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Existence and blow up of solutions to a Petrovsky equation with memory and nonlinear source term

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Abstract

We consider the semilinear Petrovsky equation

$$u_{tt} + \Delta^2 u - \int_0^t g(t-s)\Delta^2 u(s)ds = |u|^p u$$

in a bounded domain and prove the existence of weak solutions. Furthermore, we show that there are solutions under some conditions on initial data which blow up in finite time with non-positive initial energy as well as positive initial energy. Estimates of the lifespan of solutions are also given.

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1 Introduction

In this article, we concerned with the problem

$$\begin{aligned} &u_{tt} + \Delta^2 u - \int_0^t g(t-s)\Delta^2 u(s)ds = |u|^p u, \qquad x \in \Omega, \tau > 0 \\ &u(x,t) = \partial_\nu u(x,t) = 0, \qquad x \in \partial\Omega, t \geq 0 \\ &u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \qquad x \in \Omega \end{aligned} \tag{1.1}$$

where $\Omega \subset R^n$ is a bounded domain with smooth boundary $\partial\Omega$ in order that the divergence theorem can be applied. ν is the unit normal vector pointing toward the exterior of Ω and $p > 0$. Here, g represents the kernel of the memory term satisfying some conditions to be specified later.

In the absence of the viscoelastic term, i.e., ($g = 0$), we motivate our article by presenting some results related to initial-boundary value Petrovsky problem

$$\begin{aligned} &u_{tt} + \Delta^2{}_u = f(u, u_t), \qquad x \in \Omega, \quad t > 0 \\ &u(x,t) = \partial_\nu u(x,t) = 0, \quad x \in \partial\Omega, t \geq 0 \\ &u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \qquad x \in \Omega. \end{aligned} \tag{1.2}$$

Research of global existence, blow-up and energy decay of solutions for the initial boundary value problem (1.2) has attracted a lot of articles (see [1-4] and references there in).

Amroun and Benaissa [1] investigated (1.2) with $f(u, u_t) = b|u|^{p-2}u - h(u_t)$ and proved the global existence of solutions by means of the stable set method in $H_0^2(\Omega)$ combined with the Faedo-Galerkin procedure. In [3], Messaoudi studied problem (1.2) with $f(u, u_t) = b|u|^{p-2}u - a|u_t|^{m-2}u_t$. He proved the existence of a local weak solution and showed that this solution blows up in finite time with negative initial energy if $p > m$.

In the presence of the viscoelastic terms, Rivera et al. [5] considered the plate model:

$$u_{tt} + \Delta^2 u - \int_0^t g(t-s)\Delta^2 u(s)ds = 0$$

in a bounded domain $\Omega \subset R^N$ and showed that the energy of solution decay exponentially provided the kernel function also decay exponentially. For more related results about the existence, finite time blow-up and asymptotic properties, we refer the reader to [5-16].

In the present article, we devote our study to problem (1.1). We will prove the existence of weak solutions under some appropriate assumptions on the function g and blow-up behavior of solutions. In order to obtain the existence of solutions, we use the Faedo-Galerkin method and to get the blow-up properties of solutions with non-positive and positive initial energy, we modify the method in [17]. Estimates for the blow-up time T^* are also given.

2 Preliminaries

We define the energy function related with problem (1.1) is given by

$$E(t) = \frac{1}{2}\left[\|u_t\|^2 + \left(1 - \int_0^t g(s)ds\right)\|\Delta u\|^2 + (g \odot \Delta u)(t)\right] - \frac{1}{p+2}\|u\|_{p+2}^{p+2}, \quad (2.1)$$

where

$$(g \odot v)(t) = \int_0^t g(t-s)\left\|v(t) - v(s)\right\|_2^2 ds.$$

We denote by $\|.\|_k$, the L^k-norm over Ω. In particular, the L^2-norm is denoted $\|.\|_2$. We use the familiar function spaces H_0^2, H^4 and throughout this article we assume $u_0 \in H_0^2(\Omega) \cap H^4(\Omega)$ and $u_1 \in H_0^2(\Omega) \cap L^2(\Omega)$.

In the sequel, we state some hypotheses and three well-known lemmas that will be needed later.

$(A1)$ p satisfies

$$0 < p \le \infty \quad (N \le 4),$$
$$0 < p \le \frac{2(N-2)}{N-4} \quad (N \ge 5).$$

$(A2)$ g is a positive bounded C^1 function satisfying $g(0) > 0$, and for all $t > 0$

$$1 - \int_0^\infty g(t)ds = l > 0,$$

also there exists positive constants L_0, L_1 such that
$(A3)$

$$-L_0 \le g'(t) \le 0, \quad 0 \le g''(t) \le L_1.$$

Lemma 1 *(Sobolev-Poincare's inequality). Let p be a number that satisfies $(A1)$, then there is a constant $C_* = C(\Omega, p)$ such that*

$$\|u\|_p \le C_* \|\Delta u\|_2, \quad u \in H_0^2(\Omega) \tag{2.2}$$

Lemma 2 [4]. *Let $\delta > 0$ and $B(t) \in C^2(0, \infty)$ be a nonnegative function satisfying*

$$B''(t) - 4(\delta + 1)B'(t) + 4(\delta + 1)B(t) \ge 0. \tag{2.3}$$

If

$$B'(0) > r_2 B(0) + K_0, \tag{2.4}$$

with $r_2 = 2(\delta + 1) - 2\sqrt{\delta(\delta + 1)}$, then $B'(t) > K_0$ for $t > 0$, where K_0 is a constant.

