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Existence of positive solutions for singular fourth-order three-point boundary value problems

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Abstract

In this article, we consider the boundary value problem $u^{(4)}(t) + f(t, u(t)) = 0$, $0 < t < 1$, subject to the boundary conditions $u(0) = u'(0) = u''(0) = 0$ and $u''(1) - \alpha u''(\eta) = \lambda$. In this setting, $0 < \eta < 1$ and $\alpha \in [0, \frac{1}{\eta})$ are constants and $\lambda \in [0, +\infty)$ is a parameter. By imposing a sufficient structure on the nonlinearity $f(t, u)$, we deduce the existence of at least one positive solution to the problem. The novelty in our setting lies in the fact that $f(t, u)$ may be singular at $t = 0$ and $t = 1$. Our results here are achieved by making use of the Krasnosel'skii fixed point theorem. We conclude with examples illustrating our results and the improvements that they present.

MSC: 34B15; 34B25; 34B18

Keywords: cone; positive solution; existence; boundary value problems

1 Introduction

In this paper, we consider the following nonlinear singular fourth-order three-point boundary value problem:

$$\begin{cases} u^{(4)}(t) + f(t, u(t)) = 0, & 0 < t < 1, \\ u(0) = u'(0) = u''(0) = 0, \\ u''(1) - \alpha u''(\eta) = \lambda, \end{cases} \tag{1.1}$$

where $0 < \eta < 1$, $\alpha \in [0, \frac{1}{\eta})$ are constants, $\lambda \in [0, +\infty)$ is a parameter, $f(t, u(t))$ may be singular at $t = 0$ and/or $t = 1$. Here, by a positive solution we mean a function $u^*(t)$ which is positive on $(0, 1)$ and satisfies problem (1.1).

The theory of boundary value problems for ordinary differential equations arises in different areas of applied mathematics, physics and so on. The existence of positive solutions for boundary value problems has become an important area of investigation and received a great deal of attention in recent years (see [1–20] and the references cited therein). In [14], by making use of the fixed point theorem and degree theory, Bai and Wang proved the existence, uniqueness and multiplicity of positive solutions for the following fourth-order two-point boundary value problem:

$$\begin{cases} u^{(4)}(t) - \lambda f(t, u(t)) = 0, & 0 < t < 1, \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases}$$

In [8], Yao studied the following nonlinear fourth-order ordinary differential equation:

$$\begin{cases} u^{(4)}(t) = f(t, u(t), u'(t)), & t \in [0,1]\backslash E, \\ u(0) = u'(0) = u''(1) = u'''(1) = 0, \end{cases}$$

where $E \subset [0,1]$ is a closed set with measure zero and the nonlinear term $f(t,x,y)$ may be singular for $t \in E$. The author showed the existence of n positive solutions by constructing a suitable integral equation and applying fixed point theorems on a cone.

In [9], Sun considered the following third-order boundary value problems:

$$\begin{cases} u'''(t) + a(t)f(u(t)) = 0, & 0 < t < 1, \\ u(0) = u'(0) = 0, \\ u'(1) - \alpha u'(\eta) = \lambda. \end{cases}$$

The author obtained the existence and nonexistence of positive solutions by applying the Guo-Krasnosel'skii fixed point theorem and Schauder's fixed point theorem.

In [10], Zhang and Wang studied the following nonlinear singular fourth-order boundary value problem:

$$\begin{cases} u^{(4)}(t) = f(t, u(t)), & 0 < t < 1, \\ u(0) = u(1) = u'(0) = u'(1) = 0, \end{cases}$$

where the nonlinear term $f(t,u)$ may be singular at $t = 0$, $t = 1$ and $u = 0$. The author presented the existence of a positive solution by using the fixed point index theorem and the properties of Green's function.

