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An iterative algorithm to approximate a common element of the set of common fixed points for a finite family of strict pseudo-contractions and of the set of solutions for a modified system of variational inequalities

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Abstract

In this paper, we introduce a new iterative algorithm for finding a common element of the set of fixed points of a finite family of κ_i-strictly pseudo-contractive mappings and the set of solutions of new variational inequalities problems in Hilbert space. By using our main results, we obtain an interesting theorem involving a finite family of κ-strictly pseudo-contractive mappings and two sets of solutions of the variational inequalities problem.

Keywords: pseudo-contractive mapping; modification of a general system of variational inequalities; *S*-mapping

1 Introduction

Let H be a real Hilbert space whose inner product and norm are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. Let C be a nonempty closed convex subset of H. A mapping $S : C \to C$ is called *nonexpansive* if

$$\|Sx - Sy\| \leq \|x - y\|,$$

for all $x, y \in C$.

A mapping S is called a κ*-strictly pseudo-contractive* mapping if there exists $\kappa \in [0,1)$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + \kappa \big\|(I - T)x - (I - T)y\big\|^2,$$

for all $x, y \in C$.

It is easy to see that every noexpansive mapping is a κ-strictly pseudo-contractive mapping.

Let $A : C \to H$. The *variational inequality problem* is to find a point $u \in C$ such that

$$\langle Au, v - u \rangle \geq 0 \tag{1.1}$$

for all $v \in C$. The set of solutions of (1.1) is denoted by $\mathrm{VI}(C, A)$.

Variational inequalities were initially studied by Kinderlehrer and Stampacchia [1] and Lions and Stampacchia [2]. Such a problem has been studied by many researchers, and it is connected with a wide range of applications in industry, finance, economics, social sciences, ecology, regional, pure and applied sciences; see, *e.g.*, [3–9].

A mapping A of C into H is called *α-inverse-strongly monotone*, see [10], if there exists a positive real number α such that

$$\langle x - y, Ax - Ay \rangle \geq \alpha \|Ax - Ay\|^2$$

for all $x, y \in C$.

Let $D_1, D_2 : C \to H$ be two mappings. In 2008, Ceng *et al.* [11] introduced a problem for finding $(x^*, z^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda_1 D_1 z^* + x^* - z^*, x - x^* \rangle \geq 0, & \forall x \in C, \\ \langle \lambda_2 D_2 x^* + z^* - x^*, x - z^* \rangle \geq 0, & \forall x \in C, \end{cases} \tag{1.2}$$

which is called *a system of variational inequalities* where $\lambda_1, \lambda_2 > 0$. By a modification of (1.2), we consider the problem for finding $(x^*, z^*) \in C \times C$ such that

$$\begin{cases} \langle x^* - (I - \lambda_1 D_1)(ax^* + (1 - a)z^*), x - x^* \rangle \geq 0, & \forall x \in C, \\ \langle z^* - (I - \lambda_2 D_2)x^*, x - z^* \rangle \geq 0, & \forall x \in C, \end{cases} \tag{1.3}$$

which is called *a modification of system of variational inequalities*, for every $\lambda_1, \lambda_2 > 0$ and $a \in [0, 1]$. If $a = 0$, (1.3) reduce to (1.2).

In 2008, Ceng *et al.* [11] introduce and studied a relaxed extragradient method for finding solutions of a general system of variational inequalities with inverse-strongly monotone mappings in a real Hilbert space as follows.

Theorem 1.1 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let the mappings $A, B : C \to H$ be α-inverse-strongly monotone and β-inverse-strongly monotone, respectively. Let $S : C \to C$ be a nonexpansive mapping such that $F(S) \cap \Omega$, where Ω is the set of fixed points of the mapping $G : C \to C$, defined by $G(x) = P_C(P_C(x - \mu Bx) - \lambda AP_C(x - \mu Bx))$, for all $x \in C$. Suppose that $x_1 = u \in C$ and $\{x_n\}$ is generated by*

$$\begin{cases} y_n = P_C(x_n - \mu Bx_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n P_C(x_n - \lambda Ax_n), \end{cases} \tag{1.4}$$

where $\lambda \in (0, 2\alpha)$, $\mu \in (0, 2\beta)$ *and* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ *are three sequences in* $[0,1]$ *such that*

(i) $\alpha_n + \beta_n + \gamma_n = 1, \quad \forall n \geq 1,$

(ii) $\lim_{n\to\infty} \alpha_n = 0 \quad \textit{and} \quad \sum_{n=1}^{\infty} \alpha_n = \infty,$

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1.$

Then $\{x_n\}$ *converges strongly to* $\widetilde{x} = P_{F(S)\cap\Omega}u$ *and* $(\widetilde{x}, \widetilde{y})$ *is a solution of problem* (1.2), *where* $\widetilde{y} = P_C(\widetilde{x} - \mu B\widetilde{x})$.

In the last decade, many author studied the problem for finding an element of the set of fixed points of a nonlinear mapping; see, for instance, [12–14].

From the motivation of [11] and the research in the same direction, we prove a strong convergence theorem for finding a common element of the set of fixed points of a finite family of κ_i-strictly pseudo-contractive mappings and the set of solutions of a modified general system of variational inequalities problems. Moreover, in the last section, we prove an interesting theorem involving the set of a finite family of κ_i-strictly pseudo-contractive mappings and two sets of solutions of variational inequalities problems by using our main results.

2 Preliminaries

In this section, we collect and give some useful lemmas that will be used for our main result in the next section.

Let C be a closed convex subset of a real Hilbert space H, let P_C be the metric projection of H onto C, *i.e.*, for $x \in H$, $P_C x$ satisfies the property

$$\|x - P_C x\| = \min_{y\in C} \|x - y\|.$$

It is well known that P_C is a nonexpansive mapping and satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H.$$

Obviously, this immediately implies that

$$\big\|(x - y) - (P_C x - P_C y)\big\|^2 \leq \|x - y\|^2 - \|P_C x - P_C y\|^2, \quad \forall x, y \in H.$$

The following characterizes the projection P_C.

Lemma 2.1 (See [15]) *Given* $x \in H$ *and* $y \in C$. *Then* $P_C x = y$ *if and only if the following inequality holds*:

$$\langle x - y, y - z\rangle \geq 0, \quad \forall z \in C.$$

Lemma 2.2 (See [16]) *Let* $\{s_n\}$ *be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} = (1 - \alpha_n)s_n + \alpha_n\beta_n, \quad \forall n \geq 0,$$

where $\{\alpha_n\}$, $\{\beta_n\}$ *satisfy the conditions*

$$(1)\quad \{\alpha_n\} \subset [0,1], \quad \sum_{n=1}^{\infty} \alpha_n = \infty,$$

$$(2)\quad \limsup_{n\to\infty} \beta_n \le 0 \quad \textit{or} \quad \sum_{n=1}^{\infty} |\alpha_n \beta_n| < \infty.$$

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.3 (See [17]) *Let* $\{x_n\}$ *and* $\{z_n\}$ *be bounded sequences in a Banach space* X *and let* $\{\beta_n\}$ *be a sequence in* $[0,1]$ *with* $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$. *Suppose that*

$$x_{n+1} = \beta_n x_n + (1-\beta_n) z_n$$

for all integer $n \ge 0$ *and*

$$\limsup_{n\to\infty} \bigl(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|\bigr) \le 0.$$

Then $\lim_{n\to\infty} \|x_n - z_n\| = 0$.