Lemma 3 [4]. *If $Y(t)$ is a non-increasing function on $[t_0, \infty)$ and satisfies the differential inequality*

$$Y'(t)^2 \ge a + bY(t)^{2+\delta^{-1}} \quad \text{for} \quad t \ge t_0 \ge 0, \tag{2.5}$$

where $a > 0$, $\delta > 0$ and $b \in R$, then there exists a finite time T^* such that

$$\lim_{t \to T^{*-}} Y(t) = 0.$$

Upper bounds for T^* is estimated as follows:
(i) If $b < 0$, then

$$T^* \le t_0 + \frac{1}{\sqrt{-b}} ln \frac{\sqrt{\frac{-a}{b}}}{\sqrt{\frac{-a}{b} - Y(t_0)}}.$$

(ii) If $b = 0$, then

$$T^* \le t_0 + \frac{Y(t_0)}{Y'(t_0)}.$$

(iii) If $b > 0$, then

$$T^* \le \frac{Y(t_0)}{\sqrt{a}},$$

or

$$T^* \le t_0 + 2^{\frac{3\delta+1}{2\delta}} \frac{c\delta}{\sqrt{a}} \left\{ 1 - [1 + cY(t_0)]^{\frac{-1}{2\delta}} \right\},$$

where $c = \left(\frac{a}{b}\right)^{2+\frac{1}{\delta}}$.

3 Existence of solutions

In this section, we are going to obtain the existence of weak solutions to the problem (1.1) using Faedo-Galerkin's approximation.

Theorem 1 *Let the assumptions* (*A*1)-(*A*3) *hold. Then there exists at least a solution u of (1.1) satisfying*

$$\begin{gathered} u \in L^\infty(0,\infty; H_0^2(\Omega) \cap H^4(\Omega)), \quad u' \in L^\infty(0,\infty; H_0^2(\Omega) \cap L^2(\Omega)), \\ u'' \in L^\infty(0,\infty; L^2(\Omega)) \end{gathered} \tag{3.1}$$

and

$$\begin{aligned} &u(x,t) \to u_0(x) \quad \text{in } H_0^2(\Omega) \cap H^4(\Omega) \\ &u'(x,t) \to u_1(x) \quad \text{in } H_0^2(\Omega) \cap L^2(\Omega) \end{aligned}$$

as $t \to 0$.

Proof We choose a basis $\{\omega_k\}$ (k = 1, 2, ...) in $H_0^2(\Omega) \cap H^4(\Omega)$ which is orthonormal in $L^2(\Omega)$ and ω_k being the eigenfunctions of biharmonic operator subject to the homogeneous Dirichlet boundary condition.

Let V_m be the subspace of $H_0^2(\Omega) \cap H^4(\Omega)$ generated by the first m vectors. Define

$$u_m(t) = \sum_{k=1}^{m} d_m^k(t)\omega_k, \tag{3.2}$$

where $u_m(t)$ is the solution of the following Cauchy problem

$$\begin{gathered} \left(u_m''(t), \omega_k\right) + (\Delta u_m(t), \Delta\omega_k) - \int_0^t (t-s)(\Delta u_m(s), \Delta\omega_k)ds \\ - \left(\left|u_m(t)\right|^p u_m(t), \omega_k\right) = 0 \qquad \forall k = 1, m. \end{gathered} \tag{3.3}$$

with the initial conditions (when $m \to \infty$)

$$\begin{cases} u_m(0) = \sum_{k=1}^{m} (u_m(0), \omega_k)\omega_k \to u_0 & \text{in } H_0^2(\Omega) \cap H^4(\Omega) \\ u_m'(0) = \sum_{k=1}^{m} \left(u_m'(0), \omega_k\right)\omega_k \to u_1 & \text{in } H_0^2(\Omega) \cap L^2(\Omega) \end{cases} \tag{3.4}$$

The approximate systems (3.3) and (3.4) are the normal one of differential equations which has a solution in [0, T_m) for some $T_m > 0$. The solution can be extended to the [0, T] for any given $T > 0$ by the first estimate below.

First estimation. Substituting $u_m'(t)$ instead of ω_k in (3.3), we find

$$\frac{d}{dt}\left(\frac{1}{2}\left\|u_m'\right\|^2 + \frac{1}{2}\|\Delta u_m\|^2 - \frac{\|u_m\|_{p+2}^{p+2}}{p+2}\right) - \int_0^t g(t-s)(\Delta u_m(s), \Delta u_m'(t))ds = 0. \tag{3.5}$$