In [15], by applying the Krasnosel'skii fixed point theorem, Graef, Qian and Yang established the existence and nonexistence of positive solutions for the following fourth-order three-point boundary value problem:

$$\begin{cases} u^{(4)}(t) = \lambda g(t)f(u(t)), & 0 < t < 1, \\ u(0) = u'(1) = u''(0) = u''(p) - u''(1) = 0, \end{cases}$$

where $p \in (0,1)$ is a constant.

Inspired and motivated by the works mentioned above, we deal with the existence and nonexistence of positive solutions to problem (1.1) by making use of the fixed point theorem together with the properties of Green's function. The main features of the paper are as follows. Firstly, we apply the Taylor expansion formula to prove a lemma, and then we give a comparison lemma and construct a special cone. Secondly, we present the existence of positive solutions for problem (1.1). To our best knowledge, no paper has considered problem (1.1). The arguments are based upon the fixed point theorem for the special cone.

The paper is organized as follows. In Section 2, we give some properties of Green's function associated with problem (1.1) and construct a suitable cone and transform problem (1.1) into an integral equation. In Section 3, we discuss the existence of at least one positive solution for problem (1.1).

2 Preliminary lemmas

Let $E = C[0,1]$ be a Banach space of all continuous functions with the norm $\|u\| = \max_{0\leq t\leq 1}|u(t)|$, $C^+[0,1] = \{u \in C[0,1] : u(t) > 0, t \in [0,1]\}$.

Throughout the paper, we assume that

(H_1) $f : (0,1) \times [0,+\infty) \to [0,+\infty)$ is continuous.

(H_2) There exists a continuous function $q : (0,1) \to [0,+\infty)$ such that

$$0 < \int_a^b s(1-s)q(s)\,ds \leq \int_0^1 s(1-s)q(s)\,ds < +\infty \quad \text{for } [a,b] \subset (0,1).$$

(H_3) There exists a continuous function $g : [0,1] \times [0,+\infty) \to [0,+\infty)$ such that

$$f(t,u) \leq q(t)g(t,u), \quad (t,u) \in (0,1) \times [0,+\infty).$$

Lemma 2.1 *Suppose that $p(t) \in L^1(0,1)$ and $p(t) > 0$. Then the linear boundary value problem*

$$\begin{cases} u^{(4)}(t) + p(t) = 0, \\ u(0) = u'(0) = u''(0) = 0, \\ u''(1) - \alpha u''(\eta) = \lambda \end{cases} \tag{2.1}$$

has a unique positive solution, which can be expressed by

$$u(t) = \int_0^1 G(t,s)p(s)\,ds + \frac{\alpha t^3}{6(1-\alpha\eta)}\int_0^1 K(\eta,s)p(s)\,ds + \frac{\lambda t^3}{6(1-\alpha\eta)},$$

where

$$G(t,s) = \begin{cases} \frac{1}{6}t^3(1-s) - \frac{1}{6}(t-s)^3, & 0 \leq s \leq t \leq 1, \\ \frac{1}{6}t^3(1-s), & 0 \leq t \leq s \leq 1, \end{cases}$$

and

$$\frac{\partial}{\partial t}G(t,s) = \begin{cases} \frac{1}{2}t^2(1-s) - \frac{1}{2}(t-s)^2, & 0 \leq s \leq t \leq 1, \\ \frac{1}{2}t^2(1-s), & 0 \leq t \leq s \leq 1, \end{cases}$$

and

$$K(t,s) = \frac{\partial^2}{\partial t^2}G(t,s) = \begin{cases} s(1-t), & 0 \leq s \leq t \leq 1, \\ t(1-s), & 0 \leq t \leq s \leq 1. \end{cases}$$