Definition 2.1 (See [18]) Let C be a nonempty convex subset of a real Hilbert space. Let $\{T_i\}_{i=1}^N$ be a finite family of κ_i-strict pseudo-contractions of C into itself. For each $j = 1,2,\ldots,N$, let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I \in [0,1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. Define the mapping $S : C \to C$ as follows:

$$\begin{aligned}
U_0 &= I,\\
U_1 &= \alpha_1^1 T_1 U_0 + \alpha_2^1 U_0 + \alpha_3^1 I,\\
U_2 &= \alpha_1^2 T_2 U_1 + \alpha_2^2 U_1 + \alpha_3^2 I,\\
U_3 &= \alpha_1^3 T_3 U_2 + \alpha_2^3 U_2 + \alpha_3^3 I,\\
&\vdots\\
U_{N-1} &= \alpha_1^{N-1} T_{N-1} U_{N-2} + \alpha_2^{N-1} U_{N-2} + \alpha_3^{N-1} I,\\
S = U_N &= \alpha_1^N T_N U_{N-1} + \alpha_2^N U_{N-1} + \alpha_3^N I.
\end{aligned} \tag{2.1}$$

This mapping is called *S-mapping* generated by $T_1, T_2, \ldots, T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$.

Lemma 2.4 (See [18]) *Let* C *be a nonempty closed convex subset of a real Hilbert space. Let* $\{T_i\}_{i=1}^N$ *be a finite family of* κ*-strict pseudo-contractive mappings of* C *into* C *with* $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ *and* $\kappa = \max\{\kappa_i : i = 1,2,\ldots,N\}$ *and let* $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1,2,3,\ldots,N$, *where* $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j, \alpha_3^j \in (\kappa, 1)$ *for all* $j = 1,2,\ldots,N-1$ *and* $\alpha_1^N \in (\kappa, 1]$, $\alpha_3^N \in [\kappa, 1)$, $\alpha_2^j \in [\kappa, 1)$ *for all* $j = 1,2,\ldots,N$. *Let* S *be a mapping generated by* $T_1, T_2, \ldots, T_N$ *and* $\alpha_1, \alpha_2, \ldots, \alpha_N$. *Then* $F(S) = \bigcap_{i=1}^N F(T_i)$ *and* S *is a nonexpansive mapping.*

Lemma 2.5 (See [19]) *Let E be a uniformly convex Banach space, C be a nonempty closed convex subset of E and let $S : C \to C$ be a nonexpansive mapping. Then $I - S$ is demi-closed at zero.*

Lemma 2.6 *In a real Hilbert space H, the following inequality holds*:

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$$

for all $x, y \in H$.

Lemma 2.7 *Let C be a nonempty closed convex subset of a Hilbert space H and let $D_1, D_2 : C \to H$ be mappings. For every $\lambda_1, \lambda_2 > 0$ and $a \in [0,1]$, the following statements are equivalent*:

(a) $(x^*, z^*) \in C \times C$ *is a solution of problem* (1.3),

(b) x^* *is a fixed point of the mapping $G : C \to C$, i.e., $x^* \in F(G)$, defined by*

$$G(x) = P_C(I - \lambda_1 D_1)\big(ax + (1-a)P_C(I - \lambda_2 D_2)x\big),$$

where $z^* = P_C(I - \lambda_2 D_2)x^*$.

Proof (a) $\Rightarrow$ (b) Let $(x^*, z^*) \in C \times C$ be a solution of problem (1.3). For every $\lambda_1, \lambda_2 > 0$ and $a \in [0,1]$, we have

$$\begin{cases} \langle x^* - (I - \lambda_1 D_1)(ax^* + (1-a)z^*), x - x^*\rangle \geq 0, & \forall x \in C, \\ \langle z^* - (I - \lambda_2 D_2)x^*, x - z^*\rangle \geq 0, & \forall x \in C. \end{cases}$$

From the properties of P_C, we have

$$\begin{cases} x^* = P_C(I - \lambda_1 D_1)(ax^* + (1-a)z^*), \\ z^* = P_C(I - \lambda_2 D_2)x^*. \end{cases}$$

It implies that

$$x^* = P_C(I - \lambda_1 D_1)\big(ax^* + (1-a)P_C(I - \lambda_2 D_2)x^*\big) = G\big(x^*\big).$$

Hence, we have $x^* \in F(G)$, where $z^* = P_C(I - \lambda_2 D_2)x^*$.

(b) $\Rightarrow$ (a) Let $x^* \in F(G)$ and $z^* = P_C(I - \lambda_2 D_2)x^*$. Then, we have

$$\begin{aligned} x^* = G\big(x^*\big) &= P_C(I - \lambda_1 D_1)\big(ax^* + (1-a)P_C(I - \lambda_2 D_2)x^*\big) \\ &= P_C(I - \lambda_1 D_1)\big(ax^* + (1-a)z^*\big). \end{aligned}$$

From the properties of P_C, we have

$$\begin{cases} \langle x^* - (I - \lambda_1 D_1)(ax^* + (1-a)z^*), x - x^*\rangle \geq 0, & \forall x \in C, \\ \langle z^* - (I - \lambda_2 D_2)x^*, x - z^*\rangle \geq 0, & \forall x \in C. \end{cases}$$

Hence, we have $(x^*, z^*) \in C \times C$ is a solution of (1.3). □

3 Main results

Theorem 3.1 *Let C be a nonempty closed convex subset of a real Hilbert space H and let $D_1, D_2 : C \to H$ be d_1, d_2-inverse strongly monotone mappings, respectively. Define the mapping $G : C \to C$ by $G(x) = P_C(I - \lambda_1 D_1)(ax + (1-a)P_C(I - \lambda_2 D_2)x)$ for all $x \in C$, $\lambda_1, \lambda_2 > 0$ and $a \in [0,1)$. Let $\{T_i\}_{i=1}^N$ be a finite family of κ-strict pseudo-contractive mappings of C into C with $\mathcal{F} = \bigcap_{i=1}^N F(T_i) \cap F(G) \neq \emptyset$ and $\kappa = \max\{\kappa_i : i = 1,2,\ldots,N\}$ and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1,2,3,\ldots,N$, where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j, \alpha_3^j \in (\kappa, 1)$ for all $j = 1,2,\ldots,N-1$ and $\alpha_1^N \in (\kappa, 1]$, $\alpha_3^N \in [\kappa, 1)$, $\alpha_2^j \in [\kappa, 1)$ for all $j = 1,2,\ldots,N$. Let S be a mapping generated by $T_1, T_2, \ldots, T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$. Suppose that $x_1, u \in C$ and let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} y_n = P_C(I - \lambda_2 D_2)x_n, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S P_C(a x_n + (1-a)y_n - \lambda_1 D_1(a x_n + (1-a)y_n)), \\ \qquad \forall n \geq 1, \end{cases} \tag{3.1}$$

where $\lambda_1 \in (0, 2d_1)$, $\lambda_2 \in (0, 2d_2)$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in $[0,1]$. Assume that the following conditions hold:

(i) $\alpha_n + \beta_n + \gamma_n = 1$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then $\{x_n\}$ converges strongly to $x_0 = P_{\mathcal{F}} u$ and (x_0, y_0) is a solution of (1.3), *where $y_0 = P_C(I - \lambda_2 D_2)x_0$.*