Simple calculation similar to [11] yield

$$\begin{aligned}
-\int_0^t g(t-s)(\Delta u_m(s), \Delta u_m'(t))ds &= -\int_0^t g(t-s)\int_\Omega \Delta u_m(t)\Delta u_m'(t)dxds \\
&\quad -\int_0^t g(t-s)\int_\Omega (\Delta u_m(s)-\Delta u_m(t))\Delta u_m'(t)dxds \\
&= \frac{1}{2}\int_0^t g(t-s)\frac{d}{dt}\|\Delta u_m(s)-\Delta u_m(t)\|^2 ds - \frac{1}{2}\int_0^t g(t-s)\frac{d}{dt}\|\Delta u_m(t)\|^2 ds \\
&= \frac{1}{2}\frac{d}{dt}(g\odot\Delta u_m)(t) - \frac{1}{2}(g'\odot\Delta u_m)(t) - \frac{1}{2}\frac{d}{dt}\int_0^t g(s)ds\|\Delta u_m(t)\|^2 ds \\
&\quad + \frac{1}{2}g(t)\|\Delta u_m(t)\|^2.
\end{aligned} \tag{3.6}$$

Combining (3.5) and (3.6), we find

$$\begin{aligned}
\frac{d}{dt}\left(\frac{1}{2}\|u_m'\|^2 + \frac{1}{2}\left(1-\int_0^t g(s)ds\right)\|\Delta u_m\|^2 + \frac{1}{2}(g\odot\Delta u_m)(t) - \frac{\|u_m\|_{p+2}^{p+2}}{p+2}\right) \\
= \frac{1}{2}(g'\odot\Delta u_m)(t) - \frac{1}{2}g(t)\|\Delta u_m(t)\|^2,
\end{aligned} \tag{3.7}$$

integrating (3.7) over (0, t) and using assumption (A3) we infer that

$$\|u_m'\|^2 + \|\Delta u_m\|^2 + (g\odot\Delta u_m)(t) - \|u_m\|_{p+2}^{p+2} \le C_1, \tag{3.8}$$

where C_1 is a positive constant depending only on $\|u_0\|$, $\|u_1\|$, p, and l. It follows from (3.8) that

$$\begin{cases} \{u_m\} \text{ is uniformly bounded in } L^\infty(0,T;H_0^2(\Omega)) \\ \{u_m'\} \text{ is uniformly bounded in } L^\infty(0,T;L^2(\Omega)) \end{cases} \tag{3.9}$$

Second estimation. Differentiating (3.3) with respect to t, we get

$$\begin{aligned}
\left(u_m'''(t), \omega_k\right) + (\Delta u_m'(t), \Delta\omega_k) - \int_0^t g'(t-s)(\Delta u_m(s), \Delta\omega_k)ds \\
- g(0)(\Delta u_m(t), \Delta\omega_k) - (p+1)\left(|u_m(t)|^p u_m'(t), \omega_k\right) = 0.
\end{aligned} \tag{3.10}$$

If we substitute $u_m''(t)$ instead of ω_k in (3.10), it holds that

$$\begin{aligned}
\frac{d}{dt}\left(\frac{1}{2}\|u_m''\|^2 + \frac{1}{2}\|\Delta u_m'\|^2\right) - \frac{d}{dt}\int_0^t g'(t-s)\left(\Delta u_m(s), \Delta u_m'(t)\right)ds \\
+ \int_0^t g''(t-s)\left(\Delta u_m(s), \Delta u_m'(t)\right)ds + g'(0)\left(\Delta u_m(t), \Delta u_m'(t)\right) \\
- g(0)\frac{d}{dt}\left(\Delta u_m(t), \Delta u_m'(t)\right) + g(0)\left(\Delta u_m'(t), \Delta u_m'(t)\right) \\
- (p+1)\left(|u_m(t)|^p u_m'(t), u_m''(t)\right) = 0.
\end{aligned} \tag{3.11}$$

Since $H^2(\Omega) \hookrightarrow L^{2p+2}(\Omega)$, using Lemma 2, Hölder and Young's inequalities and (3.8)

$$\begin{aligned}\left|(p+1)\left(\left|u_m(t)\right|^p u_m'(t), u_m''(t)\right)\right| &\le (p+1)\left\|u_m(t)\right\|_{2p+2}^p \cdot \left\|u_m'(t)\right\|_{2p+2} \cdot \left\|u_m''(t)\right\|_2 \\ &\le C(\gamma)\left\|\Delta u_m'(t)\right\|^2 + \gamma\left\|u_m''(t)\right\|^2.\end{aligned} \tag{3.12}$$

Combining the relations (3.11), (3.12) and integrating over (0, t) for all $t \in [0, T]$ with arbitrary fixed T, we obtain

$$\begin{aligned}\frac{1}{2}\left\|u_m''\right\|^2 + \frac{1}{2}\left\|\Delta u_m'\right\|^2 \le \frac{1}{2}\left\|u_m''(0)\right\|^2 &+ \int_0^t g'(t-s)(\Delta u_m(s), \Delta u_m'(t))ds \\ &+\frac{1}{2}\left\|\Delta u_m'(0)\right\|^2 - \int_0^t \int_0^\tau g''(\tau - s)(\Delta u_m(s), \Delta u_m'(\tau))dsd\tau \\ &-g'(0)\int_0^t \left(\Delta u_m(s), \Delta u_m'(s)\right) + g(0)\left(\Delta u_m(t), \Delta u_m'(t)\right) \\ &-g(0)\left(\Delta u_m(0), \Delta u_m'(0)\right) - g(0)\int_0^t \left\|\Delta u_m'(s)\right\|^2 ds \\ &+C(\gamma)\int_0^t \left\|\Delta u_m'(s)\right\|^2 ds + \gamma\int_0^t \left\|u_m''(s)\right\|^2 ds.\end{aligned} \tag{3.13}$$