Proof In fact, if $u(t)$ is a solution of problem (2.1), by the Taylor expansion formula, we have

$$u(t) = a_0 + a_1 t + \frac{a_2}{2!}t^2 + \frac{a_3}{3!}t^3 - \frac{1}{6}\int_0^t (t-s)^3 p(s)\,ds,$$

then

$$u'(t) = a_1 + a_2 t + \frac{a_3}{2}t^2 - \frac{1}{2}\int_0^t (t-s)^2 p(s)\,ds,$$
$$u''(t) = a_2 + a_3 t - \int_0^t (t-s)p(s)\,ds,$$

which together with the boundary condition implies $a_0 = a_1 = a_2 = 0$ and

$$a_3 = \frac{1}{1-\alpha\eta}\int_0^1 (1-s)p(s)\,ds + \frac{\alpha}{1-\alpha\eta}\int_0^\eta (s-\eta)p(s)\,ds + \frac{\lambda}{1-\alpha\eta}.$$

Therefore

$$\begin{aligned} u(t) &= -\frac{1}{6}\int_0^t (t-s)^3 p(s)\,ds + \frac{t^3}{6(1-\alpha\eta)}\int_0^1 (1-s)p(s)\,ds \\ &\quad + \frac{\alpha t^3}{6(1-\alpha\eta)}\int_0^\eta (s-\eta)p(s)\,ds + \frac{\lambda t^3}{6(1-\alpha\eta)} \\ &= \frac{1}{6}\int_0^t \big[t^3(1-s) - (t-s)^3\big]p(s)\,ds + \frac{1}{6}\int_t^1 t^3(1-s)p(s)\,ds \\ &\quad + \frac{\alpha t^3}{6(1-\alpha\eta)}\left[\int_0^\eta s(1-\eta)p(s)\,ds + \int_\eta^1 \eta(1-s)p(s)\,ds\right] + \frac{\lambda t^3}{6(1-\alpha\eta)} \\ &= \int_0^1 G(t,s)p(s)\,ds + \frac{\alpha t^3}{6(1-\alpha\eta)}\int_0^1 K(\eta,s)p(s)\,ds + \frac{\lambda t^3}{6(1-\alpha\eta)}. \end{aligned}$$

The proof is complete. □

Lemma 2.2 *For all* $(t,s) \in [0,1] \times [0,1]$, *we have*

$$\frac{1}{6}t^3 s(1-s) \le G(t,s) \le s(1-s).$$

Proof If $0 \le t \le s \le 1$, then

$$G(t,s) = \frac{1}{6}t^3(1-s) \le \frac{1}{6}s^3(1-s) \le s(1-s),$$

and

$$G(t,s) = \frac{1}{6}t^3(1-s) \ge \frac{1}{6}t^3 s(1-s).$$

If $0 \le s \le t \le 1$, then

$$\begin{aligned} G(t,s) &= \frac{1}{6}t^3(1-s) - \frac{1}{6}(t-s)^3 \\ &\le \frac{1}{6}s\big[t^2 - t^3 + 3t(t-s)\big] \\ &\le \frac{1}{6}s\big[t^2(1-s) + 3t(1-s)\big] \le s(1-s), \end{aligned}$$

and

$$\begin{aligned} G(t,s) &= \frac{1}{6}t^3(1-s) - \frac{1}{6}(t-s)^3 \\ &\geq \frac{1}{6}t^3 s(1-s) + \frac{1}{6}t^3(1-s)^3 - \frac{1}{6}(t-s)^3 \\ &\geq \frac{1}{6}t^3 s(1-s) + \frac{1}{6}s(1-t)\big[t^2(1-s)^2 + t(1-s)(t-s) + (t-s)^2\big] \\ &\geq \frac{1}{6}t^3 s(1-s). \end{aligned}$$

Therefore

$$\frac{1}{6}t^3 s(1-s) \leq G(t,s) \leq s(1-s). \qquad \square$$

Define a cone $K \subset C[0,1]$ by

$$K = \left\{ u(t) \in C^+[0,1] : u(t) \geq \frac{1}{6}t^3\|u\|, 0 \leq t \leq 1 \right\},$$

then K is a positive cone in $C[0,1]$. Denote

$$\Omega_r = \{u \in K : \|u\| < r\}, \qquad \partial\Omega_r = \{u \in K : \|u\| = r\}.$$

