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Iterative scheme for a nonexpansive mapping, an η-strictly pseudo-contractive mapping and variational inequality problems in a uniformly convex and 2-uniformly smooth Banach space

Atid Kangtunyakarn*

*Correspondence:
beawrock@hotmail.com
Department of Mathematics, Faculty of Science, King Mongkut's Institute of Technology Ladkrabang, Bangkok 10520, Thailand

Abstract

In this paper, we introduce an iterative scheme by the modification of Mann's iteration process for finding a common element of the set of solutions of a finite family of variational inequality problems and the set of fixed points of an η-strictly pseudo-contractive mapping and a nonexpansive mapping. Moreover, we prove a strong convergence theorem for finding a common element of the set of fixed points of a finite family of η_i-strictly pseudo-contractive mappings for every $i = 1, 2, \dots, N$ in uniformly convex and 2-uniformly smooth Banach spaces.

Keywords: nonexpansive mapping; strictly pseudo-contractive mapping; variational inequality problem

1 Introduction

Let E be a Banach space with its dual space E^* and let C be a nonempty closed convex subset of E. Throughout this paper, we denote the norm of E and E^* by the same symbol $\|\cdot\|$. We use the symbol $\to$ to denote the strong convergence. Recall the following definition.

Definition 1.1 A Banach space E is said to be *uniformly convex* iff for any ϵ, $0 < \epsilon \le 2$, the inequalities $\|x\| \le 1$, $\|y\| \le 1$ and $\|x - y\| \ge \epsilon$ imply there exists a $\delta > 0$ such that $\|\frac{x+y}{2}\| \le 1 - \delta$.

Definition 1.2 Let E be a Banach space. Then a function $\rho_E : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be *the modulus of smoothness of E* if

$$\rho_E(t) = \sup\left\{\frac{\|x+y\| + \|x-y\|}{2} - 1 : \|x\| = 1, \|y\| = t\right\}.$$

A Banach space E is said to be *uniformly smooth* if

$$\lim_{t\to 0} \frac{\rho_E(t)}{t} = 0.$$

Let $q > 1$. A Banach space E is said to be q-uniformly smooth if there exists a fixed constant $c > 0$ such that $\rho_E(t) \leq ct^q$. It is easy to see that if E is q-uniformly smooth, then $q \leq 2$ and E is uniformly smooth.

Definition 1.3 A mapping J from E onto E^* satisfying the condition

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 \text{ and } \|f\| = \|x\|\}$$

is called the normalized duality mapping of E. The duality pair $\langle x, f\rangle$ represents $f(x)$ for $f \in E^*$ and $x \in E$.

Definition 1.4 Let C be a nonempty subset of a Banach space E and $T : C \to C$ be a self-mapping. T is called a nonexpansive mapping if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$.

T is called an η-strictly pseudo-contractive mapping if there exists a constant $\eta \in (0, 1)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \leq \|x - y\|^2 - \eta \|(I - T)x - (I - T)y\|^2 \tag{1.1}$$

for every $x, y \in C$ and for some $j(x - y) \in J(x - y)$. It is clear that (1.1) is equivalent to the following:

$$\langle (I - T)x - (I - T)y, j(x - y)\rangle \geq \eta \|(I - T)x - (I - T)y\|^2 \tag{1.2}$$

for every $x, y \in C$ and for some $j(x - y) \in J(x - y)$.

Let C and D be nonempty subsets of a Banach space E such that C is nonempty closed convex and $D \subset C$, then a mapping $P : C \to D$ is sunny [1] provided $P(x + t(x - P(x))) = P(x)$ for all $x \in C$ and $t \geq 0$, whenever $x + t(x - P(x)) \in C$. The mapping $P : C \to D$ is called a retraction if $Px = x$ for all $x \in D$. Furthermore, P is a sunny nonexpansive retraction from C onto D if P is a retraction from C onto D which is also sunny and nonexpansive. The subset D of C is called a sunny nonexpansive retraction of C if there exists a sunny nonexpansive retraction from C onto D.

An operator A of C into E is said to be *accretive* if there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y)\rangle \geq 0, \quad \forall x, y \in C.$$

A mapping $A : C \to E$ is said to be α-*inverse strongly accretive* if there exists $j(x - y) \in J(x - y)$ and $\alpha > 0$ such that

$$\langle Ax - Ay, j(x - y)\rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

Remark 1.1 From (1.1) and (1.2), if T is an η-strictly pseudo-contractive mapping, then $I - T$ is η-inverse strongly accretive.

The variational inequality problem in a Banach space is to find a point $x^* \in C$ such that for some $j(x - x^*) \in J(x - x^*)$,

$$\langle Ax^*, j(x - x^*)\rangle \geq 0, \quad \forall x \in C. \tag{1.3}$$

This problem was considered by Aoyama *et al.* [2]. The set of solutions of the variational inequality in a Banach space is denoted by $S(C, A)$, that is,

$$S(C, A) = \{u \in C : \langle Au, J(v - u)\rangle \geq 0, \forall v \in C\}. \tag{1.4}$$

Numerous problems in physics, optimization, variational inequalities, minimax problems, the Nash equilibrium problem in noncooperative games reduce to find an element of (1.4); see [3, 4].

Recall that the normal Mann's iterative process was introduced by Mann [5] in 1953. The normal Mann's iterative process generates a sequence $\{x_n\}$ in the following manner:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n, \quad \forall n \geq 1, \end{cases} \tag{1.5}$$

where the sequence $\{\alpha_n\} \subset (0, 1)$. If T is a nonexpansive mapping with a fixed point and the control sequence $\{\alpha_n\}$ is chosen so that $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$, then the sequence $\{x_n\}$ generated by normal Mann's iterative process (1.5) converges weakly to a fixed point of T.

In 2008, Cho *et al.* [6] modified the normal Mann's iterative process and proved strong convergence for a finite family of nonexpansive mappings in the framework of Banach spaces without any commutative assumption as follows.

Theorem 1.2 *Let C be a closed convex subset of a uniformly smooth and strictly convex Banach space E. Let $\{T_i\}$ be a nonexpansive mapping from C into itself for $i = 1, 2, \dots, N$. Assume that $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Given a point $u \in C$ and given sequences $\{\alpha_n\}, \{\beta_n\} \in (0, 1)$, the following conditions are satisfied*:

$$\begin{aligned} &\text{(i)} \quad \lim_{n\to\infty} \alpha_n = 0 \quad \textit{and} \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \\ &\text{(ii)} \quad \lim_{n\to\infty} |\gamma_{ni} - \gamma_{n-1i}| = 0 \quad \textit{for all } i = 1, 2, \dots, N, \\ &\text{(iii)} \quad 0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1. \end{aligned}$$

Let $\{x_n\}$ be a sequence generated by $u, x_0 = x \in C$ and

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n) W_n x_n, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n) y_n, \quad n \geq 0, \end{cases} \tag{1.6}$$

where W_n is the W-mapping generated by $T_1, T_2, \dots, T_N$ and $\gamma_{n1}, \gamma_{n2}, \dots, \gamma_{nN}$. Then $\{x_n\}$ converges strongly to $x^* \in F$, where $x^* = Q(u)$ and $Q : C \to F$ is the unique sunny nonexpansive retraction from C onto F.

In 2008, Zhou [7] proved a strong convergence theorem for the modification of normal Mann's iteration algorithm generated by a strict pseudo-contraction in a real 2-uniformly smooth Banach space as follows.

Theorem 1.3 *Let C be a closed convex subset of a real 2-uniformly smooth Banach space E and let $T : C \to C$ be a λ-strict pseudo-contraction such that $F(T) \neq \emptyset$. Given $u, x_0 \in C$ and the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ in $(0,1)$, the following control conditions are satisfied*:

(i) $a \leq \alpha_n \leq \frac{\lambda}{K^2}$ *for some* $a > 0$ *and for all* $n \geq 0$,

(ii) $\beta_n + \gamma_n + \delta_n = 1$ *for all* $n \geq 0$,

(iii) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=1}^{\infty} \beta_n = \infty$,

(iv) $\alpha_{n+1} - \alpha_n \to 0$ *as* $n \to \infty$,

(v) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$.

Let a sequence $\{x_n\}$ be generated by

$$\begin{cases} y_n = \alpha_n T x_n + (1 - \alpha_n) x_n, \\ x_{n+1} = \beta_n u + \gamma_n x_n + \delta_n y_n, \quad n \geq 0. \end{cases} \tag{1.7}$$

Then $\{x_n\}$ converges strongly to $x^* \in F(T)$, where $x^* = Q_{F(T)}(u)$ and $Q_{F(T)} : C \to F(T)$ is the unique sunny nonexpansive retraction from C onto $F(T)$.