From (3.4) and (3.8), we deduce that

$$\left|\frac{1}{2}\left\|\Delta u_m'(0)\right\|^2 - g(0)(\Delta u_m(0), \Delta u_m'(0))\right| \le L_2, \tag{3.14}$$

where L_2 is a positive constant independent of m. In the following, we find the upper bound for $\left\|u_m''(0)\right\|^2$. Again we substitute $u_m''(t)$ instead of ω_k in (3.3), and choosing t = 0, we arrive at

$$\left(u_m''(0), u_m''(0)\right) + \left(\Delta u_m(0), \Delta u_m''(0)\right) - \left(\left|u_m(0)\right|^p u_m(0), u_m''(0)\right) = 0,$$

which combined with the Green's formula imply

$$\left\|u_m''(0)\right\|^2 + \left(\Delta^2 u_m(0), u_m''(0)\right) - \left(\left|u_m(0)\right|^p u_m(0), u_m''(0)\right) = 0. \tag{3.15}$$

By using (A1), (3.4) and Young's inequality, we deduce that

$$\left\|u_m''(0)\right\| \le L_3, \tag{3.16}$$

where $L_3 > 0$ is a constant independent of m.

Owing to (3.8), (3.5) and Young's inequality with (A3), we deduce that

$$\begin{aligned}\left|\int_0^t g'(t-s)(\Delta u_m(s),\Delta u'_m(t))ds\right| &= \left|\left(\Delta u'_m(t),\int_0^t g'(t-s)\Delta u_m(s)ds\right)\right|\\ &\le \gamma\left\|\Delta u'_m(t)\right\|^2+\frac{1}{4\gamma}\int_\Omega\left(\int_0^t g'(t-s)\Delta u_m(s)ds\right)^2dx\\ &\le \gamma\left\|\Delta u'_m(t)\right\|^2+\frac{L_0^2}{4\gamma}\int_0^t\left\|\Delta u_m(s)\right\|^2ds\\ &\le \gamma\left\|\Delta u'_m(t)\right\|^2+L_4(T),\end{aligned}\tag{3.17}$$

$$\begin{aligned}&\left|-\int_0^t\int_0^\tau g''(\tau-s)(\Delta u_m(s),\Delta u'_m(\tau))dsd\tau\right|\\ &\quad=\int_0^t\left(\Delta u'_m(\tau),\int_0^\tau g''(\tau-s)\Delta u_m(s)ds\right)d\tau\\ &\quad\le\frac{1}{2}\int_0^t\left\|\Delta u'_m(s)\right\|^2ds+\frac{1}{2}\int_0^t\int_\Omega\left(\int_0^\tau g''(\tau-s)\Delta u_m(s)ds\right)^2dxd\tau\\ &\quad\le\frac{1}{2}\int_0^t\left\|\Delta u'_m(s)\right\|^2ds+\frac{TL_1^2}{2}\int_0^t\left\|\Delta u_m(s)\right\|^2ds\\ &\quad\le\frac{1}{2}\int_0^t\left\|\Delta u'_m(s)\right\|^2ds+L_5(T),\end{aligned}\tag{3.18}$$

$$\left|-g'(0)\int_0^t(\Delta u_m(s),\Delta u'_m(s))ds\right|\le L_0\int_0^t\left\|\Delta u'_m(s)\right\|^2ds+L_6(T),\tag{3.19}$$

and

$$\left|g(0)(\Delta u_m(t),\Delta u'_m(t))\right|\le\gamma\left\|\Delta u'_m(t)\right\|^2+L_7(\gamma).\tag{3.20}$$

Now we choose $\gamma > 0$ small enough and combining (A3), (3.8), (3.13), (3.14), and (3.16)-(3.20), we get

$$\frac{1}{2}\left\|u''_m\right\|^2+\frac{1}{2}\left\|\Delta u'_m\right\|^2\le L_8\left(\int_0^t\left\|u''_m(s)\right\|^2ds+\int_0^t\left\|\Delta u'_m(s)\right\|^2ds\right)+L_9.\tag{3.21}$$

By using Gronwall's lemma we arrive at

$$\frac{1}{2}\left\|u''_m\right\|^2+\frac{1}{2}\left\|\Delta u'_m\right\|^2\le L_{10},\tag{3.22}$$

for all $t \in [0, T]$, and L_{10} is a positive constant independent of m. Estimate (3.22) implies

$$\begin{cases}\{u''_m\}\ \textit{is uniformly bounded in}\ L^\infty(0,T;L^2(\Omega))\\ \{u'_m\}\ \textit{is uniformly bounded in}\ L^\infty(0,T;H_0^2(\Omega))\end{cases}\tag{3.23}$$

By attention to (3.9) and (3.23), there exists a subsequence $\{u_i\}$ of $\{u_m\}$ and a function u such that

$$\begin{cases} u_i \rightharpoonup u \textit{ weakly star in } L^\infty(0,T;H_0^2(\Omega)) \\ u_i' \rightharpoonup u' \textit{ weakly star in } L^\infty(0,T;H_0^2(\Omega)) \\ u_i'' \rightharpoonup u'' \textit{weakly star in } L^\infty(0,T;L^2(\Omega)) \end{cases} \tag{3.24}$$