Fix $R > r > 0$. Define an operator $A : (\overline{\Omega}_R \backslash \Omega_r) \cap K \to K$ by

$$Au(t) = \int_0^1 G(t,s)f\big(s,u(s)\big)\,ds + \frac{\alpha t^3}{6(1-\alpha\eta)}\int_0^1 K(\eta,s)f\big(s,u(s)\big)\,ds + \frac{\lambda t^3}{6(1-\alpha\eta)}.$$

It is well known that problem (1.1) has a positive solution $u = u(t)$ if and only if u is a fixed point of A.

Lemma 2.3 *Suppose that* $(\mathrm{H}_1)\sim(\mathrm{H}_3)$ *hold. Then* $A(K) \subseteq K$.

Proof From (H_2) and (H_3), we know that

$$\begin{aligned} 0 &\leq (Au)(t) \\ &\leq \int_0^1 s(1-s)f\big(s,u(s)\big)\,ds + \frac{\alpha}{6(1-\alpha\eta)}\int_0^1 K(\eta,s)f\big(s,u(s)\big)\,ds + \frac{\lambda}{6(1-\alpha\eta)} \\ &\leq \int_0^1 s(1-s)q(s)g\big(s,u(s)\big)\,ds + \frac{\alpha}{6(1-\alpha\eta)}\int_0^1 K(\eta,s)q(s)g\big(s,u(s)\big)\,ds + \frac{\lambda}{6(1-\alpha\eta)} \\ &< +\infty. \end{aligned}$$

On the other hand, for any $u \in K$, we have $u(t) \geq \frac{1}{6}t^3\|u\|$, $t \in [0,1]$, and

$$\|Au\| \leq \int_0^1 s(1-s)f\big(s,u(s)\big)\,ds + \frac{\alpha}{1-\alpha\eta}\int_0^1 K(\eta,s)f\big(s,u(s)\big)\,ds + \frac{\lambda}{1-\alpha\eta}.$$

Therefore

$$
\begin{aligned}
(Au)(t) &\geq \frac{1}{6}t^3\left[\int_0^1 s(1-s)f\bigl(s,u(s)\bigr)\,ds + \frac{\alpha}{1-\alpha\eta}\int_0^1 K(\eta,s)f\bigl(s,u(s)\bigr)\,ds + \frac{\lambda}{1-\alpha\eta}\right] \\
&\geq \frac{1}{6}t^3\|Au\|.
\end{aligned}
$$

The proof is complete. □

Lemma 2.4 *Suppose that* $(\mathrm{H}_1)\sim(\mathrm{H}_3)$ *hold. Then* $A:(\overline{\Omega}_R\backslash\Omega_r)\cap K\to K$ *is completely continuous.*

Proof For any $u\in(\overline{\Omega}_R\backslash\Omega_r)\cap K$, we have $0\leq\frac{1}{6}t^3r\leq\frac{1}{6}t^3\|u\|\leq u(t)\leq R$.

Let

$$
q_n(t)=\begin{cases}\inf_{t\leq s\leq\frac{1}{n}}q(s), & 0\leq t\leq\frac{1}{n},\\ q(t), & \frac{1}{n}\leq t\leq\frac{n-1}{n},\\ \inf_{\frac{n-1}{n}\leq s\leq t}q(s), & \frac{n-1}{n}\leq t\leq 1.\end{cases}
$$

Then, from (H_2) and (H_3), we have $\lim_{n\to\infty}\int_0^1(q(t)-q_n(t))\,dt=0$ for $0\leq t\leq 1$ and $u\in K$. Let

$$
f_n(t,u)=\begin{cases}f(t,u), & f(t,u)\leq q_n(t)g(t,u),\\ q_n(t)g(t,u), & f(t,u)>q_n(t)g(t,u).\end{cases}
$$