Proof First, we show that $P_C(I - \lambda_1 D_1)$ and $P_C(I - \lambda_2 D_2)$ are nonexpansive mappings for every $\lambda_1 \in (0, 2d_1)$, $\lambda_2 \in (0, 2d_2)$. Let $x, y \in C$. Since D_1 is d_1-inverse strongly monotone and $\lambda_1 < 2d_1$, we have

$$\begin{aligned} \bigl\|(I - \lambda_1 D_1)x - (I - \lambda_1 D_1)y\bigr\|^2 &= \bigl\|x - y - \lambda_1(D_1 x - D_1 y)\bigr\|^2 \\ &= \|x - y\|^2 - 2\lambda_1 \langle x - y, D_1 x - D_1 y\rangle + \lambda_1^2 \|D_1 x - D_1 y\|^2 \\ &\leq \|x - y\|^2 - 2d_1\lambda_1 \|D_1 x - D_1 y\|^2 + \lambda_1^2 \|D_1 x - D_1 y\|^2 \\ &= \|x - y\|^2 + \lambda_1(\lambda_1 - 2d_1)\|D_1 x - D_1 y\|^2 \\ &\leq \|x - y\|^2. \end{aligned} \tag{3.2}$$

Thus $(I - \lambda_1 D_1)$ is a nonexpansive mapping. By using the same method as (3.2), we have $(I - \lambda_2 D_2)$ is a nonexpansive mapping. Hence, $P_C(I - \lambda_1 D_1)$, $P_C(I - \lambda_2 D_2)$ are nonexpansive mappings. It is easy to see that the mapping G is a nonexpansive mapping. Let $x^* \in \mathcal{F}$. Then we have $x^* = Sx^*$ and

$$x^* = G(x^*) = P_C(I - \lambda_1 D_1)\bigl(a x^* + (1-a)P_C(I - \lambda_2 D_2)x^*\bigr).$$

Put $w_n = P_C(I-\lambda_1 D_1)(ax_n + (1-a)y_n)$ and $y^* = P_C(I-\lambda_2 D_2)x^*$, we can rewrite (3.1) by

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S w_n, \quad \forall n \geq 1,$$

and $x^* = P_C(I-\lambda_1 D_1)(ax^* + (1-a)y^*)$.

From the definition of x_n, we have

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|\alpha_n(u-x^*) + \beta_n(x_n - x^*) + \gamma_n(Sw_n - x^*)\| \\
&\leq \alpha_n\|u-x^*\| + \beta_n\|x_n - x^*\| + \gamma_n\|w_n - x^*\| \\
&= \alpha_n\|u-x^*\| + \beta_n\|x_n - x^*\| + \gamma_n\|P_C(I-\lambda_1 D_1)(ax_n + (1-a)y_n) \\
&\quad - P_C(I-\lambda_1 D_1)(ax^* + (1-a)P_C(I-\lambda_2 D_2)x^*)\| \\
&\leq \alpha_n\|u-x^*\| + \beta_n\|x_n - x^*\| + \gamma_n\|a(x_n - x^*) \\
&\quad + (1-a)(P_C(I-\lambda_2 D_2)x_n - P_C(I-\lambda_2 D_2)x^*)\| \\
&\leq \alpha_n\|u-x^*\| + \beta_n\|x_n - x^*\| + \gamma_n(a\|x_n - x^*\| + (1-a)\|x_n - x^*\|) \\
&= \alpha_n\|u-x^*\| + (1-\alpha_n)\|x_n - x^*\| \\
&\leq \max\{\|u-x^*\|, \|x_1 - x^*\|\}.
\end{aligned}$$

By induction we can conclude that $\|x_n - x^*\| \leq \max\{\|u-x^*\|, \|x_1 - x^*\|\}$ for all $n \in \mathbb{N}$. It implies that $\{x_n\}$ is bounded and so are $\{y_n\}$ and $\{w_n\}$.

Next, we show that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.

Let

$$x_{n+1} = (1-\beta_n)z_n + \beta_n x_n, \tag{3.3}$$

where $z_n = \frac{x_{n+1} - \beta_n x_n}{1-\beta_n}$.

Since $x_{n+1} - \beta_n x_n = \alpha_n u + \gamma_n S w_n$ and (3.3), we have

$$\begin{aligned}
z_{n+1} - z_n &= \frac{x_{n+2} - \beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1-\beta_n} \\
&= \frac{\alpha_{n+1}u + \gamma_{n+1}Sw_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n u + \gamma_n S w_n}{1-\beta_n} \\
&\quad - \frac{\gamma_{n+1}Sw_n}{1-\beta_{n+1}} + \frac{\gamma_{n+1}Sw_n}{1-\beta_{n+1}} \\
&= \left(\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right)u + \frac{\gamma_{n+1}}{1-\beta_{n+1}}(Sw_{n+1} - Sw_n) \\
&\quad + \left(\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}\right)Sw_n \\
&= \left(\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right)u + \frac{\gamma_{n+1}}{1-\beta_{n+1}}(Sw_{n+1} - Sw_n) \\
&\quad + \left(\frac{\alpha_n}{1-\beta_n} - \frac{\alpha_{n+1}}{1-\beta_{n+1}}\right)Sw_n.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|z_{n+1}-z_n\| &\le \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right|\|u\| + \frac{\gamma_{n+1}}{1-\beta_{n+1}}\|Sw_{n+1}-Sw_n\| \\
&\quad + \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right|\|Sw_n\| \\
&= \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right|\bigl(\|u\|+\|Sw_n\|\bigr) + \frac{\gamma_{n+1}}{1-\beta_{n+1}}\|w_{n+1}-w_n\| \\
&= \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right|\bigl(\|u\|+\|Sw_n\|\bigr) \\
&\quad + \frac{\gamma_{n+1}}{1-\beta_{n+1}}\bigl\|P_C(I-\lambda_1 D_1)\bigl(ax_{n+1}+(1-a)y_{n+1}\bigr) \\
&\quad - P_C(I-\lambda_1 D_1)\bigl(ax_n+(1-a)y_n\bigr)\bigr\| \\
&\le \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right|\bigl(\|u\|+\|Sw_n\|\bigr) \\
&\quad + \frac{\gamma_{n+1}}{1-\beta_{n+1}}\bigl\|a(x_{n+1}-x_n)+(1-a)(y_{n+1}-y_n)\bigr\| \\
&\le \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right|\bigl(\|u\|+\|Sw_n\|\bigr) \\
&\quad + \frac{\gamma_{n+1}}{1-\beta_{n+1}}\bigl(a\|x_{n+1}-x_n\|+(1-a)\bigl\|P_C(I-\lambda_2 D_2)x_{n+1}-P_C(I-\lambda_2 D_2)x_n\bigr\|\bigr) \\
&\le \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right|\bigl(\|u\|+\|Sw_n\|\bigr) \\
&\quad + \|x_{n+1}-x_n\|.
\end{aligned}$$

From conditions (ii) and (iii), we have

$$\limsup_{n\to\infty}\bigl(\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|\bigr)\le 0.$$

From Lemma 2.3 and (3.3) we have $\lim_{n\to\infty}\|z_n-x_n\|=0$. Since $x_{n+1}-x_n=(1-\beta_n)(z_n-x_n)$, then we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.4}$$

From the definition of w_n, we have

$$\begin{aligned}
\|w_{n+1}-w_n\| &\le \bigl\|P_C(I-\lambda_1 D_1)\bigl(ax_{n+1}+(1-a)y_{n+1}\bigr)-P_C(I-\lambda_1 D_1)\bigl(ax_n+(1-a)y_n\bigr)\bigr\| \\
&\le a\|x_{n+1}-x_n\|+(1-a)\|y_{n+1}-y_n\| \\
&= a\|x_{n+1}-x_n\|+(1-a)\bigl\|P_C(I-\lambda_2 D_2)x_{n+1}-P_C(I-\lambda_2 D_2)x_n\bigr\| \\
&\le a\|x_{n+1}-x_n\|+(1-a)\|x_{n+1}-x_n\| \\
&= \|x_{n+1}-x_n\|.
\end{aligned}$$