In 2005, Aoyama *et al.* [2] proved a weak convergence theorem for finding a solution of problem (1.3) as follows.

Theorem 1.4 *Let E be a uniformly convex and 2-uniformly smooth Banach space and let C be a nonempty closed convex subset of E. Let Q_C be a sunny nonexpansive retraction from E onto C, let $\alpha > 0$ and let A be an α-inverse strongly accretive operator of C into E with $S(C,A) \neq \emptyset$. Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) Q_C(x_n - \lambda_n A x_n)$$

for every $n = 1, 2, \ldots$, where $\{\lambda_n\}$ is a sequence of positive real numbers and $\{\alpha_n\}$ is a sequence in $[0,1]$. If $\{\lambda_n\}$ and $\{\alpha_n\}$ are chosen so that $\lambda_n \in [a, \frac{\alpha}{K^2}]$ for some $a > 0$ and $\alpha_n \in [b,c]$ for some b, c with $0 < b < c < 1$, then $\{x_n\}$ converges weakly to some element z of $S(C,A)$, where K is the 2-uniformly smoothness constant of E.

In this paper, motivated by Theorems 1.2, 1.3 and 1.4, we prove a strong convergence theorem for finding a common element of the set of solutions of a finite family of variational inequality problems and the set of fixed points of a nonexpansive mapping and an η-strictly pseudo-contractive mapping in uniformly convex and 2-uniformly smooth spaces. Moreover, by using our main result, we prove a strong convergence theorem for finding a common element of the set of fixed points of a finite family of η_i-strictly pseudo-contractive mappings for every $i = 1, 2, \dots, N$ in uniformly convex and 2-uniformly smooth Banach spaces.

2 Preliminaries

In this section, we collect and prove the following lemmas to use in our main result.

Lemma 2.1 (See [8]) *Let E be a real 2-uniformly smooth Banach space with the best smooth constant K. Then the following inequality holds*:

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, J(x)\rangle + 2\|Ky\|^2$$

for any $x, y \in E$.

Definition 2.1 (See [9]) Let C be a nonempty convex subset of a real Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpanxive mappings of C into itself and let $\lambda_1, \dots, \lambda_N$ be real numbers such that $0 \le \lambda_i \le 1$ for every $i = 1, \dots, N$. Define a mapping $K : C \to C$ as follows:

$$\begin{aligned}
U_1 &= \lambda_1 T_1 + (1-\lambda_1)I, \\
U_2 &= \lambda_2 T_2 U_1 + (1-\lambda_2)U_1, \\
U_3 &= \lambda_3 T_3 U_2 + (1-\lambda_3)U_2, \\
&\vdots \\
U_{N-1} &= \lambda_{N-1} T_{N-1} U_{N-2} + (1-\lambda_{N-1})U_{N-2}, \\
K = U_N &= \lambda_N T_N U_{N-1} + (1-\lambda_N)U_{N-1}.
\end{aligned} \tag{2.1}$$

Such a mapping K is called the *K-mapping* generated by $T_1, \dots, T_N$ and $\lambda_1, \dots, \lambda_N$.

Lemma 2.2 (See [9]) *Let C be a nonempty closed convex subset of a strictly convex Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpanxive mappings of C into itself with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ and let $\lambda_1, \dots, \lambda_N$ be real numbers such that $0 < \lambda_i < 1$ for every $i = 1, \dots, N-1$ and $0 < \lambda_N \le 1$. Let K be the K-mapping generated by $T_1, \dots, T_N$ and $\lambda_1, \dots, \lambda_N$. Then $F(K) = \bigcap_{i=1}^N F(T_i)$.*

Remark 2.3 From Lemma 2.2, it is easy to see that the K mapping is a nonexpansive mapping.

Lemma 2.4 (See [10]) *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$. Suppose*

$$x_{n+1} = \beta_n x_n + (1-\beta_n) z_n$$

for all integer $n \ge 0$ and

$$\limsup_{n\to\infty} \bigl(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|\bigr) \le 0.$$

Then $\lim_{n\to\infty} \|x_n - z_n\| = 0$.

Lemma 2.5 (See [11]) *Let X be a uniformly convex Banach space and $B_r = \{x \in X : \|x\| \leq r\}$, $r > 0$. Then there exists a continuous, strictly increasing and convex function $g : [0, \infty] \rightarrow [0, \infty]$, $g(0) = 0$ such that*

$$\|\alpha x + \beta y + \gamma z\|^2 \leq \alpha \|x\|^2 + \beta \|y\|^2 + \gamma \|z\|^2 - \alpha\beta g\big(\|x - y\|\big)$$

for all $x, y, z \in B_r$ and all $\alpha, \beta, \gamma \in [0,1]$ with $\alpha + \beta + \gamma = 1$.

Lemma 2.6 (See [2]) *Let C be a nonempty closed convex subset of a smooth Banach space E. Let Q_C be a sunny nonexpansive retraction from E onto C and let A be an accretive operator of C into E. Then for all $\lambda > 0$,*

$$S(C, A) = F\big(Q_C(I - \lambda A)\big).$$

Lemma 2.7 (See [12]) *Let C be a closed convex subset of a strictly convex Banach space X. Let $\{T_n : n \in \mathbb{N}\}$ be a sequence of nonexpansive mappings on C. Suppose $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ is nonempty. Let $\{\lambda_n\}$ be a sequence of positive numbers with $\sum_{n=1}^{\infty} \lambda_n = 1$. Then a mapping S on C defined by $Sx = \sum_{n=1}^{\infty} \lambda_n T_n x$ for $x \in C$ is well defined, non-expansive and $F(S) = \bigcap_{n=1}^{\infty} F(T_n)$ holds.*

Lemma 2.8 (See [8]) *Let $r > 0$. If E is uniformly convex, then there exists a continuous, strictly increasing and convex function $g : [0, \infty) \rightarrow [0, \infty)$, $g(0) = 0$ such that for all $x, y \in B_r(0) = \{x \in E : \|x\| \leq r\}$ and for any $\alpha \in [0,1]$, we have $\|\alpha x + (1 - \alpha)y\|^2 \leq \alpha \|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)g(\|x - y\|)$.*

Lemma 2.9 (See [13]) *Let X be a uniformly smooth Banach space, C be a closed convex subset of X, $T : C \rightarrow C$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and let $f \in \prod_C$ where $\prod_C$ is to denote the collection of all contractions on C. Then the sequence $\{x_t\}$ defined by $x_t = tf(x_t) + (1 - t)Tx_t$ converses strongly to a point in $F(T)$. If we define a mapping $Q : \prod_C \rightarrow F(T)$ by $Q(f) = \lim_{t \rightarrow 0} x_t$ for all $f \in \prod_C$, then $Q(f)$ solves the following variational inequality*:

$$\big\langle (I - f)Q(f), j\big(Q(f) - p\big)\big\rangle \leq 0$$

for all $f \in \prod_C$, $p \in F(T)$.

Lemma 2.10 (See [14]) *In a Banach space E, the following inequality holds*:

$$\|x + y\|^2 \leq \|x\|^2 + 2\big\langle y, j(x + y)\big\rangle, \quad \forall x, y \in E,$$

where $j(x + y) \in J(x + y)$.