It is easy to see that f_n is a continuous function on $[0,1]\times[0,+\infty)$ and f_n is bounded on any bounded set. Define

$$
(A_nu)(t)=\int_0^1 G(t,s)f_n\bigl(s,u(s)\bigr)\,ds+\frac{\alpha t^3}{6(1-\alpha\eta)}\int_0^1 K(\eta,s)f_n\bigl(s,u(s)\bigr)\,ds+\frac{\lambda t^3}{6(1-\alpha\eta)}.
$$

By the Arzela-Ascoli theorem, we know that $A_n:\overline{\Omega}_R\backslash\Omega_r\to C[0,1]$ is completely continuous.

Let $M(R)=\max\{g(t,u):(t,u)\in[0,1]\times[0,R]\}$. For $u\in(\overline{\Omega}_R\backslash\Omega_r)\cap K$, we know that

$$
\begin{aligned}
\|Au-A_nu\| &= \max_{0\leq t\leq 1}\left\{\int_0^1 G(t,s)\bigl[f\bigl(s,u(s)\bigr)-f_n\bigl(s,u(s)\bigr)\bigr]\,ds\right\} \\
&\quad+\max_{0\leq t\leq 1}\left\{\frac{\alpha t^3}{6(1-\alpha\eta)}\int_0^1 K(\eta,s)\bigl[f\bigl(s,u(s)\bigr)-f_n\bigl(s,u(s)\bigr)\bigr]\,ds\right\} \\
&\leq \max_{0\leq t\leq 1}\left\{\int_0^1 G(t,s)\bigl[q(s)g\bigl(s,u(s)\bigr)-q_n(s)g\bigl(s,u(s)\bigr)\bigr]\,ds\right\} \\
&\quad+\max_{0\leq t\leq 1}\left\{\frac{\alpha t^3}{6(1-\alpha\eta)}\int_0^1 K(\eta,s)\bigl[q(s)g\bigl(s,u(s)\bigr)-q_n(s)g\bigl(s,u(s)\bigr)\bigr]\,ds\right\} \\
&\leq M(R)\max_{0\leq t\leq 1}\left\{\int_0^1 G(t,s)\bigl[q(s)-q_n(s)\bigr]\,ds\right\}
\end{aligned}
$$

$$\begin{aligned}&+ M(R)\max_{0\le t\le 1}\left\{\frac{\alpha t^3}{6(1-\alpha\eta)}\int_0^1 K(\eta,s)\bigl[q(s)-q_n(s)\bigr]\,ds\right\}\\ &\to 0 \quad \text{as } n\to\infty.\end{aligned}$$

It shows that a completely continuous operator A_n converges to an operator A uniformly on $(\overline{\Omega}_R\backslash\Omega_r)\cap K$. Hence A is continuous.

Suppose that $D\subset K$ is a bounded set, then there exists $d>0$ such that $\|u\|\le d$ for any $u\in D$. From (H_3), we know that $|f(t,u)|\le q(t)g(t,u)\le M(d)q(t)$ for $(t,u)\in(0,1)\times[0,d]$. Then we have

$$\begin{aligned}\|Au\| &\le \int_0^1 s(1-s)f\bigl(s,u(s)\bigr)\,ds+\frac{\alpha}{6(1-\alpha\eta)}\int_0^1 K(\eta,s)f\bigl(s,u(s)\bigr)\,ds+\frac{\lambda}{6(1-\alpha\eta)}\\ &\le \int_0^1 s(1-s)q(s)g\bigl(s,u(s)\bigr)\,ds+\frac{\alpha}{6(1-\alpha\eta)}\int_0^1 K(\eta,s)q(s)g\bigl(s,u(s)\bigr)\,ds+\frac{\lambda}{6(1-\alpha\eta)}\\ &\le M(d)\int_0^1 s(1-s)q(s)\,ds+M(d)\frac{\alpha}{6(1-\alpha\eta)}\int_0^1 K(\eta,s)q(s)\,ds+\frac{\lambda}{6(1-\alpha\eta)}\\ &< +\infty.\end{aligned}$$

Hence A is uniformly bounded.