By Aubin-Lions compactness lemma [18], it follows from (3.24) that

$$\begin{cases} u_i \to u \textit{ strongly in } C([0,T];H_0^2(\Omega)) \\ u_i' \to u \textit{ strongly in } C([0,T]);L^2(\Omega)) \end{cases} \tag{3.25}$$

In the sequel we will deal with the nonlinear term. By (3.9) and Sobolev embedding theorem, we obtain

$$\left\{|u_m|^p u_m\right\} \textit{ is uniformly bounded in } L^\infty(0,T;L^2(\Omega)), \tag{3.26}$$

and therefore we can extract a subsequence $\{u_i\}$ of $\{u_m\}$ such that

$$|u_i|^p u_i \rightharpoonup |u|^p u \textit{ weakly star in } L^\infty(0,T;L^2(\Omega)). \tag{3.27}$$

Applying (3.24), (3.27) and letting $i \to \infty$ in (3.3), we see that u satisfies the equation. For the initial conditions by using (3.4), (3.25) and the simple inequality

$$\|u-u_0\|_{H_0^2(\Omega)} \le \|u-u_i\|_{H_0^2(\Omega)} + \left\|u_i-u_i(0)\right\|_{H_0^2(\Omega)} + \left\|u_i(0)-u_0\right\|_{H_0^2(\Omega)},$$

we get the first initial condition immediately. In the similar way, we can show the second initial condition and the proof is complete.

4 Blow-up of solutions

In this section, we study blow-up property of solutions with non-positive initial energy as well as positive initial energy, and estimate the lifespan of solutions. For this purpose, we assume that g is positive and C^1 function satisfying

($A4$)

$$g(0) > 0, \quad g'(s) \le 0, \quad 1 - \int_0^\infty g(s)ds = l > 0,$$

and we make the following extra assumption on g

($A5$)

$$\int_0^\infty g(s)ds < \frac{p}{1+p}.$$

From (2.1), (A4) and Lemma 1, we have

$$\begin{aligned} E(t) &\ge \frac{1}{2}\left[\left(1-\int_0^t g(s)ds\right)\|\Delta u\|^2 + (g \odot \Delta u)(t)\right] - \frac{1}{p+2}\|u\|_{p+2}^{p+2} \\ &\ge \frac{1}{2}\left[l\|\Delta u\|^2 + g \odot \Delta u)(t)\right] - \frac{C_1^{p+2} l^{\frac{p+2}{2}}}{p+2}\|\Delta u\|^{p+2} \\ &\ge G\left(\sqrt{l\|\Delta u\|^2 + (g \odot \Delta u)(t)}\right), \quad t \ge 0, \end{aligned}$$

where $G(\lambda) = \frac{1}{2}\lambda^2 - \frac{C_1^{p+2}}{p+2}\lambda^{p+2}, \quad C_1 = \frac{C_*}{\sqrt{l}}$. It is easy to verify that $G(\lambda)$ has a maximum at $\lambda_1 = C_1^{-\frac{p+2}{p}}$ and the maximum value is $E_1 = \frac{p}{2p+4}C_1^{-\frac{2p+4}{p}}$.

Lemma 4 *Let (A4) hold andu be a local solution of (1.1). Then E(t) is a non-increasing function on* $[0,\ T]$ *and*

$$\frac{d}{dt}E(t) = \frac{1}{2}(g' \odot \Delta u)(t) - \frac{1}{2}g(t)\|\Delta u\|^2 \leq 0, \tag{4.2}$$

for almost every $t \in\ [0,\ T]$.

Proof Multiplying (1.1) by u_t, integrating over Ω, and finally integrating by parts, we obtain (4.2) for any regular solution. Then by density arguments, we have the result.

Lemma 5 *Let (A4) hold and u be a local solution of (1.1) with initial data satisfying* $E(0) < E_1$ *and* $l^{\frac{1}{2}}\ \|\Delta u_0\| > \lambda_1$. *Then there exists* $\lambda_2 > \lambda_1$ *such that*

$$l\|\Delta u\|^2 + (g \odot \Delta u)(t) \geq \lambda_2^2. \tag{4.3}$$

Proof See Li and Tsai [11].

The choice of the functional is standard (see [19])

$$\psi(t) = \|u\|^2. \tag{4.4}$$

It is clear that

$$\psi'(t) = 2(u, u_t), \tag{4.5}$$

and from (1.1)

$$\psi''(t) = 2\|u_t\|^2 - 2\|\Delta u\|^2 + 2\ \|u\|_{p+2}^{p+2} + 2\int_0^t g(t-s)(\Delta u(t), \Delta u(s))ds. \tag{4.6}$$

Lemma 6 *Let u be a solution of (1.1) and (A4), (A5) hold, then we have*

$$\psi''(t) - (4+p)\int_\Omega u_t^2 dx \geq m(l\|\Delta u\|^2 + (g \odot \Delta u)(t)) - (4+2p)E(0), \tag{4.7}$$

where $m = (1+p) - \frac{1}{l} > 0$.