Lemma 2.11 (See [15]) *Let $\{s_n\}$ be a sequence of nonnegative real number satisfying*

$$s_{n+1} = (1 - \alpha_n)s_n + \alpha_n \beta_n, \quad \forall n \geq 0,$$

where $\{\alpha_n\}$, $\{\beta_n\}$ *satisfy the conditions*

$$(1)\quad \{\alpha_n\} \subset [0,1], \qquad \sum_{n=1}^{\infty} \alpha_n = \infty;$$

$$(2)\quad \limsup_{n\to\infty} \beta_n \le 0 \quad \textit{or} \quad \sum_{n=1}^{\infty} |\alpha_n \beta_n| < \infty.$$

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.12 *Let C be a nonempty closed convex subset of a 2-uniformly smooth Banach space E and let $T : C \to C$ be a nonexpansive mapping and $S : C \to C$ be an η-strictly pseudocontractive mapping with $F(S) \cap F(T) \neq \emptyset$. Define a mapping $B_A : C \to C$ by $B_A x = T((1-\alpha)I + \alpha S)x$ for all $x \in C$ and $\alpha \in (0, \frac{\eta}{K^2})$, where K is the 2-uniformly smooth constant of E. Then $F(B_A) = F(S) \cap F(T)$.*

Proof It is easy to see that $F(T) \cap F(S) \subseteq F(B_A)$. Let $x_0 \in F(B_A)$ and $x^* \in F(T) \cap F(S)$, we have

$$\begin{aligned}
\|x_0 - x^*\|^2 &= \|T((1-\alpha)x_0 + \alpha S x_0) - x^*\|^2 \\
&\le \|(1-\alpha)x_0 + \alpha S x_0 - x^*\|^2 \\
&= \|x_0 - x^* + \alpha(Sx_0 - x_0)\|^2 \\
&\le \|x_0 - x^*\|^2 + 2\alpha\langle Sx_0 - x_0, j(x_0 - x^*)\rangle + 2K^2\alpha^2\|Sx_0 - x_0\|^2 \\
&= \|x_0 - x^*\|^2 + 2\alpha\langle Sx_0 - x^*, j(x_0 - x^*)\rangle + 2\alpha\langle x^* - x_0, j(x_0 - x^*)\rangle \\
&\quad + 2K^2\alpha^2\|Sx_0 - x_0\|^2 \\
&= \|x_0 - x^*\|^2 + 2\alpha\langle Sx_0 - x^*, j(x_0 - x^*)\rangle - 2\alpha\|x_0 - x^*\|^2 + 2K^2\alpha^2\|Sx_0 - x_0\|^2 \\
&\le \|x_0 - x^*\|^2 + 2\alpha(\|x_0 - x^*\|^2 - \eta\|(I-S)x_0\|^2) - 2\alpha\|x_0 - x^*\|^2 \\
&\quad + 2K^2\alpha^2\|Sx_0 - x_0\|^2 \\
&= \|x_0 - x^*\|^2 - 2\alpha\eta\|x_0 - Sx_0\|^2 + 2K^2\alpha^2\|Sx_0 - x_0\|^2 \\
&= \|x_0 - x^*\|^2 - 2\alpha(\eta - K^2\alpha)\|x_0 - Sx_0\|^2. \qquad (2.2)
\end{aligned}$$

(2.2) implies that

$$2\alpha(\eta - K^2\alpha)\|x_0 - Sx_0\|^2 \le \|x_0 - x^*\|^2 - \|x_0 - x^*\|^2 = 0.$$

Then we have $Sx_0 = x_0$, that is, $x_0 \in F(S)$.

Since $x_0 \in F(B_A)$, from the definition of B_A, we have

$$x_0 = B_A x_0 = T((1-\alpha)x_0 + \alpha S x_0) = Tx_0.$$

Then we have $x_0 \in F(T)$. Therefore, $x_0 \in F(T) \cap F(S)$. It follows that $F(B_A) \subseteq F(T) \cap F(S)$. Hence, $F(B_A) = F(T) \cap F(S)$. □

Remark 2.13 Applying (2.2), we have that the mapping B_A is nonexpansive.

3 Main results

Theorem 3.1 *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space E. Let Q_C be the sunny nonexpansive retraction from E onto C. For every $i = 1,2,\dots,N$, let $A_i : C \to E$ be an α_i-inverse strongly accretive mapping. Define a mapping $G_i : C \to C$ by $Q_C(I - \lambda_i A_i)x = G_i x$ for all $x \in C$ and $i = 1,2,\dots,N$, where $\lambda_i \in (0, \frac{\alpha_i}{K^2})$, K is the 2-uniformly smooth constant of E. Let $B : C \to C$ be the K-mapping generated by $G_1, G_2,\dots,G_N$ and $\rho_1,\rho_2,\dots,\rho_N$, where $\rho_i \in (0,1)$, $\forall i = 1,2,\dots,N-1$ and $\rho_N \in (0,1]$. Let $T : C \to C$ be a nonexpansive mapping and $S : C \to C$ be an η-strictly pseudo-contractive mapping with $\mathcal{F} = F(S) \cap F(T) \cap \bigcap_{i=1}^N S(C,A_i) \neq \emptyset$. Define a mapping $B_A : C \to C$ by $T((1-\alpha)I + \alpha S)x = B_A x$, $\forall x \in C$ and $\alpha \in (0, \frac{\eta}{K^2})$. Let $\{x_n\}$ be the sequence generated by $x_1 \in C$ and*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n B x_n + \delta_n B_A x_n, \quad \forall n \geq 1, \tag{3.1}$$

where $f : C \to C$ is a contractive mapping and $\{\alpha_n\},\{\beta_n\},\{\gamma_n\},\{\delta_n\} \subseteq [0,1]$, $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ and satisfy the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\{\gamma_n\},\{\delta_n\} \subseteq [c,d] \subset (0,1)$ *for some* $c,d > 0$ *and* $\forall n \geq 1$,

(iii) $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n|$, $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n|$, $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$,

(iv) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then the sequence $\{x_n\}$ converses strongly to $q \in \mathcal{F}$, which solves the following variational inequality:

$$\langle q - f(q), j(q-p) \rangle \leq 0, \quad \forall p \in \mathcal{F}.$$

Proof First, we will show that G_i is a nonexpansive mapping for every $i = 1,2,\dots,N$.

Let $x,y \in C$. From nonexpansiveness of Q_C, we have

$$\begin{aligned}
\|G_i x - G_i y\|^2 &= \left\| Q_C(I - \lambda_i A_i)x - Q_C(I - \lambda_i A_i)y \right\|^2 \\
&\leq \left\| (I - \lambda_i A_i)x - (I - \lambda_i A_i)y \right\|^2 \\
&= \left\| x - y - \lambda_i(A_i x - A_i y) \right\|^2 \\
&\leq \|x-y\|^2 - 2\lambda_i \langle A_i x - A_i y, j(x-y) \rangle + 2K^2\lambda_i^2 \|A_i x - A_i y\|^2 \\
&\leq \|x-y\|^2 - 2\lambda_i \alpha_i \|A_i x - A_i y\|^2 + 2K^2\lambda_i^2 \|A_i x - A_i y\|^2 \\
&= \|x-y\|^2 - 2\lambda_i\left(\alpha_i - K^2\lambda_i\right)\|A_i x - A_i y\|^2 \\
&\leq \|x-y\|^2.
\end{aligned}$$

Then we have G_i is a nonexpansive mapping for every $i = 1,2,\dots,N$. Since $B : C \to C$ is the K-mapping generated by $G_1, G_2,\dots,G_N$ and $\rho_1,\rho_2,\dots,\rho_N$ and Lemma 2.2, we can conclude that $F(B)=\bigcap_{i=1}^{N}F(G_i)$. From Lemma 2.6 and the definition of G_i, we have $F(G_i)=S(C,A_i)$ for every $i=1,2,\dots,N$. Hence, we have

$$F(B)=\bigcap_{i=1}^{N}F(G_i)=\bigcap_{i=1}^{N}S(C,A_i). \tag{3.2}$$

Next, we will show that the sequence $\{x_n\}$ is bounded.

Let $z\in\mathcal{F}$; from the definition of x_n, we have

$$\begin{aligned}
\|x_{n+1}-z\| &\le \alpha_n\big\|f(x_n)-z\big\|+\beta_n\|x_n-z\|+\gamma_n\|Bx_n-z\|+\delta_n\|B_Ax_n-z\| \\
&\le \alpha_n\big\|f(x_n)-z\big\|+(1-\alpha_n)\|x_n-z\| \\
&\le \alpha_n\big\|f(x_n)-f(z)\big\|+\alpha_n\big\|f(z)-z\big\|+(1-\alpha_n)\|x_n-z\| \\
&\le \alpha_n a\|x_n-z\|+\alpha_n\big\|f(z)-z\big\|+(1-\alpha_n)\|x_n-z\| \\
&= \big(1-\alpha_n(1-a)\big)\|x_n-z\|+\alpha_n\big\|f(z)-z\big\| \\
&\le \max\left\{\|x_1-z\|,\frac{\|f(z)-z\|}{1-a}\right\}.
\end{aligned}$$

By induction, we can conclude that the sequence $\{x_n\}$ is bounded and so are $\{f(x_n)\}$, $\{Bx_n\}$, $\{B_Ax_n\}$.

Next, we will show that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.3}$$

From the definition of x_n, we can rewrite x_n by

$$x_{n+1}=\beta_nx_n+(1-\beta_n)z_n, \tag{3.4}$$

where $z_n=\frac{\alpha_nf(x_n)+\gamma_nBx_n+\delta_nB_Ax_n}{1-\beta_n}$.