From (3.4), we obtain

$$\lim_{n\to\infty}\|w_{n+1}-w_n\|=0. \tag{3.5}$$

From the definition of x_n, we have

$$x_{n+1} - x_n = \alpha_n(u - x_n) + \gamma_n(Sw_n - x_n).$$

From (3.4), conditions (ii) and (iii), we have

$$\lim_{n\to\infty} \|Sw_n - x_n\| = 0. \tag{3.6}$$

From the definition of y_n, we have

$$\|y_{n+1} - y_n\| = \big\|P_C(I - \lambda_2 D_2)x_{n+1} - P_C(I - \lambda_2 D_2)x_n\big\| \le \|x_{n+1} - x_n\|. \tag{3.7}$$

From (3.4) and (3.7), we derive

$$\lim_{n\to\infty} \|y_{n+1} - y_n\| = 0. \tag{3.8}$$

From the nonexpansiveness of $P_C(I - \lambda_1 D_1)$ and $P_C(I - \lambda_2 D_2)$, we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le \alpha_n\|u - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\|Sw_n - x^*\|^2 \\
&\le \alpha_n\|u - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\|w_n - x^*\|^2 \\
&= \alpha_n\|u - x^*\|^2 + \beta_n\|x_n - x^*\|^2 \\
&\quad + \gamma_n\big\|P_C(I - \lambda_1 D_1)\big(ax_n + (1-a)y_n\big) - P_C(I - \lambda_1 D_1)\big(ax^* + (1-a)y^*\big)\big\|^2 \\
&\le \alpha_n\|u - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\big(a\|x_n - x^*\|^2 + (1-a)\|y_n - y^*\|^2\big) \\
&= \alpha_n\|u - x^*\|^2 + \beta_n\|x_n - x^*\|^2 \\
&\quad + \gamma_n\big(a\|x_n - x^*\|^2 + (1-a)\|P_C(I - \lambda_2 D_2)x_n - P_C(I - \lambda_2 D_2)x^*\|^2\big) \\
&\le \alpha_n\|u - x^*\|^2 + \beta_n\|x_n - x^*\|^2 \\
&\quad + \gamma_n\big(a\|x_n - x^*\|^2 + (1-a)\|(I - \lambda_2 D_2)x_n - (I - \lambda_2 D_2)x^*\|^2\big) \\
&= \alpha_n\|u - x^*\|^2 + \beta_n\|x_n - x^*\|^2 \\
&\quad + \gamma_n\big(a\|x_n - x^*\|^2 + (1-a)\big\|\big(x_n - x^*\big) - \lambda_2\big(D_2 x_n - D_2 x^*\big)\big\|^2\big) \\
&= \alpha_n\|u - x^*\|^2 + \beta_n\|x_n - x^*\|^2 \\
&\quad + \gamma_n\big(a\|x_n - x^*\|^2 + (1-a)\big(\|x_n - x^*\|^2 - 2\lambda_2\langle x_n - x^*, D_2 x_n - D_2 x^*\rangle \\
&\quad + \lambda_2^2\|Dx_n - Dx^*\|^2\big)\big) \\
&\le \alpha_n\|u - x^*\|^2 + \beta_n\|x_n - x^*\|^2 \\
&\quad + \gamma_n\big(a\|x_n - x^*\|^2 + (1-a)\big(\|x_n - x^*\|^2 - 2\lambda_2 d_2\|D_2 x_n - D_2 x^*\|^2 \\
&\quad + \lambda_2^2\|Dx_n - Dx^*\|^2\big)\big) \\
&= \alpha_n\|u - x^*\|^2 + \beta_n\|x_n - x^*\|^2 \\
&\quad + \gamma_n\big(a\|x_n - x^*\|^2 + (1-a)\big(\|x_n - x^*\|^2 \\
&\quad - \lambda_2(2d_2 - \lambda_2)\|D_2 x_n - D_2 x^*\|^2\big)\big)
\end{aligned}$$

$$
\begin{aligned}
&= \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 \\
&\quad + \gamma_n \left(\|x_n - x^*\|^2 - \lambda_2 (1-a)(2d_2 - \lambda_2) \|D_2 x_n - D_2 x^*\|^2 \right) \\
&\le \alpha_n \|u - x^*\|^2 + \|x_n - x^*\|^2 - \lambda_2 \gamma_n (1-a)(2d_2 - \lambda_2) \|D_2 x_n - D_2 x^*\|^2.
\end{aligned}
$$

It implies that

$$
\begin{aligned}
\lambda_2 \gamma_n (1-a)(2d_2 - \lambda_2) \|D_2 x_n - D_2 x^*\|^2 &\le \alpha_n \|u - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
&\le \alpha_n \|u - x^*\|^2 + \left(\|x_n - x^*\| + \|x_{n+1} - x^*\| \right) \\
&\quad \times \|x_{n+1} - x_n\|.
\end{aligned} \tag{3.9}
$$

From (3.4), (3.9) conditions (ii) and (iii), we have

$$
\lim_{n\to\infty} \|D_2 x_n - D_2 x^*\| = 0. \tag{3.10}
$$

Put $h^* = ax^* + (1-a)y^*$ and $h_n = ax_n + (1-a)y_n$. From the definition of x_n, we have

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|w_n - x^*\|^2 \\
&= \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|P_C(I - \lambda_1 D_1)h_n - P_C(I - \lambda_1 D_1)h^*\|^2 \\
&\le \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|(I - \lambda_1 D_1)h_n - (I - \lambda_1 D_1)h^*\|^2 \\
&= \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \left\| \left(h_n - h^*\right) - \lambda_1 \left(D_1 h_n - D_1 h^*\right) \right\|^2 \\
&= \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 \\
&\quad + \gamma_n \left(\|h_n - h^*\|^2 - 2\lambda_1 \langle h_n - h^*, D_1 h_n - D_1 h^* \rangle + \lambda_1^2 \|D_1 h_n - D_1 h^*\|^2 \right) \\
&\le \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \left(\|h_n - h^*\|^2 - 2\lambda_1 d_1 \|D_1 h_n - D_1 h^*\|^2 \right. \\
&\quad \left. + \lambda_1^2 \|D_1 h_n - D_1 h^*\|^2 \right) \\
&= \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 \\
&\quad + \gamma_n \left(\|h_n - h^*\|^2 - \lambda_1 (2d_1 - \lambda_1) \|D_1 h_n - D_1 h^*\|^2 \right) \\
&= \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \left(\left\| a\left(x_n - x^*\right) + (1-a)\left(y_n - y^*\right) \right\|^2 \right. \\
&\quad \left. - \lambda_1 (2d_1 - \lambda_1) \|D_1 h_n - D_1 h^*\|^2 \right) \\
&\le \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \left(a \|x_n - x^*\|^2 \right. \\
&\quad + (1-a) \|P_C(I - \lambda_2 D_2)x_n - P_C(I - \lambda_2 D_2)x^*\|^2 \\
&\quad \left. - \lambda_1 (2d_1 - \lambda_1) \|D_1 h_n - D_1 h^*\|^2 \right) \\
&\le \alpha_n \|u - x^*\|^2 + \|x_n - x^*\|^2 - \lambda_1 \gamma_n (2d_1 - \lambda_1) \|D_1 h_n - D_1 h^*\|^2,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\lambda_1 \gamma_n (2d_1 - \lambda_1) \|D_1 h_n - D_1 h^*\|^2 &\le \alpha_n \|u - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
&\le \alpha_n \|u - x^*\|^2 + \left(\|x_n - x^*\| + \|x_{n+1} - x^*\| \right) \\
&\quad \times \|x_{n+1} - x_n\|.
\end{aligned} \tag{3.11}
$$