Proof Using the Hölder and Young's inequalities, we arrive at

$$\begin{aligned}\int_0^t g(t-s)(\Delta u(t), \Delta u(s))ds \geq -&\left[\frac{1}{2}(g \odot \Delta u)(t) + \frac{1}{2}\int_0^t g(s)ds\|\Delta u\|^2\right] \\ &+ \int_0^t g(s)ds\|\Delta u\|^2,\end{aligned}$$

therefore (4.6) becomes

$$\psi''(t) - (4+p)\|u_t\|^2 \geq -(2+p)\|u_t\|^2 - 2\|\Delta u\|^2 - (g \odot \Delta u)(t) + \int_0^t g(s)ds\|\Delta u\|^2 + 2\,\|u\|_{p+2}^{p+2}.$$

Then, using (4.2), we obtain

$$\psi''(t) - (4+p)\|u_t\|^2 \geq -(4+2p)E(0) + p\|\Delta u\|^2 + (1+p)(g \odot \Delta u)(t) -(1+p)\int_0^t g(s)ds\|\Delta u\|^2 - (2+p)\int_0^t (g' \odot \Delta u)(s)ds,$$

and so by (2.5) and (A5), we deduce

$$\psi''(t) - (4+p)\|u_t\|^2 \geq -(4+2p)E(0) + (p-(1+p)(1-l))\|\Delta u\|^2 +(1+p)(g \odot \Delta u)(t), \tag{4.8}$$

if we set $m := (1+p) - \frac{1}{l}$ then inequality (4.8) yields the desired result.

Consequently, we have the following result.

Lemma 7 *Assume that (A4) and (A5) hold. u be a local solution of (1.1) and that either one of the following four conditions is satisfied:*

(*i*) $E(0) < 0$
(*ii*) $E(0) = 0$ *and* $\psi'(0) > 0$
(*iii*) $0 < E(0) < \frac{m}{p}E_1$ *and* $l^{\frac{1}{2}}\,\|\Delta u_0\| > \lambda_1$
(*iv*) $\frac{m}{p}E_1 \leq E(0)$ *and* $\psi'(0) > r_2\left[\psi(0) + \frac{(4+2p)E(0)}{4+p}\right]$.

Then $\psi'(t) > 0$ *for* $t > t^*$, *where*

in case (i)

$$t^* = \max\left\{0, \frac{\psi'(0)}{(4+2p)E(0)}\right\}, \tag{4.9}$$

in cases (ii), (iv)

$$t^* = 0, \tag{4.10}$$

and in case (iii)

$$t^* = \max\left\{0, \frac{-\psi'(0)}{(4+2p)\left(\frac{m}{p}E_1 - E(0)\right)}\right\}. \tag{4.11}$$

Proof Suppose that condition (i) is satisfied. Then from (4.5), we have

$$\psi'(t) \geq \psi'(0) - (4+2p)E(0)t, \qquad t \geq 0.$$

Thus $\psi'(t) > 0$ for $t > t^*$, and it is easy to see that t^* satisfies (4.9).
If $E(0) = 0$, then by using (4.3) we have $\psi''(t) \geq 0$, and since $\psi'(0) > 0$ we arrive at

$$\psi'(t) > 0, \quad for \quad t > 0.$$

If $0 < E(0) < \frac{m}{p}E_1$ and $l^{\frac{1}{2}} \|\Delta u_0\| > \lambda_1$ then by Lemma 4, we see that

$$\begin{aligned} m(l\|\Delta u\|^2 + (g \odot \Delta u)(t)) - (4+2p)E(0) &\geq m\lambda_2^2 - (4+2p)E(0) \\ &> m\frac{4+2p}{p}E_1 - (4+2p)E(0) \\ &= (4+2p)\left[\frac{m}{p}E_1 - E(0)\right]. \end{aligned}$$

Thus from (4.5), we have

$$\begin{aligned} \psi''(t) &\geq m(l\|\Delta u\|^2 + (g \odot \Delta u)(t)) - (4+2p)E(0) \\ &> (4+2p)\left[\frac{m}{p}E_1 - E(0)\right] > 0, \end{aligned} \tag{4.12}$$

and integrating (4.12) from 0 to t gives

$$\psi'(t) > 0, \quad for \quad t \geq t^*,$$

where t^* satisfies (4.11).
Let $\frac{m}{p}E_1 \leq E(0)$, this assumption causes that

$$\psi''(t) - (4+p)\|u_t\|^2 + (4+2p)E(0) \geq 0,$$

and by using Hölder and Young's inequalities, we get

$$\|u_t\|^2 \geq \psi'(t) - \psi(t),$$

thus

$$\psi''(t) - (4+p)\psi'(t) + (4+p)\psi(t) + (4+2p)E(0) \geq 0. \tag{4.13}$$

We see that the hypotheses of Lemma 2 are fulfilled with

$$\delta = \frac{p}{4} \quad and \quad B(t) = \psi(t) + \frac{(4+2p)E(0)}{4+p}$$

and the conclusion of Lemma 2.2 gives us

$$\psi'(t) > 0, \quad for \quad t > 0.$$

Therefore the proof is complete.
To estimate the life-span of $\psi(t)$, we define the following functional

$$Y(t) = \psi(t)^{-\frac{p}{4}}, \quad for \quad t \geq 0. \tag{4.14}$$

Then we have

$$Y'(t) = \frac{p}{4} Y(t)^{1+\frac{4}{p}} \psi'(t), \tag{4.15}$$

$$Y''(t) = -\frac{p}{4} Y(t)^{1+\frac{8}{p}} \left[\psi''(t)\psi(t) - \left(1 + \frac{p}{4}\right) (\psi'(t))^2 \right]. \tag{4.16}$$