Since

$$\begin{aligned}
\|z_{n+1}-z_n\| &= \left\|\frac{\alpha_{n+1}f(x_{n+1})+\gamma_{n+1}Bx_{n+1}+\delta_{n+1}B_Ax_{n+1}}{1-\beta_{n+1}}\right. \\
&\qquad \left.-\left(\frac{\alpha_nf(x_n)+\gamma_nBx_n+\delta_nB_Ax_n}{1-\beta_n}\right)\right\| \\
&= \left\|\frac{x_{n+2}-\beta_{n+1}x_{n+1}}{1-\beta_{n+1}}-\frac{x_{n+1}-\beta_nx_n}{1-\beta_n}\right\| \\
&= \left\|\frac{x_{n+2}-\beta_{n+1}x_{n+1}}{1-\beta_{n+1}}-\frac{x_{n+1}-\beta_nx_n}{1-\beta_{n+1}}+\frac{x_{n+1}-\beta_nx_n}{1-\beta_{n+1}}-\frac{x_{n+1}-\beta_nx_n}{1-\beta_n}\right\| \\
&\le \left\|\frac{x_{n+2}-\beta_{n+1}x_{n+1}}{1-\beta_{n+1}}-\frac{x_{n+1}-\beta_nx_n}{1-\beta_{n+1}}\right\|+\left\|\frac{x_{n+1}-\beta_nx_n}{1-\beta_{n+1}}-\frac{x_{n+1}-\beta_nx_n}{1-\beta_n}\right\| \\
&= \frac{1}{1-\beta_{n+1}}\big\|x_{n+2}-\beta_{n+1}x_{n+1}-(x_{n+1}-\beta_nx_n)\big\| \\
&\qquad +\left|\frac{1}{1-\beta_{n+1}}-\frac{1}{1-\beta_n}\right|\|x_{n+1}-\beta_nx_n\|
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{1-\beta_{n+1}}\big\|x_{n+2}-\beta_{n+1}x_{n+1}-(x_{n+1}-\beta_n x_n)\big\| \\
&\quad + \frac{|\beta_{n+1}-\beta_n|}{(1-\beta_n)(1-\beta_{n+1})}\|x_{n+1}-\beta_n x_n\| \\
&= \frac{1}{1-\beta_{n+1}}\big\|\alpha_{n+1}f(x_{n+1})+\gamma_{n+1}Bx_{n+1}+\delta_{n+1}B_A x_{n+1} \\
&\quad -\big(\alpha_n f(x_n)+\gamma_n Bx_n+\delta_n B_A x_n\big)\big\| + \frac{|\beta_{n+1}-\beta_n|}{(1-\beta_n)(1-\beta_{n+1})}\|x_{n+1}-\beta_n x_n\| \\
&= \frac{1}{1-\beta_{n+1}}\big(\big\|\alpha_{n+1}f(x_{n+1})-\alpha_n f(x_n)\big\| + \gamma_{n+1}\|Bx_{n+1}-Bx_n\| \\
&\quad + \delta_{n+1}\|B_A x_{n+1}-B_A x_n\| + |\gamma_{n+1}-\gamma_n|\|Bx_n\| + |\delta_{n+1}-\delta_n|\|B_A x_n\|\big) \\
&\quad + \frac{|\beta_{n+1}-\beta_n|}{(1-\beta_n)(1-\beta_{n+1})}\|x_{n+1}-\beta_n x_n\| \\
&\le \frac{1}{1-\beta_{n+1}}\big(\alpha_{n+1}\big\|f(x_{n+1})\big\| + \alpha_n\big\|f(x_n)\big\| + (\gamma_{n+1}+\delta_{n+1})\|x_{n+1}-x_n\| \\
&\quad + |\gamma_{n+1}-\gamma_n|\|Bx_n\| + |\delta_{n+1}-\delta_n|\|B_A x_n\|\big) \\
&\quad + \frac{|\beta_{n+1}-\beta_n|}{(1-\beta_n)(1-\beta_{n+1})}\|x_{n+1}-\beta_n x_n\| \\
&= \frac{\alpha_{n+1}}{1-\beta_{n+1}}\big\|f(x_{n+1})\big\| + \frac{\alpha_n}{1-\beta_{n+1}}\big\|f(x_n)\big\| + \frac{\gamma_{n+1}+\delta_{n+1}}{1-\beta_{n+1}}\|x_{n+1}-x_n\| \\
&\quad + \frac{|\gamma_{n+1}-\gamma_n|}{1-\beta_{n+1}}\|Bx_n\| + \frac{|\delta_{n+1}-\delta_n|}{1-\beta_{n+1}}\|B_A x_n\| \\
&\quad + \frac{|\beta_{n+1}-\beta_n|}{(1-\beta_n)(1-\beta_{n+1})}\|x_{n+1}-\beta_n x_n\| \\
&\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\big\|f(x_{n+1})\big\| + \frac{\alpha_n}{1-\beta_{n+1}}\big\|f(x_n)\big\| + \|x_{n+1}-x_n\| + \frac{|\gamma_{n+1}-\gamma_n|}{1-\beta_{n+1}}\|Bx_n\| \\
&\quad + \frac{|\delta_{n+1}-\delta_n|}{1-\beta_{n+1}}\|B_A x_n\| + \frac{|\beta_{n+1}-\beta_n|}{(1-\beta_n)(1-\beta_{n+1})}\|x_{n+1}-\beta_n x_n\|. \qquad (3.5)
\end{aligned}$$

From (3.5) and the conditions (i)-(iv), we have

$$\limsup_{n\to\infty}\big(\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|\big) \le 0. \qquad (3.6)$$

From Lemma 2.4 and (3.4), we have

$$\lim_{n\to\infty}\|z_n-x_n\| = 0. \qquad (3.7)$$

From (3.4), we have

$$\|x_{n+1}-x_n\| = (1-\beta_n)\|z_n-x_n\|,$$

and from the condition (iv) and (3.7), we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0.$$

Next, we will show that

$$\lim_{n\to\infty} \|Bx_n - x_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|B_A x_n - x_n\| = 0.$$

From the definition of x_n, we can rewrite x_{n+1} by

$$\begin{aligned} x_{n+1} &= \alpha_n f(x_n) + \beta_n x_n + \gamma_n Bx_n + \delta_n B_A x_n \\ &= \alpha_n f(x_n) + \beta_n x_n + (\gamma_n + \delta_n)\frac{(\gamma_n Bx_n + \delta_n B_A x_n)}{\gamma_n + \delta_n} \\ &= \alpha_n f(x_n) + \beta_n x_n + e_n z'_n, \end{aligned} \tag{3.8}$$

where $e_n = \gamma_n + \delta_n$ and $z'_n = \frac{(\gamma_n Bx_n + \delta_n B_A x_n)}{\gamma_n + \delta_n}$.

From Lemma 2.5 and (3.8), we have

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \left\|\alpha_n\big(f(x_n) - z\big) + \beta_n(x_n - z) + e_n\big(z'_n - z\big)\right\|^2 \\ &\leq \alpha_n \left\|f(x_n) - z\right\|^2 + \beta_n\|x_n - z\|^2 + e_n\left\|z'_n - z\right\|^2 - \beta_n e_n g_1\left(\left\|z'_n - x_n\right\|\right) \\ &= \alpha_n \left\|f(x_n) - z\right\|^2 + \beta_n\|x_n - z\|^2 - \beta_n e_n g_1\left(\left\|z'_n - x_n\right\|\right) \\ &\quad + e_n\left\|\frac{(\gamma_n Bx_n + \delta_n B_A x_n)}{\gamma_n + \delta_n} - z\right\|^2 \\ &= \alpha_n \left\|f(x_n) - z\right\|^2 + \beta_n\|x_n - z\|^2 - \beta_n e_n g_1\left(\left\|z'_n - x_n\right\|\right) \\ &\quad + e_n\left\|\left(1 - \frac{\delta_n}{\gamma_n + \delta_n}\right)(Bx_n - z) + \frac{\delta_n}{\gamma_n + \delta_n}(B_A x_n - z)\right\|^2 \\ &\leq \alpha_n \left\|f(x_n) - z\right\|^2 + \beta_n\|x_n - z\|^2 - \beta_n e_n g_1\left(\left\|z'_n - x_n\right\|\right) \\ &\quad + e_n\left(\left(1 - \frac{\delta_n}{\gamma_n + \delta_n}\right)\|Bx_n - z\| + \frac{\delta_n}{\gamma_n + \delta_n}\|B_A x_n - z\|\right)^2 \\ &\leq \alpha_n \left\|f(x_n) - z\right\|^2 + \beta_n\|x_n - z\|^2 - \beta_n e_n g_1\left(\left\|z'_n - x_n\right\|\right) + e_n\|x_n - z\|^2 \\ &\leq \alpha_n \left\|f(x_n) - z\right\|^2 + \|x_n - z\|^2 - \beta_n e_n g_1\left(\left\|z'_n - x_n\right\|\right), \end{aligned}$$

which implies that

$$\begin{aligned} \beta_n e_n g_1\left(\left\|z'_n - x_n\right\|\right) &\leq \alpha_n \left\|f(x_n) - z\right\|^2 + \|x_n - z\|^2 - \|x_{n+1} - z\|^2 \\ &\leq \alpha_n \left\|f(x_n) - z\right\|^2 + \big(\|x_n - z\| + \|x_{n+1} - z\|\big)\|x_{n+1} - x_n\|. \end{aligned} \tag{3.9}$$