From (3.4), (3.11), conditions (ii) and (iii), we can conclude

$$\lim_{n\to\infty}\|D_1h_n - D_1h^*\| = 0. \tag{3.12}$$

Next, we show that

$$\lim_{n\to\infty}\|Sw_n - w_n\| = 0. \tag{3.13}$$

From the definition of y_n, we have

$$\begin{aligned}
\|y_n - y^*\|^2 &= \|P_C(I-\lambda_2D_2)x_n - P_C(I-\lambda_2D_2)x^*\|^2 \\
&\le \langle x_n - \lambda_2D_2x_n - (x^* - \lambda_2D_2x^*), y_n - y^*\rangle \\
&= \frac{1}{2}\big(\|x_n - \lambda_2D_2x_n - (x^* - \lambda_2D_2x^*)\|^2 + \|y_n - y^*\|^2 \\
&\quad - \|x_n - \lambda_2D_2x_n - (x^* - \lambda_2D_2x^*) - (y_n - y^*)\|^2\big) \\
&= \frac{1}{2}\big(\|x_n - \lambda_2D_2x_n - (x^* - \lambda_2D_2x^*)\|^2 + \|y_n - y^*\|^2 \\
&\quad - \|x_n - y_n - (x^* - y^*) - \lambda_2(D_2x_n - D_2x^*)\|^2\big) \\
&= \frac{1}{2}\big(\|x_n - \lambda_2D_2x_n - (x^* - \lambda_2D_2x^*)\|^2 + \|y_n - y^*\|^2 \\
&\quad - \|x_n - y_n - (x^* - y^*)\|^2 + 2\lambda_2\langle x_n - y_n - (x^* - y^*), D_2x_n - D_2x^*\rangle \\
&\quad - \lambda_1^2\|D_2x_n - D_2x^*\|^2\big).
\end{aligned}$$

It implies that

$$\begin{aligned}
\|y_n - y^*\| &\le \|x_n - \lambda_2D_2x_n - (x^* - \lambda_2D_2x^*)\|^2 - \|x_n - y_n - (x^* - y^*)\|^2 \\
&\quad + 2\lambda_2\langle x_n - y_n - (x^* - y^*), D_2x_n - D_2x^*\rangle - \lambda_1^2\|D_2x_n - D_2x^*\|^2 \\
&\le \|x_n - x^*\|^2 - \|x_n - y_n - (x^* - y^*)\|^2 \\
&\quad + 2\lambda_2\langle x_n - y_n - (x^* - y^*), D_2x_n - D_2x^*\rangle \\
&\quad - \lambda_1^2\|D_2x_n - D_2x^*\|^2.
\end{aligned} \tag{3.14}$$

From the nonexpansiveness of $P_C(I-\lambda_1D_1)$ and (3.14), we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le \alpha_n\|u - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\|Sw_n - x^*\|^2 \\
&\le \alpha_n\|u - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\|w_n - x^*\|^2 \\
&= \alpha_n\|u - x^*\|^2 + \beta_n\|x_n - x^*\|^2 \\
&\quad + \gamma_n\|P_C(I-\lambda_1D_1)(ax_n + (1-a)y_n) - P_C(I-\lambda_1D_1)(ax^* + (1-a)y^*)\|^2 \\
&\le \alpha_n\|u - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\big(a\|x_n - x^*\|^2 + (1-a)\|y_n - y^*\|^2\big) \\
&\le \alpha_n\|u - x^*\|^2 + \beta_n\|x_n - x^*\|^2 \\
&\quad + \gamma_n\big(a\|x_n - x^*\|^2 + (1-a)\big(\|x_n - x^*\|^2 - \|x_n - y_n - (x^* - y^*)\|^2
\end{aligned}$$

$$
\begin{aligned}
&\quad + 2\lambda_2\langle x_n - y_n - (x^* - y^*), D_2x_n - D_2x^*\rangle - \lambda_1^2\|D_2x_n - D_2x^*\|^2)\big) \\
&\le \alpha_n\|u - x^*\|^2 + \beta_n\|x_n - x^*\|^2 \\
&\quad + \gamma_n\big(a\|x_n - x^*\|^2 + (1-a)\|x_n - x^*\|^2 - (1-a)\|x_n - y_n - (x^* - y^*)\|^2 \\
&\quad + 2\lambda_2\|x_n - y_n - (x^* - y^*)\|\|D_2x_n - D_2x^*\|\big) \\
&\le \alpha_n\|u - x^*\|^2 + \|x_n - x^*\|^2 - \gamma_n(1-a)\|x_n - y_n - (x^* - y^*)\|^2 \\
&\quad + 2\lambda_2\|x_n - y_n - (x^* - y^*)\|\|D_2x_n - D_2x^*\|.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\gamma_n(1-a)\|x_n - y_n - (x^* - y^*)\|^2 &\le \alpha_n\|u - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
&\quad + 2\lambda_2\|x_n - y_n - (x^* - y^*)\|\|D_2x_n - D_2x^*\| \\
&\le \alpha_n\|u - x^*\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_{n+1} - x_n\| \\
&\quad + 2\lambda_2\|x_n - y_n - (x^* - y^*)\|\|D_2x_n - D_2x^*\|.
\end{aligned}
$$

From condition (ii), (3.4) and (3.10), we have

$$
\lim_{n\to\infty}\|x_n - y_n - (x^* - y^*)\| = 0. \tag{3.15}
$$

From the definition of w_n, x^*, h_n, h^*, we have

$$
w_n = P_C(I - \lambda_1D_1)(ax_n + (1-a)y_n) = P_C(I - \lambda_1D_1)h_n
$$

and

$$
x^* = P_C(I - \lambda_1D_1)(ax^* + (1-a)y^*) = P_C(I - \lambda_1D_1)h^*.
$$

From the properties of P_C, we have

$$
\begin{aligned}
\|y_n - w_n + (x^* - y^*)\|^2 &= \|y_n - y^* - (w_n - x^*)\|^2 \\
&= \|y_n - ax_n + ax_n - ay_n + ay_n - \lambda_1D_1(ax_n + (1-a)y_n) \\
&\quad + \lambda_1D_1(ax_n + (1-a)y_n - y^* + ax^* - ax^* + ay^* - ay^* \\
&\quad + \lambda_1D_1(ax^* + (1-a)y^*) \\
&\quad - \lambda_1D_1(ax^* + (1-a)y^*) - (w_n - x^*)\|^2 \\
&= \|ax_n + (1-a)y_n - \lambda_1D_1(ax_n + (1-a)y_n) \\
&\quad - (ax^* + (1-a)y^* - \lambda_1D_1(ax^* + (1-a)y^*)) - (w_n - x^*) \\
&\quad + \lambda_1(D_1(ax_n + (1-a)y_n) - D_1(ax^* + (1-a)y^*)) \\
&\quad + a(y_n - x_n - y^* + x^*)\|^2 \\
&= \|(I - \lambda_1D_1)(ax_n + (1-a)y_n) - (I - \lambda_1D_1)(ax^* + (1-a)y^*) \\
&\quad - (w_n - x^*) + \lambda_1(D_1(ax_n + (1-a)y_n) - D_1(ax^* + (1-a)y^*))
\end{aligned}
$$