Using (4.4)-(4.6) and exploiting Holder's inequality on $\psi'(t)$, we get

$$\begin{aligned}
&\psi''(t)\psi(t) - \left(1 + \frac{p}{4}\right)(\psi'(t))^2 \\
&\quad \geq [(l\|\Delta u\|^2 + (g \odot \Delta u)(t)) - (4 + 2p)E(0) + (4 + p)\|u_t\|^2]\psi(t) \\
&\quad\quad -4\left(1 + \frac{p}{4}\right)\|u_t\|^2\psi(t) \\
&\quad = [(l\|\Delta u\|^2 + (g \odot \Delta u)(t)) - (4 + 2p)E(0)]Y(t)^{\frac{-4}{p}}.
\end{aligned}$$

Utilizing the last inequality into (4.16) yields

$$Y''(t) \leq -\frac{p}{4}[(l\|\Delta u\|^2 + (g \odot \Delta u)(t)) - (4 + 2p)E(0)]Y(t)^{1+\frac{4}{p}}. \tag{4.17}$$

Now we should assume different values for initial energy $E(0)$.

(1) At first if $E(0) \leq 0$ then from (4.17) we have

$$Y''(t) \leq \frac{p}{4}(4 + 2p)E(0)Y(t)^{1+\frac{4}{p}}, \tag{4.18}$$

on the other hand by Lemma 7, $Y'(t) < 0$ for $t > t^*$. Multiplying (4.18) by $Y'(t)$ and integrating from t^* to t, we deduce that

$$Y'(t)^2 \geq \alpha + \beta Y(t)^{2+\frac{4}{p}} \quad \textit{for} \quad t \geq t^*,$$

where

$$\alpha = \frac{p^2}{16} Y(t^*)^{2+\frac{8}{p}} \left[\psi'(t^*)^2 - 8E(0)Y(t^*)^{-\frac{4}{p}} \right] > 0, \tag{4.19}$$

and

$$\beta = \frac{p^2}{2} E(0). \tag{4.20}$$

Then the hypotheses of Lemma 3 are fulfilled with $\delta = \frac{p}{4}$, $t_0 = t^*$ and using the conclusion of Lemma 3, there exists a finite time T^* such that $\lim_{t \to T^{*-}} Y(t) = 0$, i.e., in this case some solutions blow up in finite time T^*.

(2) If $0 < E(0) < \frac{m}{p} E_1$, then from (4.17) and (4.12) we have

$$Y''(t) \leq -\frac{p}{4}(4 + 2p)\left[\frac{m}{p}E_1 - E(0)\right] Y(t)^{1+\frac{4}{p}}.$$

Then using the same arguments as in (1), we get

$$Y'(t)^2 \geq \alpha_1 + \beta_1 Y(t)^{2+\frac{4}{p}} \quad \textit{for} \quad t \geq t^*,$$

where

$$\alpha_1 = \frac{p^2}{16} Y(t^*)^{2+\frac{8}{p}} (\psi'(t^*)^2 + 8\left[\frac{m}{p}E_1 - E(0)\right] Y(t^*)^{-\frac{4}{p}}) > 0, \tag{4.21}$$

and

$$\beta_1 = \frac{p^2}{2}\left[E(0) - \frac{m}{p}E_1\right]. \tag{4.22}$$

Thus by Lemma 3, there exists a finite time T^* such that

$$\lim_{t \to T*-} \psi(t) = \infty.$$

(3) $\frac{m}{p}E_1 \le E(0)$. In this case, it is easy to see that by using (4.19) and (4.20) into discussion in part (1), we obtain

$$\alpha > 0 \quad \textit{if and only if} \quad E(0) < \frac{\psi'(t^*)^2}{8\psi(t^*)}.$$

Hence, Lemma 3 yields the blow-up property in this case.

Therefore, we proved the following theorem.

Theorem 2 *Assume that (A4) and (A5) hold. u be a local solution of (1.1) and that either one of the following four conditions is satisfied:*

(i) $E(0) < 0$
(ii) $E(0) = 0$ *and* $\psi'(0) > 0$
(iii) $0 < E(0) < \frac{m}{p}E_1$*and* $l^{\frac{1}{2}} \, \|\Delta u_0\| > \lambda_1$
(iv) $\frac{m}{p}E_1 \le E(0)$*and* $\psi'(0) > r_2\left[\psi(0) + \frac{(4+2p)E(0)}{4+p}\right]$*holds.*