From the conditions (i), (ii), (iv) and (3.3), we have

$$\lim_{n\to\infty} g_1\left(\left\|z'_n - x_n\right\|\right) = 0.$$

From the properties of g_1, we have

$$\lim_{n\to\infty} \left\|z'_n - x_n\right\| = 0. \tag{3.10}$$

From Lemma 2.8 and the definition of z'_n, we have

$$\begin{aligned}
\|z'_n - z\|^2 &= \left\| \frac{(\gamma_n Bx_n + \delta_n B_A x_n)}{\gamma_n + \delta_n} - z \right\|^2 \\
&= \left\| \left(1 - \frac{\delta_n}{\delta_n + \gamma_n}\right)(Bx_n - z) + \frac{\delta_n}{\delta_n + \gamma_n}(B_A x_n - z) \right\|^2 \\
&\le \left(1 - \frac{\delta_n}{\delta_n + \gamma_n}\right)\|Bx_n - z\|^2 + \frac{\delta_n}{\delta_n + \gamma_n}\|B_A x_n - z\|^2 \\
&\quad - \left(1 - \frac{\delta_n}{\delta_n + \gamma_n}\right)\frac{\delta_n}{\delta_n + \gamma_n} g_2\big(\|Bx_n - B_A x_n\|\big) \\
&\le \|x_n - z\|^2 - \left(1 - \frac{\delta_n}{\delta_n + \gamma_n}\right)\frac{\delta_n}{\delta_n + \gamma_n} g_2\big(\|Bx_n - B_A x_n\|\big),
\end{aligned}$$

which implies that

$$\begin{aligned}
\left(1 - \frac{\delta_n}{\delta_n + \gamma_n}\right)\frac{\delta_n}{\delta_n + \gamma_n} g_2\big(\|Bx_n - B_A x_n\|\big) &\le \|x_n - z\|^2 - \|z'_n - z\|^2 \\
&\le \big(\|x_n - z\| + \|z'_n - z\|\big)\|z'_n - x_n\|.
\end{aligned}$$

From the condition (iii) and (3.10), we have

$$\lim_{n\to\infty} g_2\big(\|Bx_n - B_A x_n\|\big) = 0.$$

From the properties of g_2, we have

$$\lim_{n\to\infty} \|Bx_n - B_A x_n\| = 0. \tag{3.11}$$

From the definition of x_n, we can rewrite x_{n+1} by

$$\begin{aligned}
x_{n+1} &= \alpha_n f(x_n) + \beta_n x_n + \gamma_n Bx_n + \delta_n B_A x_n \\
&= \beta_n x_n + \gamma_n Bx_n + (\alpha_n + \delta_n)\frac{\alpha_n f(x_n) + \delta_n B_A x_n}{\alpha_n + \delta_n} \\
&= \beta_n x_n + \gamma_n Bx_n + d_n z''_n,
\end{aligned} \tag{3.12}$$

where $d_n = \alpha_n + \delta_n$ and $z''_n = \frac{\alpha_n f(x_n) + \delta_n B_A x_n}{\alpha_n + \delta_n}$.

From Lemma 2.5 and the convexity of $\|\cdot\|^2$, we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \big\|\beta_n(x_n - z) + \gamma_n(Bx_n - z) + d_n\big(z''_n - z\big)\big\|^2 \\
&\le \beta_n\|x_n - z\|^2 + \gamma_n\|Bx_n - z\|^2 + d_n\|z''_n - z\|^2 - \beta_n\gamma_n g_3\big(\|x_n - Bx_n\|\big) \\
&= \beta_n\|x_n - z\|^2 + \gamma_n\|Bx_n - z\|^2 + d_n\left\|\frac{\alpha_n f(x_n) + \delta_n B_A x_n}{\alpha_n + \delta_n} - z\right\|^2 \\
&\quad - \beta_n\gamma_n g_3\big(\|x_n - Bx_n\|\big) \\
&= \beta_n\|x_n - z\|^2 + \gamma_n\|Bx_n - z\|^2 + d_n\left\|\frac{\alpha_n}{\alpha_n + \delta_n}\big(f(x_n) - z\big)\right. \\
&\quad \left. + \left(1 - \frac{\alpha_n}{\alpha_n + \delta_n}\right)(B_A x_n - z)\right\|^2 - \beta_n\gamma_n g_3\big(\|x_n - Bx_n\|\big)
\end{aligned}$$

$$\begin{aligned}
&\le \beta_n\|x_n - z\|^2 + \gamma_n\|Bx_n - z\|^2 + d_n\left(\frac{\alpha_n}{\alpha_n+\delta_n}\left\|f(x_n) - z\right\|^2 \right.\\
&\quad \left. + \left(1 - \frac{\alpha_n}{\alpha_n+\delta_n}\right)\|B_A x_n - z\|^2\right) - \beta_n\gamma_n g_3\big(\|x_n - Bx_n\|\big)\\
&= \beta_n\|x_n - z\|^2 + \gamma_n\|Bx_n - z\|^2 + d_n\frac{\alpha_n}{\alpha_n+\delta_n}\left\|f(x_n) - z\right\|^2\\
&\quad + d_n\left(1 - \frac{\alpha_n}{\alpha_n+\delta_n}\right)\|B_A x_n - z\|^2 - \beta_n\gamma_n g_3\big(\|x_n - Bx_n\|\big)\\
&\le \beta_n\|x_n - z\|^2 + \gamma_n\|x_n - z\|^2 + d_n\frac{\alpha_n}{\alpha_n+\delta_n}\left\|f(x_n) - z\right\|^2\\
&\quad + d_n\|x_n - z\|^2 - \beta_n\gamma_n g_3\big(\|x_n - Bx_n\|\big)\\
&\le \|x_n - z\|^2 + d_n\frac{\alpha_n}{\alpha_n+\delta_n}\left\|f(x_n) - z\right\|^2 - \beta_n\gamma_n g_3\big(\|x_n - Bx_n\|\big),
\end{aligned} \tag{3.13}$$

which implies that

$$\begin{aligned}
\beta_n\gamma_n g_3\big(\|x_n - Bx_n\|\big) &\le \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + d_n\frac{\alpha_n}{\alpha_n+\delta_n}\left\|f(x_n) - z\right\|^2\\
&\le \big(\|x_n - z\| + \|x_{n+1} - z\|\big)\|x_{n+1} - x_n\|\\
&\quad + d_n\frac{\alpha_n}{\alpha_n+\delta_n}\left\|f(x_n) - z\right\|^2.
\end{aligned} \tag{3.14}$$

From the conditions (i), (ii), (iv) (3.14) and (3.3), we have

$$\lim_{n\to\infty} g_3\big(\|x_n - Bx_n\|\big) = 0.$$

From the properties of g_3, we have

$$\lim_{n\to\infty} \|x_n - Bx_n\| = 0. \tag{3.15}$$

From (3.11), (3.15) and

$$\|x_n - B_A x_n\| \le \|x_n - Bx_n\| + \|Bx_n - B_A x_n\|,$$

we have

$$\lim_{n\to\infty} \|x_n - B_A x_n\| = 0. \tag{3.16}$$

Define a mapping $L : C \to C$ by $Lx = (1-\epsilon)Bx + \epsilon B_A x$ for all $x \in C$ and $\epsilon \in (0,1)$. From Lemma 2.7, 2.12 and (3.2), we have $F(L) = F(B) \cap F(B_A) = \bigcap_{i=1}^N S(C, A_i) \cap F(S) \cap F(T) = \mathcal{F}$.