$$
\begin{aligned}
&+ a\big(y_n - x_n - y^* + x^*\big)\big\|^2 \\
&= \big\|(I - \lambda_1 D_1)h_n - (I - \lambda_1 D_1)h^* \\
&\quad - \big(P_C(I - \lambda_1 D_1)h_n - P_C(I - \lambda_1 D_1)h^*\big) + \lambda_1\big(D_1 h_n - D_1 h^*\big) \\
&\quad + a\big(y_n - x_n - y^* + x^*\big)\big\|^2 \\
&\le \big\|(I - \lambda_1 D_1)h_n - (I - \lambda_1 D_1)h^* - \big(P_C(I - \lambda_1 D_1)h_n \\
&\quad - P_C(I - \lambda_1 D_1)h^*\big)\big\|^2 \\
&\quad + 2\big\langle \lambda_1\big(D_1 h_n - D_1 h^*\big) + a\big(y_n - x_n - y^* + x^*\big), \\
&\quad y_n - w_n + \big(x^* - y^*\big)\big\rangle \\
&\le \big\|(I - \lambda_1 D_1)h_n - (I - \lambda_1 D_1)h^*\big\|^2 \\
&\quad - \big\|P_C(I - \lambda_1 D_1)h_n - P_C(I - \lambda_1 D_1)h^*\big\|^2 \\
&\quad + 2\big(\lambda_1\big\|D_1 h_n - D_1 h^*\big\| + a\big\|y_n - x_n - y^* + x^*\big\|\big) \\
&\quad \times \big\|y_n - w_n + \big(x^* - y^*\big)\big\| \\
&= \big\|(I - \lambda_1 D_1)h_n - (I - \lambda_1 D_1)h^*\big\|^2 - \big\|w_n - x^*\big\|^2 \\
&\quad + 2\big(\lambda_1\big\|D_1 h_n - D_1 h^*\big\| + a\big\|y_n - x_n - y^* + x^*\big\|\big) \\
&\quad \times \big\|y_n - w_n + \big(x^* - y^*\big)\big\| \\
&\le \big\|(I - \lambda_1 D_1)h_n - (I - \lambda_1 D_1)h^*\big\|^2 - \big\|Sw_n - Sx^*\big\|^2 \\
&\quad + 2\big(\lambda_1\big\|D_1 h_n - D_1 h^*\big\| + a\big\|y_n - x_n - y^* + x^*\big\|\big) \\
&\quad \times \big\|y_n - w_n + \big(x^* - y^*\big)\big\| \\
&\le \big(\big\|(I - \lambda_1 D_1)h_n - (I - \lambda_1 D_1)h^*\big\| + \big\|Sw_n - Sx^*\big\|\big) \\
&\quad \times \big\|(I - \lambda_1 D_1)h_n - (I - \lambda_1 D_1)h^* - \big(Sw_n - x^*\big)\big\| \\
&\quad + 2\big(\lambda_1\big\|D_1 h_n - D_1 h^*\big\| + a\big\|y_n - x_n - y^* + x^*\big\|\big) \\
&\quad \times \big\|y_n - w_n + \big(x^* - y^*\big)\big\| \\
&= \big(\big\|(I - \lambda_1 D_1)h_n - (I - \lambda_1 D_1)h^*\big\| + \big\|Sw_n - Sx^*\big\|\big) \\
&\quad \times \big\|h_n - h^* - \lambda_1\big(D_1 h_n - D_1 h^*\big) - \big(Sw_n - x^*\big)\big\| \\
&\quad + 2\big(\lambda_1\big\|D_1 h_n - D_1 h^*\big\| + a\big\|y_n - x_n - y^* + x^*\big\|\big) \\
&\quad \times \big\|y_n - w_n + \big(x^* - y^*\big)\big\| \\
&= \big(\big\|(I - \lambda_1 D_1)h_n - (I - \lambda_1 D_1)h^*\big\| + \big\|Sw_n - Sx^*\big\|\big) \\
&\quad \times \big\|x_n - Sw_n + \big(x^* - h^*\big) - (x_n - h_n) - \lambda_1\big(D_1 h_n - D_1 h^*\big)\big)\big\| \\
&\quad + 2\big(\lambda_1\big\|D_1 h_n - D_1 h^*\big\| + a\big\|y_n - x_n - y^* + x^*\big\|\big) \\
&\quad \times \big\|y_n - w_n + \big(x^* - y^*\big)\big\| \\
&\le \big(\big\|(I - \lambda_1 D_1)h_n - (I - \lambda_1 D_1)h^*\big\| + \big\|Sw_n - Sx^*\big\|\big) \\
&\quad \times \big(\|x_n - Sw_n\| + \big\|\big(x^* - h^*\big) - (x_n - h_n)\big\| \\
&\quad + \lambda_1\big\|D_1 h_n - D_1 h^*\big\|\big)
\end{aligned}
$$

$$
\begin{aligned}
&+ 2\big(\lambda_1\|D_1h_n - D_1h^*\| + a\|y_n - x_n - y^* + x^*\|\big)\\
&\times \|y_n - w_n + \big(x^* - y^*\big)\|\\
= {} & \big(\|(I-\lambda_1 D_1)h_n - (I-\lambda_1 D_1)h^*\| + \|Sw_n - Sx^*\|\big)\\
&\times \big(\|x_n - Sw_n\| + (1-a)\|x^* - y^* - x_n + y_n\|\\
&+ \lambda_1\|D_1h_n - D_1h^*\big)\|\\
&+ 2\big(\lambda_1\|D_1h_n - D_1h^*\| + a\|y_n - x_n - y^* + x^*\|\big)\\
&\times \|y_n - w_n + \big(x^* - y^*\big)\|.
\end{aligned}
$$

From (3.6), (3.12) and (3.15), we have

$$\lim_{n\to\infty}\|y_n - w_n + \big(x^* - y^*\big)\| = 0. \tag{3.16}$$

Since

$$\|x_n - w_n\| \leq \|x_n - y_n - \big(x^* - y^*\big)\| + \|y_n + \big(x^* - y^*\big) - w_n\|$$

and (3.15), (3.16), then we have

$$\lim_{n\to\infty}\|x_n - w_n\| = 0. \tag{3.17}$$

From (3.6) and (3.17), we can conclude that

$$\lim_{n\to\infty}\|Sw_n - w_n\| = 0.$$

Next we show that

$$\limsup_{n\to\infty}\langle u - x_0, x_n - x_0\rangle \leq 0, \tag{3.18}$$

where $x_0 = P_{\mathcal{F}}u$. To show this inequality, take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle u - x_0, x_n - x_0\rangle = \lim_{k\to\infty}\langle u - x_0, x_{n_k} - x_0\rangle.$$

Without loss of generality, we may assume that $x_{n_k} \rightharpoonup \omega$ as $k \to \infty$, where $\omega \in C$. From (3.17), we have $w_{n_k} \rightharpoonup \omega$ as $k \to \infty$. From Lemma 2.5 and (3.13), we have

$$\omega \in F(S).$$

From Lemma 2.4, we have $F(S) = \bigcap_{i=1}^N F(T_i)$. Then we obtain

$$\omega \in \bigcap_{i=1}^N F(T_i).$$

From the nonexpansiveness of the mapping G and the definition of w_n, we have

$$\begin{aligned}\|w_n - Gw_n\| &= \bigl\|P_C(I-\lambda_1 D_1)\bigl(ax_n + (1-a)P_C(I-\lambda_2 D_2)x_n\bigr) - G(w_n)\bigr\| \\ &= \|Gx_n - Gw_n\| \\ &\le \|x_n - w_n\|.\end{aligned}$$

From (3.17), we have

$$\lim_{n\to\infty} \|w_n - Gw_n\| = 0. \tag{3.19}$$

From $w_{n_k} \rightharpoonup \omega$ as $k \to \infty$, (3.19) and Lemma 2.5, we have

$$\omega \in F(G).$$

Hence, we can conclude that $\omega \in \mathcal{F}$.