Then the solution u blows up at finite time T^*. Moreover, the upper bounds for T^* can be estimated according to the sign of $E(0)$:

in case (i)

$$T^* \le t^* - \frac{Y(t^*)}{Y'(t^*)}.$$

Furthermore, if $Y(t^*) < \min\{1, \sqrt{\frac{\alpha}{-\beta}}\}$, *then*

$$T^* \le t^* + \frac{1}{\sqrt{-\beta}} ln \frac{\sqrt{\frac{\alpha}{-\beta}}}{\sqrt{\frac{\alpha}{-\beta}} - Y(t^*)}$$

in cases (ii)

$$T^* \le t^* - \frac{Y(t^*)}{Y'(t^*)} \quad or \quad T^* \le t^* + \frac{Y(t^*)}{\sqrt{\alpha}}$$

in case (iii)

$$T^* \leq t^* - \frac{Y(t^*)}{Y'(t^*)}.$$

Furthermore, if $Y(t^*) < \min\{1, \sqrt{\frac{\alpha}{-\beta}}\}$, *then*

$$T^* \leq t^* + \frac{1}{\sqrt{-\beta_1}} ln \frac{\sqrt{\frac{\alpha_1}{-\beta_1}}}{\sqrt{\frac{\alpha_1}{-\beta_1}} - Y(t^*)}$$

and in case (iv)

$$T^* \leq \frac{Y(t^*)}{\sqrt{\alpha}} \quad or \quad T^* \leq t^* + 2^{\frac{3p+4}{2p}} \frac{pc}{4\sqrt{\alpha}} \left[1 - \left(1 + cY(t^*)\right)^{\frac{-2}{p}}\right],$$

where $d = \left(\frac{\beta}{\alpha}\right)^{\frac{p}{p+8}}$. *Here* α, β, α_1, *and* β_1 *are given in (4.19)-(4.22), respectively. Note that each* t^* *in the above cases satisfy the same case in Lemma 7.*

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Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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References

1. Bayrak, V, Can, M, Aliyev, FA: Nonexistence of global solutions of a quasilinear hyperbolic equations. Math Inequal Appl. **1**, 367–374 (1998)
2. Kalantarov, VK, Ladyzhenskaya, OA: Formation of collapses in quasilinear equations of parabolic and hyperbolic types. zap Nauchn Semin LOMI. **61**, 77–102 (1977)
3. Messaoudi, SA: Blow-up of positive-initial-energy solutions of a nonlinear viscoelastic hyperbolic equation. J Math Anal Appl. **320**, 902–915 (2006). doi:10.1016/j.jmaa.2005.07.022
4. Wu, ST: Energy decay rates via convexity for some second-order evolution equation with memory and nonlinear time-dependent dissipation. Nonlinear Anal. **74**, 532–543 (2011). doi:10.1016/j.na.2010.09.007
5. Munoz Rivera, JE, Lapa, EC, Baretto, R: Decay rates for viscoelastic plates with memory. J Elasticity. **44**, 61–87 (1996). doi:10.1007/BF00042192
6. Amroun, NE, Benaissa, A: Global existence and energy decay of solutions to a Petrovsky equation with general nonlinear dissipation and source term. Georg Math J. **13**, 397–410 (2006)
7. Andrade, D, Fatori, LH, Rivera, JM: Nonlinear transmission problem with a dissipative boundary condition of memory type. Electron J Diff Equ. **53**, 1–16 (2006)
8. Berrimi, S, Messaoudi, SA: Existence and decay of solutions of a viscoelastic equation with a nonlinear source. Nonlinear Anal. **64**, 2314–2331 (2006). doi:10.1016/j.na.2005.08.015
9. Cavalcanti, MM, Domingos Cavalcanti, VN, Ma, TF, Soriano, JA: Global existence and asymptotic stability for viscoelastic problems. Diff Integ Equ. **15**, 731–748 (2002)
10. Chen, CS, Ren, L: Weak solution for a fourth order nonlinear wave equation. J Southeast Univ (English Ed). **21**, 369–374 (2005)
11. Li, MR, Tsai, LY: Existence and nonexistence of global solutions of some systems of semilinear wave equations. Nonlinear Anal. **54**, 1397–1415 (2003). doi:10.1016/S0362-546X(03)00192-5
12. Messaoudi, SA: Global existence and nonexistence in a system of Petrovsky. J Math Anal Appl. **265**, 296–308 (2002). doi:10.1006/jmaa.2001.7697

13. Messaoudi, SA, Tatar, N: Exponential and polynomial decay for a quasilinear viscoelastic equation. Nonlinear Anal. **68**, 785–793 (2008). doi:10.1016/j.na.2006.11.036
14. Wang, Y: A global nonexistence theorem for viscoelastic equations with arbitrary positive initial energy. Appl Math Lett. **22**, 1394–1400 (2009). doi:10.1016/j.aml.2009.01.052
15. Han, X, Wang, M: Global existence and uniform decay for a nonlinear viscoelastic equation with damping. Nonlinear Anal. **70**, 3090–3098 (2009). doi:10.1016/j.na.2008.04.011
16. Messaoudi, SA, Tatar, N: Global existence and uniform decay of solutions for a quasilinear viscoelastic problem. Math Methods Appl Sci. **30**, 665–680 (2007). doi:10.1002/mma.804
17. Li, G, Sun, Y, Liu, W: Global existence, uniform decay and blow-up of solutions for a system of Petrovsky equations. Nonlinear Anal. **74**, 1523–1538 (2011). doi:10.1016/j.na.2010.10.025
18. Adams, RA, Fournier, JJF: Sobolev Spaces, of Pure and Applied Mathematics (Amsterdam). Elsevier/Academic Press, Amsterdam, 2**140** (2003)
19. Wu, ST, Tsai, LY: On global solutions and blow-up of solutions for a nonlinearly damped Petrovsky system. Taiwanese J Math. **13**, 545–558 (2009)

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