From (3.15) and (3.16) and

$$\begin{aligned}
\|x_n - Lx_n\| &= \left\|(1-\epsilon)(x_n - Bx_n) + \epsilon(x_n - B_A x_n)\right\|\\
&\le (1-\epsilon)\|x_n - Bx_n\| + \epsilon\|x_n - B_A x_n\|,
\end{aligned}$$

we have

$$\lim_{n\to\infty} \|x_n - Lx_n\| = 0. \tag{3.17}$$

Next, we will show that

$$\limsup_{n\to\infty}\langle f(q)-q, j(x_n-q)\rangle \le 0, \tag{3.18}$$

where $\lim_{t\to 0} x_t = q \in \mathcal{F}$ and x_t begins the fixed point of the contraction

$$x \mapsto tf(x) + (1-t)Lx.$$

Then x_t solves the fixed point equation $x_t = tf(x_t) + (1-t)Lx_t$.

From the definition of x_t, we have

$$\begin{aligned}
\|x_t - x_n\|^2 &= \big\|t\big(f(x_t)-x_n\big) + (1-t)(Lx_t - x_n)\big\|^2 \\
&\le (1-t)^2\|Lx_t - x_n\|^2 + 2t\langle f(x_t)-x_n, j(x_t-x_n)\rangle \\
&\le (1-t)^2\big(\|Lx_t - Lx_n\| + \|Lx_n - x_n\|\big)^2 + 2t\langle f(x_t)-x_n, j(x_t-x_n)\rangle \\
&\le (1-t)^2\big(\|x_t - x_n\| + \|Lx_n - x_n\|\big)^2 + 2t\langle f(x_t)-x_n, j(x_t-x_n)\rangle \\
&= (1-t)^2\big(\|x_t - x_n\|^2 + 2\|x_t - x_n\|\|Lx_n - x_n\| + \|Lx_n - x_n\|^2\big) \\
&\quad + 2t\langle f(x_t)-x_n, j(x_t-x_n)\rangle \\
&= (1-t)^2\big(\|x_t - x_n\|^2 + 2\|x_t - x_n\|\|Lx_n - x_n\| + \|Lx_n - x_n\|^2\big) \\
&\quad + 2t\langle f(x_t)-x_t, j(x_t-x_n)\rangle + 2t\langle x_t - x_n, j(x_t-x_n)\rangle \\
&= \big(1-2t+t^2\big)\|x_t - x_n\|^2 + (1-t)^2\big(2\|x_t - x_n\|\|Lx_n - x_n\| + \|Lx_n - x_n\|^2\big) \\
&\quad + 2t\langle f(x_t)-x_t, j(x_t-x_n)\rangle + 2t\|x_t - x_n\|^2 \\
&= \big(1+t^2\big)\|x_t - x_n\|^2 + f_n(t) + 2t\langle f(x_t)-x_t, j(x_t-x_n)\rangle,
\end{aligned} \tag{3.19}$$

where $f_n(t) = (1-t)^2(2\|x_t - x_n\|\|Lx_n - x_n\| + \|Lx_n - x_n\|^2)$. From (3.17), we have

$$\lim_{n\to\infty} f_n(t) = 0. \tag{3.20}$$

(3.19) implies that

$$\begin{aligned}
\langle x_t - f(x_t), j(x_t-x_n)\rangle &\le \frac{t}{2}\|x_t - x_n\|^2 + \frac{1}{2t}f_n(t) \\
&\le \frac{t}{2}D + \frac{1}{2t}f_n(t),
\end{aligned} \tag{3.21}$$

where $D > 0$ such that $\|x_t - x_n\|^2 \le D$ for all $t \in (0,1)$ and $n \ge 1$. From (3.20) and (3.21), we have

$$\limsup_{n\to\infty}\langle x_t - f(x_t), j(x_t-x_n)\rangle \le \frac{t}{2}D. \tag{3.22}$$

From (3.22) taking $t \to 0$, we have

$$\limsup_{t\to 0}\limsup_{n\to\infty}\langle x_t - f(x_t), j(x_t-x_n)\rangle \le 0. \tag{3.23}$$

Since

$$\begin{aligned}\langle f(q)-q, j(x_n-q)\rangle &= \langle f(q)-q, j(x_n-q)\rangle - \langle f(q)-q, j(x_n-x_t)\rangle + \langle f(q)-q, j(x_n-x_t)\rangle \\ &\quad - \langle f(q)-x_t, j(x_n-x_t)\rangle + \langle f(q)-x_t, j(x_n-x_t)\rangle \\ &\quad - \langle f(x_t)-x_t, j(x_n-x_t)\rangle + \langle f(x_t)-x_t, j(x_n-x_t)\rangle \\ &= \langle f(q)-q, j(x_n-q)-j(x_n-x_t)\rangle + \langle x_t-q, j(x_n-x_t)\rangle \\ &\quad + \langle f(q)-f(x_t), j(x_n-x_t)\rangle + \langle f(x_t)-x_t, j(x_n-x_t)\rangle \\ &\le \langle f(q)-q, j(x_n-q)-j(x_n-x_t)\rangle + \|x_t-q\|\|x_n-x_t\| \\ &\quad + a\|q-x_t\|\|x_n-x_t\| + \langle f(x_t)-x_t, j(x_n-x_t)\rangle,\end{aligned}$$

it follows that

$$\begin{aligned}\limsup_{n\to\infty}\langle f(q)-q, j(x_n-q)\rangle &\le \limsup_{n\to\infty}\langle f(q)-q, j(x_n-q)-j(x_n-x_t)\rangle \\ &\quad + \|x_t-q\|\limsup_{n\to\infty}\|x_n-x_t\| + a\|q-x_t\|\limsup_{n\to\infty}\|x_n-x_t\| \\ &\quad + \limsup_{n\to\infty}\langle f(x_t)-x_t, j(x_n-x_t)\rangle. \end{aligned} \tag{3.24}$$

Since j is norm-to-norm uniformly continuous on a bounded subset of C and (3.24), then we have

$$\limsup_{n\to\infty}\langle f(q)-q, j(x_n-q)\rangle = \limsup_{t\to 0}\limsup_{n\to\infty}\langle f(q)-q, j(x_n-q)\rangle \le 0.$$

Finally, we will show the sequence $\{x_n\}$ converses strongly to $q \in \mathcal{F}$. From the definition of x_n, we have

$$\begin{aligned}\|x_{n+1}-q\|^2 &= \|\alpha_n(f(x_n)-q) + \beta_n(x_n-q) + \gamma_n(Bx_n-q) + \delta_n(B_Ax_n-q)\|^2 \\ &\le \|\beta_n(x_n-q) + \gamma_n(Bx_n-q) + \delta_n(B_Ax_n-q)\|^2 \\ &\quad + 2\alpha_n\langle f(x_n)-q, j(x_{n+1}-q)\rangle \\ &\le (\beta_n\|x_n-q\| + \gamma_n\|Bx_n-q\| + \delta_n\|B_Ax_n-q\|)^2 \\ &\quad + 2\alpha_n\langle f(x_n)-f(q), j(x_{n+1}-q)\rangle + 2\alpha_n\langle f(q)-q, j(x_{n+1}-q)\rangle \\ &\le (1-\alpha_n)^2\|x_n-q\|^2 + 2\alpha_n\langle f(x_n)-f(q), j(x_{n+1}-q)\rangle \\ &\quad + 2\alpha_n\langle f(q)-q, j(x_{n+1}-q)\rangle \\ &\le (1-\alpha_n)^2\|x_n-q\|^2 + 2a\alpha_n\|x_n-q\|\|x_{n+1}-q\| \\ &\quad + 2\alpha_n\langle f(q)-q, j(x_{n+1}-q)\rangle \\ &\le (1-\alpha_n)^2\|x_n-q\|^2 + a\alpha_n\|x_n-q\|^2 + a\alpha_n\|x_{n+1}-q\|^2 \\ &\quad + 2\alpha_n\langle f(q)-q, j(x_{n+1}-q)\rangle \\ &= (1-2\alpha_n+\alpha_n^2)\|x_n-q\|^2 + a\alpha_n\|x_n-q\|^2 + a\alpha_n\|x_{n+1}-q\|^2 \\ &\quad + 2\alpha_n\langle f(q)-q, j(x_{n+1}-q)\rangle\end{aligned}$$