Since $x_{n_k} \rightharpoonup \omega$ as $k \to \infty$ and $\omega \in \mathcal{F}$, we have

$$\limsup_{n\to\infty} \langle u - x_0, x_n - x_0\rangle = \lim_{k\to\infty} \langle u - x_0, x_{n_k} - x_0\rangle = \langle u - x_0, \omega - x_0\rangle \le 0. \tag{3.20}$$

From the definition of x_n and $x_0 = P_{\mathcal{F}}u$, we have

$$\begin{aligned}\|x_{n+1} - x_0\|^2 &= \bigl\|\alpha_n(u - x_0) + \beta_n(x_n - x_0) + \gamma_n(Sw_n - x_0)\bigr\|^2 \\ &\le \bigl\|\beta_n(x_n - x_0) + \gamma_n(Sw_n - x_0)\bigr\|^2 + 2\alpha_n\langle u - x_0, x_{n+1} - x_0\rangle \\ &\le \beta_n\|x_n - x_0\|^2 + \gamma_n\|Gx_n - x_0\|^2 + 2\alpha_n\langle u - x_0, x_{n+1} - x_0\rangle \\ &\le \beta_n\|x_n - x_0\|^2 + \gamma_n\|x_n - x_0\|^2 + 2\alpha_n\langle u - x_0, x_{n+1} - x_0\rangle \\ &\le (1-\alpha_n)\|x_n - x_0\|^2 + 2\alpha_n\langle u - x_0, x_{n+1} - x_0\rangle.\end{aligned}$$

From condition (ii), (3.18) and Lemma 2.2, we can conclude that the sequence $\{x_n\}$ converges strongly to $x_0 = P_{\mathcal{F}}u$. This completes the proof. □

Remark 3.2 (1) If we take $a = 0$, then the iterative scheme (3.1) reduces to the following scheme:

$$\begin{cases} x_1, \quad u \in C, \\ y_n = P_C(I-\lambda_2 D_2)x_n, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n SP_C(I-\lambda_1 D_1)y_n, \quad \forall n \ge 1, \end{cases} \tag{3.21}$$

which is an improvement to (1.4). From Theorem 3.1, we obtain that the sequence $\{x_n\}$ generated by (3.21) converges strongly to $x_0 = P_{\bigcap_{i=1}^N F(T_i)\cap F(G)}u$, where the mapping $G : C \to C$ defined by $Gx = P_C(I-\lambda_1 D_1)P_C(I-\lambda_2 D_2)x$ for all $x \in C$ and (x_0, y_0) is a solution of (1.2) where $y_0 = P_C(I-\lambda_2 D_2)x_0$.

(2) If we take $N = 1$, $\alpha_1^1 = 1$ and $T_1 = T$, then the iterative scheme (3.1) reduces to the following scheme:

$$\begin{cases} x_1, \quad u \in C, \\ y_n = P_C(I - \lambda_2 D_2)x_n, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n T P_C(I - \lambda_1 D_1)(ax_n + (1-a)y_n), \quad \forall n \geq 1, \end{cases} \tag{3.22}$$

From Theorem 3.1, we obtain that the sequence $\{x_n\}$ generated by (3.22) converges strongly to $x_0 = P_{F(T)\cap F(G)}u$, where the mapping $G : C \to C$ defined by $G(x) = P_C(I - \lambda_1 D_1)(ax + (1-a)P_C(I - \lambda_2 D_2)x)$ for all $x \in C$ and (x_0, y_0) is a solution of (1.3) where $y_0 = P_C(I - \lambda_2 D_2)x_0$.

4 Applications

In this section we prove a strong convergence theorem involving variational inequalities problems by using our main result. We need the following lemmas to prove the desired results.

Lemma 4.1 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T, S : C \to C$ be nonexpansive mappings. Define a mapping $B^A : C \to C$ by $B^A x = T(\alpha I + (1-\alpha)S)x$ for every $x \in C$ and $\alpha \in (0,1)$. Then $F(B^A) = F(T) \cap F(S)$ and B^A is a nonexpansive mapping.*

Proof It is easy to see that $F(T) \cap F(S) \subseteq F(B^A)$. Let $x_0 \in F(B^A)$ and $x^* \in F(T) \cap F(S)$. By the definition of B^A, we have

$$\begin{aligned} \|x_0 - x^*\|^2 &= \|Bx_0 - x^*\|^2 = \|T(\alpha I + (1-\alpha)S)x_0 - x^*\|^2 \\ &\leq \|\alpha x_0 + (1-\alpha)Sx_0 - x^*\|^2 \\ &= \alpha\|x_0 - x^*\|^2 + (1-\alpha)\|Sx_0 - x^*\|^2 - \alpha(1-\alpha)\|x_0 - Sx_0\|^2 \\ &\leq \alpha\|x_0 - x^*\|^2 + (1-\alpha)\|x_0 - x^*\|^2 - \alpha(1-\alpha)\|x_0 - Sx_0\|^2 \\ &= \|x_0 - x^*\|^2 - \alpha(1-\alpha)\|x_0 - Sx_0\|^2. \end{aligned} \tag{4.1}$$

From (4.1), it implies that

$$\alpha(1-\alpha)\|x_0 - Sx_0\|^2 \leq 0.$$

Then we have $x_0 = Sx_0$, that is, $x_0 \in F(S)$. By the definition of B^A, we have

$$x_0 = B^A x_0 = T(\alpha x_0 + (1-\alpha)Sx_0) = Tx_0.$$

It follows that $x_0 \in F(T)$. Then we have $x_0 \in F(T) \cap F(S)$. Hence $F(B^A) \subseteq F(T) \cap F(S)$.

Next, we show that B^A is a nonexpansive mapping. Let $x, y \in C$, since

$$\begin{aligned} \|B^A x - B^A y\|^2 &= \|T(\alpha I + (1-\alpha)S)x - T(\alpha I + (1-\alpha)S)y\|^2 \\ &\leq \|(\alpha I + (1-\alpha)S)x - (\alpha I + (1-\alpha)S)y\|^2 \end{aligned}$$

$$\begin{aligned} &= \left\|\alpha(x-y) + (1-\alpha)(Sx-Sy)\right\|^2 \\ &\le \alpha\|x-y\|^2 + (1-\alpha)\|Sx-Sy\|^2 \\ &\le \|x-y\|^2. \end{aligned} \tag{4.2}$$

Then we have B^A is a nonexpansive mapping. □

Lemma 4.2 (See [15]) *Let H be a real Hibert space, let C be a nonempty closed convex subset of H and let A be a mapping of C into H. Let $u \in C$. Then for $\lambda > 0$,*

$$u = P_C(I - \lambda A)u \quad \Leftrightarrow \quad u \in \mathrm{VI}(C, A),$$

where P_C is the metric projection of H onto C.

