12), 1589-1596 (2009). doi:10.1007/s10483-009-1210-x
5. Jitpeera, T, Kumam, P: Hybrid algorithms for minimization problems over the solutions of generalized mixed equilibrium and variational inclusion problems. Math. Probl. Eng. **2011**, Article ID 648617 (2011)
6. Jitpeera, T, Kumam, P: Approximating common solution of variational inclusions and generalized mixed equilibrium problems with applications to optimization problems. Dyn. Contin. Discrete Impuls. Syst. **18**(6), 813-837 (2011)
7. Liu, M, Chang, SS, Zuo, P: An algorithm for finding a common solution for a system of mixed equilibrium problem, quasivariational inclusion problems of nonexpansive semigroup. J. Inequal. Appl. **2010**, Article ID 895907 (2010)
8. Petrot, N, Wangkeeree, R, Kumam, P: A viscosity approximation method of common solutions for quasi variational inclusion and fixed point problems. Fixed Point Theory **12**(1), 165-178 (2011)
9. Zhang, SS, Lee, JHW, Chan, CK: Algorithms of common solutions to quasi variational inclusion and fixed point problems. Appl. Math. Mech. **29**(5), 571-581 (2008)
10. Tan, JF, Chang, SS: Iterative algorithms for finding common solutions to variational inclusion equilibrium and fixed point problems. Fixed Point Theory Appl. **2011**, Article ID 915629 (2011)
11. Rockafellar, RT: On the maximality of sums of nonlinear monotone operators. Trans. Am. Math. Soc. **149**, 46-55 (2000)
12. Takahashi, W: Nonlinear Functional Analysis. Yokohama Publishers, Yokohama (2000)
13. Blum, E, Oettli, W: From optimization and variational inequalities to equilibrium problems. Math. Stud. **63**, 123-145 (1994)
14. Combettes, PL, Hirstoaga, SA: Equilibrium programming in Hilbert spaces. J. Nonlinear Convex Anal. **6**, 117-136 (2005)
15. Flam, SD, Antipin, AS: Equilibrium programming using proximal-link algorithms. Math. Program. **78**, 29-41 (1997)
16. Takahashi, S, Takahashi, W: Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces. J. Math. Anal. Appl. **331**, 506-515 (2007)
17. Hartman, P, Stampacchia, G: On some nonlinear elliptic differential functional equations. Acta Math. **115**, 271-310 (1966)
18. Yao, JC, Chadli, O: Pseudomonotone complementarity problems and variational inequalities. In: Crouzeix, JP, Haddjissas, N, Schaible, S (eds.) Handbook of Generalized Convexity and Monotonicity, pp. 501-558 (2005)
19. Marino, G, Xu, HK: A general iterative method for nonexpansive mapping in Hilbert space. J. Math. Anal. Appl. **318**, 43-52 (2006)
20. Moudafi, A: Viscosity approximation methods for fixed points problems. J. Math. Anal. Appl. **241**, 46-55 (2000)
21. Iiduka, H, Takahashi, W: Strong convergence theorems for nonexpansive mapping and inverse-strong monotone mappings. Nonlinear Anal. **61**, 341-350 (2005)
22. Su, Y, Shang, M, Qin, X: An iterative method of solution for equilibrium and optimization problems. Nonlinear Anal. **69**, 2709-2719 (2008)
23. Brézis, H: Opérateur Maximaux Monotones. Mathematics Studies, vol. 5. North-Holland, Amsterdam (1973)
24. Opial, Z: Weak convergence of successive approximations for nonexpansive mappings. Bull. Am. Math. Soc. **73**, 591-597 (1967)
25. Xu, HK: Iterative algorithms for nonlinear operators. J. Lond. Math. Soc. **66**, 240-256 (2002)
26. Browder, FE: Nonlinear operators and nonlinear equations of evolution in Banach spaces. Proc. Symp. Pure Math. **18**, 78-81 (1976)
27. Peng, J-W, Liou, Y-C, Yao, J-C: An iterative algorithm combining viscosity method with parallel method for a generalized equilibrium problem and strict pseudocontractions. Fixed Point Theory Appl. **2009**, Article ID 794178 (2009). doi:10.1155/2009/794178
28. Wangkeeree, R, Bantaojai, T: A general composite algorithms for solving general equilibrium problems and fixed point problems in Hilbert spaces. Thai J. Math. **9**, 191-212 (2011)
29. Yao, Y, Liou, Y-C: Composite algorithms for minimization over the solutions of equilibrium problems and fixed point problems. Abstr. Appl. Anal. **2010**, Article ID 763506 (2010). doi:10.1155/2010/763506

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