$$
\begin{aligned}
&= (1-2\alpha_n + a\alpha_n)\|x_n - q\|^2 + \alpha_n^2\|x_n - q\|^2 + a\alpha_n\|x_{n+1} - q\|^2 \\
&\quad + 2\alpha_n\langle f(q) - q, j(x_{n+1} - q)\rangle \\
&= (1 - a\alpha_n - 2\alpha_n + 2a\alpha_n)\|x_n - q\|^2 + \alpha_n^2\|x_n - q\|^2 + a\alpha_n\|x_{n+1} - q\|^2 \\
&\quad + 2\alpha_n\langle f(q) - q, j(x_{n+1} - q)\rangle \\
&= \bigl(1 - a\alpha_n - 2\alpha_n(1-a)\bigr)\|x_n - q\|^2 + \alpha_n^2\|x_n - q\|^2 + a\alpha_n\|x_{n+1} - q\|^2 \\
&\quad + 2\alpha_n\langle f(q) - q, j(x_{n+1} - q)\rangle,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\|x_{n+1} - q\|^2 &\le \left(1 - \frac{2\alpha_n(1-a)}{1 - a\alpha_n}\right)\|x_n - q\|^2 \\
&\quad + \frac{\alpha_n}{1 - a\alpha_n}\bigl(\alpha_n\|x_n - q\|^2 + 2\langle f(q) - q, j(x_{n+1} - q)\rangle\bigr) \\
&\le \left(1 - \frac{2\alpha_n(1-a)}{1 - a\alpha_n}\right)\|x_n - q\|^2 \\
&\quad + \frac{2\alpha_n(1-a)}{1 - a\alpha_n} \cdot \frac{1}{2(1-a)}\bigl(\alpha_n\|x_n - q\|^2 + 2\langle f(q) - q, j(x_{n+1} - q)\rangle\bigr).
\end{aligned}
$$

From the condition (i) and Lemma 2.11, we can imply that $\{x_n\}$ converses strongly to $q \in \mathcal{F}$. This completes the proof. □

The following results can be obtained from Theorem 3.1. We, therefore, omit the proof.

Corollary 3.2 *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space E. Let Q_C be the sunny nonexpansive retraction from E onto C. For every $i = 1, 2, \dots, N$, let $A : C \to E$ be a ν-inverse strongly accretive mapping. Let $T : C \to C$ be a nonexpansive mapping and $S : C \to C$ be an η-strictly pseudo-contractive mapping with $\mathcal{F} = F(S) \cap F(T) \cap S(C, A) \neq \emptyset$. Define a mapping $B_A : C \to C$ by $T((1-\alpha)I + \alpha S)x = B_A x$, $\forall x \in C$ and $\alpha \in (0, \frac{\eta}{K^2})$, where K is the 2-uniformly smooth constant of E. Let $\{x_n\}$ be the sequence generated by $x_1 \in C$ and*

$$
x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n Q_C(I - \lambda A)x_n + \delta_n B_A x_n, \quad \forall n \ge 1,
$$

where $f : C \to C$ is a contractive mapping and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subseteq [0,1]$, $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$, $\lambda \in (0, \frac{\nu}{K^2})$ and satisfy the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\{\gamma_n\}, \{\delta_n\} \subseteq [c, d] \subset (0,1)$ *for some* $c, d > 0$ *and* $\forall n \ge 1$,

(iii) $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n|, \quad \sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n|, \quad \sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$,

(iv) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$.

Then the sequence $\{x_n\}$ *converses strongly to* $q \in \mathcal{F}$, *which solves the following variational inequality*:

$$\langle q - f(q), j(q-p) \rangle \le 0, \quad \forall p \in \mathcal{F}.$$

Corollary 3.3 *Let* C *be a nonempty closed convex subset of a uniformly convex and* 2-*uniformly smooth Banach space* E. *Let* Q_C *be the sunny nonexpansive retraction from* E *onto* C. *For every* $i = 1, 2, \dots, N$, *let* $A_i : C \to E$ *be an* α_i-*inverse strongly accretive mapping. Define a mapping* $G_i : C \to C$ *by* $Q_C(I - \lambda_i A_i)x = G_i x$ *for all* $x \in C$ *and* $i = 1, 2, \dots, N$, *where* $\lambda_i \in (0, \frac{\alpha_i}{K^2})$, K *is the* 2-*uniformly smooth constant of* E. *Let* $B : C \to C$ *be the* K-*mapping generated by* $G_1, G_2, \dots, G_N$ *and* $\rho_1, \rho_2, \dots, \rho_N$, *where* $\rho_i \in (0,1)$, $\forall i = 1, 2, \dots, N-1$ *and* $\rho_N \in (0,1]$. *Let* $T : C \to C$ *be a nonexpansive mapping with* $\mathcal{F} = F(T) \cap \bigcap_{i=1}^N S(C, A_i) \ne \emptyset$. *Let* $\{x_n\}$ *be the sequence generated by* $x_1 \in C$ *and*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n B x_n + \delta_n T x_n, \quad \forall n \ge 1,$$

where $f : C \to C$ *is a contractive mapping and* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subseteq [0,1]$, $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ *and satisfy the following conditions*:

$$\text{(i)} \quad \lim_{n\to\infty} \alpha_n = 0 \quad \textit{and} \quad \sum_{n=1}^{\infty} \alpha_n = \infty,$$

$$\text{(ii)} \quad \{\gamma_n\}, \{\delta_n\} \subseteq [c,d] \subset (0,1) \quad \textit{for some } c, d > 0 \textit{ and } \forall n \ge 1,$$

$$\text{(iii)} \quad \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n|, \qquad \sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n|, \qquad \sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty,$$

$$\text{(iv)} \quad 0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1.$$

Then the sequence $\{x_n\}$ *converses strongly to* $q \in \mathcal{F}$, *which solves the following variational inequality*:

$$\langle q - f(q), j(q-p) \rangle \le 0, \quad \forall p \in \mathcal{F}.$$

Corollary 3.4 *Let* C *be a nonempty closed convex subset of a uniformly convex and* 2-*uniformly smooth Banach space* E. *Let* Q_C *be the sunny nonexpansive retraction from* E *onto* C. *For every* $i = 1, 2, \dots, N$, *let* $A_i : C \to E$ *be an* α_i-*inverse strongly accretive mapping. Define a mapping* $G_i : C \to C$ *by* $Q_C(I - \lambda_i A_i)x = G_i x$ *for all* $x \in C$ *and* $i = 1, 2, \dots, N$, *where* $\lambda_i \in (0, \frac{\alpha_i}{K^2})$, K *is the* 2-*uniformly smooth constant of* E. *Let* $B : C \to C$ *be the* K-*mapping generated by* $G_1, G_2, \dots, G_N$ *and* $\rho_1, \rho_2, \dots, \rho_N$, *where* $\rho_i \in (0,1)$, $\forall i = 1, 2, \dots, N-1$ *and* $\rho_N \in (0,1]$. *Let* $S : C \to C$ *be an* η-*strictly pseudo-contractive mapping with* $\mathcal{F} = F(S) \cap \bigcap_{i=1}^N S(C, A_i) \ne \emptyset$. *Define a mapping* $B_A : C \to C$ *by* $(1-\alpha)x + \alpha S x = B_A x$, $\forall x \in C$ *and* $\alpha \in (0, \frac{\eta}{K^2})$. *Let* $\{x_n\}$ *be the sequence generated by* $x_1 \in C$ *and*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n B x_n + \delta_n B_A x_n, \quad \forall n \ge 1,$$

where $f : C \to C$ *is a contractive mapping and* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subseteq [0,1]$, $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ *and satisfy the following conditions*:

$$\text{(i)}\quad \lim_{n\to\infty} \alpha_n = 0 \quad \textit{and} \quad \sum_{n=1}^{\infty} \alpha_n = \infty,$$

$$\text{(ii)}\quad \{\gamma_n\}, \{\delta_n\} \subseteq [c,d] \subset (0,1) \quad \textit{for some } c,d > 0 \textit{ and } \forall n \geq 1,$$

$$\text{(iii)}\quad \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n|, \qquad \sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n|, \qquad \sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty,$$

$$\text{(iv)}\quad 0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1.$$

Then the sequence $\{x_n\}$ *converses strongly to* $q \in \mathcal{F}$, *which solves the following variational inequality*:

$$\langle q - f(q), j(q-p)\rangle \leq 0, \quad \forall p \in \mathcal{F}.$$

4 Applications

To prove the next theorem, we needed the following lemma.

Lemma 4.1 *Let* C *be a nonempty closed convex subset of a Banach space* E *and let* $P : C \to C$ *be an* η*-strictly pseudo-contractive mapping with* $F(P) \neq \emptyset$. *Then* $F(P) = S(C, I-P)$.