Lemma 4.3 *Let C be a nonempty closed convex subset of a real Hilbert space H and let $D_1, D_2 : C \to H$ be d_1, d_2-inverse strongly monotone mappings, respectively, which $\mathrm{VI}(C, D_1) \cap \mathrm{VI}(C, D_2) \neq \emptyset$. Define a mapping $G : C \to C$ as in Lemma 2.7 for every $\lambda_1 \in (0, 2d_1)$, $\lambda_2 \in (0, 2d_2)$ and $a \in (0, 1)$. Then $F(G) = \mathrm{VI}(C, D_1) \cap \mathrm{VI}(C, D_2)$.*

Proof First, we show that $(I - \lambda_1 D_1)$, $(I - \lambda_2 D2)$ are nonexpansive. Let $x, y \in C$. Since D_1 is d_1-inverse strongly monotone and $\lambda_1 < 2d_1$, we have

$$\begin{aligned} &\left\|(I - \lambda_1 D_1)x - (I - \lambda_1 D_1)y\right\|^2 \\ &\quad = \left\|x - y - \lambda_1(D_1 x - D_1 y)\right\|^2 \\ &\quad = \|x-y\|^2 - 2\lambda_1\langle x - y, D_1 x - D_1 y\rangle + \lambda_1^2\|D_1 x - D_1 y\|^2 \\ &\quad \le \|x-y\|^2 - 2d_1\lambda_1\|D_1 x - D_1 y\|^2 + \lambda_1^2\|D_1 x - D_1 y\|^2 \\ &\quad = \|x-y\|^2 + \lambda_1(\lambda_1 - 2d_1)\|D_1 x - D_1 y\|^2 \\ &\quad \le \|x-y\|^2. \end{aligned} \tag{4.3}$$

Thus $(I - \lambda_1 D_1)$ is nonexpansive. By using the same method as (4.3), we have $(I - \lambda_2 D_2)$ is a nonexpansive mapping. Hence $P_C(I - \lambda_1 D_1)$, $P_C(I - \lambda_2 D_2)$ are nonexpansive mappings. From

$$G(x) = P_C(I - \lambda_1 D_1)\big(ax + (1-a)P_C(I - \lambda_2 D_2)x\big),$$

for every $x \in C$ and Lemma 4.1, we have

$$F(G) = F\big(P_C(I - \lambda_1 D_1)\big) \cap F\big(P_C(I - \lambda_2 D_2)\big). \tag{4.4}$$

From Lemma 4.2, we have

$$F(G) = \mathrm{VI}(C, D_1) \cap \mathrm{VI}(C, D_2).$$ □

Theorem 4.4 *Let C be a nonempty closed convex subset of a real Hilbert space H and let $D_1, D_2 : C \to H$ be d_1, d_2-inverse strongly monotone mappings, respectively. Define the* *mapping* $G : C \to C$ *by* $G(x) = P_C(I - \lambda_1 D_1)(ax + (1-a)P_C(I - \lambda_2 D_2)x)$ *for all* $x \in C$, $\lambda_1, \lambda_2 > 0$ *and* $a \in (0,1)$. *Let* $\{T_i\}_{i=1}^N$ *be a finite family of* κ*-strict pseudo-contractive mappings of* C *into* C *with* $\mathcal{F} = \bigcap_{i=1}^N F(T_i) \cap \mathrm{VI}(C, D_1) \cap \mathrm{VI}(C, D_2) \neq \emptyset$ *and* $\kappa = \max\{\kappa_i : i = 1, 2, \ldots, N\}$ *and let* $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1, 2, 3, \ldots, N$, *where* $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j, \alpha_3^j \in (\kappa, 1)$ *for all* $j = 1, 2, \ldots, N-1$ *and* $\alpha_1^N \in (\kappa, 1]$, $\alpha_3^N \in [\kappa, 1)$, $\alpha_2^j \in [\kappa, 1)$ *for all* $j = 1, 2, \ldots, N$. *Let* S *be a mapping generated by* $T_1, T_2, \ldots, T_N$ *and* $\alpha_1, \alpha_2, \ldots, \alpha_N$. *Suppose that* $x_1, u \in C$ *and let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} y_n = P_C(I - \lambda_2 D_2)x_n, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S P_C(a x_n + (1-a) y_n - \lambda_1 D_1(a x_n + (1-a) y_n)), \\ \qquad \forall n \geq 1, \end{cases} \tag{4.5}$$

where $\lambda_1 \in (0, 2d_1)$, $\lambda_2 \in (0, 2d_2)$ *and* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ *are sequences in* $[0,1]$. *Assume that the following conditions hold*:

(i) $\alpha_n + \beta_n + \gamma_n = 1$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then $\{x_n\}$ *converges strongly to* $x_0 = P_{\mathcal{F}} u$.

Proof From Lemma 4.3 and Theorem 3.1 we can conclude the desired conclusion. □

Competing interests
The author declares that they have no competing interests.

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References

1. Kinderlehrer, D, Stampacchia, G: An Introduction to Variational Inequalities and Their Applications. Academic Press, New York (1980)
2. Lions, JL, Stampacchia, G: Variational inequalities. Commun. Pure Appl. Math. **20**, 493-517 (1967)
3. Chang, SS, Joseph Lee, HW, Chan, CK: A new method for solving equilibrium problem fixed point problem and variational inequality problem with application to optimization. Nonlinear Anal. **70**, 3307-3319 (2009)
4. Nadezhkina, N, Takahashi, W: Weak convergence theorem by an extragradientmethod for nonexpansive mappings and monotone mappings. J. Optim. Theory Appl. **128**, 191-201 (2006)
5. Yao, JC, Chadli, O: Pseudomonotone complementarity problems and variational inequalities. In: Crouzeix, JP, Haddjissas, N, Schaible, S (eds.) Handbook of Generalized Convexity and Monotonicity, pp. 501-558. Springer, New York (2005)
6. Yao, Y, Yao, JC: On modified iterative method for nonexpansive mappings and monotone mappings. Appl. Math. Comput. **186**(2), 1551-1558 (2007)
7. Ceng, LC, Yao, JC: Strong convergence theorems for variational inequalities and fixed point problems of asymptotically strict pseudocontractive mappings in the intermediate sense. Acta Appl. Math. **115**, 167-191 (2011)
8. Sahu, DR, Wong, NC, Yao, JC: A unified hybrid iterative method for solving variational inequalities involving generalized pseudo-contractive mappings. SIAM J. Control Optim. **50**, 2335-2354 (2012)
9. Zeng, LC, Ansari, QH, Wong, NC, Yao, JC: An extragradient-like approximation method for variational inequalities and fixed point problems. Fixed Point Theory Appl. **2011**, Article ID 22 (2011). doi:10.1186/1687-1812-2011-22
10. Iiduka, H, Takahashi, W: Weak convergence theorem by Ces'aro means for nonexpansive mappings and inverse-strongly monotone mappings. J. Nonlinear Convex Anal. **7**, 105-113 (2006)

11. Ceng, LC, Wang, CY, Yao, JC: Strong convergence theorems by a relaxed extragradient method for a general system of variational inequalities. Math. Methods Oper. Res. **67**, 375-390 (2008)
12. Ceng, LC, Ansari, QH, Yao, JC: Strong and weak convergence theorems for asymptotically strict pseudocontractive mappings in intermediate sense. J. Nonlinear Convex Anal. **11**, 283-308 (2010)
13. Ceng, LC, Petruşel, A, Yao, JC: Iterative approximation of fixed points for asymptotically strict pseudocontractive type mappings in the intermediate sense. Taiwan. J. Math. **15**, 587-606 (2011)
14. Ceng, LC, Shyu, DS, Yao, JC: Relaxed composite implicit iteration process for common fixed points of a finite family of strictly pseudocontractive mappings. Fixed Point Theory Appl. **2009**, Article ID 402602 (2009). doi:10.1155/2009/402602
15. Takahashi, W: Nonlinear Functional Analysis. Yokohama Publishers, Yokohama (2000)
16. Xu, HK: An iterative approach to quadratic optimization. J. Optim. Theory Appl. **116**(3), 659-678 (2003)
17. Suzuki, T: Strong convergence of Krasnoselskii and Mann's type sequences for one-parameter nonexpansive semigroups without Bochner integrals. J. Math. Anal. Appl. **305**, 227-239 (2005)
18. Kangtunyakarn, A, Suantai, S: Strong convergence of a new iterative scheme for a finite family of strict pseudo-contractions. Comput. Math. Appl. **60**, 680-694 (2010)
19. Browder, FE: Nonlinear operators and nonlinear equations of evolution in Banach spaces. Proc. Symp. Pure Math. **18**, 78-81 (1976)

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