Proof It is easy to see that $F(P) \subseteq S(C, I-P)$. Put $A = I - P$ and $z^* \in F(P)$. Let $z_0 \in S(C, I-P)$, then there exists $j(x-z_0) \in J(x-z_0)$ such that

$$\langle (I-P)z_0, j(x-z_0)\rangle \geq 0, \quad \forall x \in C. \tag{4.1}$$

Since P is an η-strictly pseudo-contractive mapping, then there exists $j(z_0 - z^*)$ such that

$$\begin{aligned}
\langle Pz_0 - Pz^*, j(z_0 - z^*)\rangle &= \langle (I-A)z_0 - (I-A)z^*, j(z_0 - z^*)\rangle \\
&= \langle z_0 - z^* - (Az_0 - Az^*), j(z_0 - z^*)\rangle \\
&= \langle z_0 - z^*, j(z_0 - z^*)\rangle - \langle Az_0 - Az^*, j(z_0 - z^*)\rangle \\
&= \|z_0 - z^*\|^2 - \langle Az_0, j(z_0 - z^*)\rangle \\
&\leq \|z_0 - z\|^2 - \eta \|(I-P)z_0\|^2.
\end{aligned} \tag{4.2}$$

From (4.1), (4.2), we have

$$\eta \|z_0 - Pz_0\|^2 \leq \langle Az_0, j(z_0 - z^*)\rangle = -\langle Az_0, j(z^* - z_0)\rangle \leq 0.$$

It implies that $z_0 = Pz_0$, that is, $z_0 \in F(P)$. Then we have $S(C, I-P) \subseteq F(P)$. Hence, we have $S(C, I-P) = F(P)$. □

Remark 4.2 If C is a closed convex subset of a smooth Banach space E and Q_C is a sunny nonexpansive retraction from E onto C, from Remark 1.1, Lemma 2.6 and 4.1, we have

$$F(P) = S(C, I-P) = F(Q_C(I - \lambda(I-P))) \tag{4.3}$$

for all $\lambda > 0$.

Theorem 4.3 *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space E. Let Q_C be the sunny nonexpansive retraction from E onto C. For every $i = 1,2,\dots,N$, let $S_i : C \to E$ be an η_i-strictly pseudo-contractive mapping. Define a mapping $G_i : C \to C$ by $Q_C(I - \lambda_i(I - S_i))x = G_i x$ for all $x \in C$ and $i = 1,2,\dots,N$, where $\lambda_i \in (0, \frac{\eta_i}{K^2})$, K is the 2-uniformly smooth constant of E. Let $B : C \to C$ be the K-mapping generated by $G_1, G_2, \dots, G_N$ and $\rho_1, \rho_2, \dots, \rho_N$, where $\rho_i \in (0,1)$, $\forall i = 1,2,\dots,N-1$ and $\rho_N \in (0,1]$. Let $T : C \to C$ be a nonexpansive mapping and $S : C \to C$ be an η-strictly pseudo-contractive mapping with $\mathcal{F} = F(S) \cap F(T) \cap \bigcap_{i=1}^N F(S_i) \neq \emptyset$. Define a mapping $B_A : C \to C$ by $T((1-\alpha)I + \alpha S)x = B_A x$, $\forall x \in C$ and $\alpha \in (0, \frac{\eta}{K^2})$. Let $\{x_n\}$ be the sequence generated by $x_1 \in C$ and*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n B x_n + \delta_n B_A x_n, \quad \forall n \geq 1,$$

where $f : C \to C$ is a contractive mapping and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subseteq [0,1]$, $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ and satisfy the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\{\gamma_n\}, \{\delta_n\} \subseteq [c,d] \subset (0,1)$ *for some* $c,d > 0$ *and* $\forall n \geq 1$,

(iii) $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n|, \quad \sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n|, \quad \sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$,

(iv) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then the sequence $\{x_n\}$ converses strongly to $q \in \mathcal{F}$, which solves the following variational inequality:

$$\langle q - f(q), j(q-p)\rangle \leq 0, \quad \forall p \in \mathcal{F}.$$

Proof Since S_i is an η_i-strictly pseudo-contractive mapping, then we have $(I - S_i)$ is an η_i-inverse strongly accretive mapping for every $i = 1,2,\dots,N$. For every $i = 1,2,\dots,N$, putting $A_i = I - S_i$ in Theorem 3.1, from Remark 4.2 and Theorem 3.1, we can conclude the desired results. □

Next corollaries are derived from Theorem 4.3. We, therefore, omit the proof.

Corollary 4.4 *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space E. Let Q_C be the sunny nonexpansive retraction from E onto C. For every $i = 1,2,\dots,N$, let $S_i : C \to E$ be an η_i-strictly pseudo contractive mapping. Define a mapping $G_i : C \to C$ by $Q_C(I - \lambda_i(I - S_i))x = G_i x$ for all $x \in C$ and $i = 1,2,\dots,N$, where $\lambda_i \in (0, \frac{\eta_i}{K^2})$, K is the 2-uniformly smooth constant of E. Let $B : C \to C$ be the K-mapping generated by $G_1, G_2, \dots, G_N$ and $\rho_1, \rho_2, \dots, \rho_N$, where $\rho_i \in (0,1)$, $\forall i = 1,2,\dots,N-1$ and $\rho_N \in (0,1]$. Let $T : C \to C$ be a nonexpansive mapping with $\mathcal{F} = F(T) \cap \bigcap_{i=1}^N F(S_i) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by $x_1 \in C$ and*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n B x_n + \delta_n T x_n, \quad \forall n \geq 1,$$

where $f : C \to C$ *is a contractive mapping and* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subseteq [0,1]$, $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ *and satisfy the following conditions*:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\{\gamma_n\}, \{\delta_n\} \subseteq [c,d] \subset (0,1)$ *for some* $c, d > 0$ *and* $\forall n \geq 1$,

(iii) $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n|$, $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n|$, $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$,

(iv) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then the sequence $\{x_n\}$ *converses strongly to* $q \in \mathcal{F}$, *which solves the following variational inequality*:

$$\langle q - f(q), j(q-p) \rangle \leq 0, \quad \forall p \in \mathcal{F}.$$

Corollary 4.5 *Let* C *be a nonempty closed convex subset of a uniformly convex and* 2-*uniformly smooth Banach space* E. *Let* Q_C *be the sunny nonexpansive retraction from* E *onto* C. *For every* $i = 1, 2, \ldots, N$, *let* $S_i : C \to E$ *be an* η_i-*strictly pseudo contractive mapping. Define a mapping* $G_i : C \to C$ *by* $Q_C(I - \lambda_i(I - S_i))x = G_i x$ *for all* $x \in C$ *and* $i = 1, 2, \ldots, N$, *where* $\lambda_i \in (0, \frac{\eta_i}{K^2})$, K *is the* 2-*uniformly smooth constant of* E. *Let* $B : C \to C$ *be the* K-*mapping generated by* $G_1, G_2, \ldots, G_N$ *and* $\rho_1, \rho_2, \ldots, \rho_N$, *where* $\rho_i \in (0,1)$, $\forall i = 1, 2, \ldots, N-1$ *and* $\rho_N \in (0,1]$. $S : C \to C$ *be an* η-*strictly pseudo contractive mapping with* $\mathcal{F} = F(S) \cap \bigcap_{i=1}^{N} F(S_i) \neq \emptyset$. *Define a mapping* $B_A : C \to C$ *by* $(1-\alpha)x + \alpha Sx = B_A x$, $\forall x \in C$ *and* $\alpha \in (0, \frac{\eta}{K^2})$. *Let* $\{x_n\}$ *be a sequence generated by* $x_1 \in C$ *and*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n B x_n + \delta_n B_A x_n, \quad \forall n \geq 1,$$

where $f : C \to C$ *is a contractive mapping and* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subseteq [0,1]$, $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ *and satisfy the following conditions*:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\{\gamma_n\}, \{\delta_n\} \subseteq [c,d] \subset (0,1)$ *for some* $c, d > 0$ *and* $\forall n \geq 1$,

(iii) $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n|$, $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n|$, $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$,

(iv) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then the sequence $\{x_n\}$ *converses strongly to* $q \in \mathcal{F}$, *which solves the following variational inequality*:

$$\langle q - f(q), j(q-p) \rangle \leq 0, \quad \forall p \in \mathcal{F}.$$

Competing interests
The author declares that they have no competing interests.

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