On the other hand, for any $u\in D$, we know that

$$\begin{aligned}\bigl|(Au)'(t)\bigr| &\le \frac{1}{2}\int_0^t s(1-s)f\bigl(s,u(s)\bigr)\,ds+\frac{1}{2}\int_t^1 s(1-s)f\bigl(s,u(s)\bigr)\,ds\\ &\quad+\frac{\alpha t^2}{2(1-\alpha\eta)}\int_0^1 K(\eta,s)f\bigl(s,u(s)\bigr)\,ds+\frac{\lambda t^2}{2(1-\alpha\eta)}\\ &\le \frac{1}{2}\int_0^t s(1-s)q(s)g\bigl(s,u(s)\bigr)\,ds+\frac{1}{2}\int_t^1 s(1-s)q(s)g\bigl(s,u(s)\bigr)\,ds\\ &\quad+\frac{\alpha t^2}{2(1-\alpha\eta)}\int_0^1 K(\eta,s)q(s)g\bigl(s,u(s)\bigr)\,ds+\frac{\lambda t^2}{2(1-\alpha\eta)}\\ &\le \frac{1}{2}M(d)\int_0^t s(1-s)q(s)\,ds+\frac{1}{2}M(d)\int_t^1 s(1-s)q(s)\,ds\\ &\quad+\frac{\alpha t^2}{2(1-\alpha\eta)}M(d)\int_0^1 K(\eta,s)q(s)\,ds+\frac{\lambda t^2}{2(1-\alpha\eta)}.\end{aligned}$$

Let

$$\begin{aligned}&\varphi(t)=\frac{1}{2}M(d)\int_0^1 s(1-s)q(s)\,ds+\frac{\alpha t^2}{2(1-\alpha\eta)}M(d)\int_0^1 K(\eta,s)q(s)\,ds+\frac{\lambda t^2}{2(1-\alpha\eta)},\\ &\int_0^1\bigl|\varphi(t)\bigr|\,dt=M(d)\int_0^1 dt\int_0^t\frac{1}{2}s(1-s)q(s)\,ds+M(d)\int_0^1 dt\int_t^1\frac{1}{2}s(1-s)q(s)\,ds\\ &\qquad+M(d)\int_0^1\frac{\alpha t^2}{2(1-\alpha\eta)}\,dt\int_0^1 K(\eta,s)q(s)\,ds+\int_0^1\frac{\lambda t^2}{2(1-\alpha\eta)}\,dt\\ &\quad=M(d)\int_0^1\frac{1}{2}s(1-s)^2q(s)\,ds+M(d)\int_0^1\frac{1}{2}s^2(1-s)q(s)\,ds\\ &\qquad+\frac{\alpha}{6(1-\alpha\eta)}M(d)\int_0^1 K(\eta,s)q(s)\,ds+\frac{\lambda}{6(1-\alpha\eta)}\end{aligned}$$

$$= \frac{1}{2}M(d)\int_0^1 s(1-s)q(s)\,ds + \frac{\alpha}{6(1-\alpha\eta)}M(d)\int_0^1 K(\eta,s)q(s)\,ds + \frac{\lambda}{6(1-\alpha\eta)}$$

$$< +\infty.$$

Therefore A is equicontinuous. Consequently, A is completely continuous. □

Lemma 2.5 [21, 22] *Let E be a Banach space, and let $P \subset E$ be a cone in E. Assume that Ω_1 and Ω_2 are open subsets of E with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Let $T : P \cap (\overline{\Omega}_2 \backslash \Omega_1) \to P$ be a completely continuous operator such that either*

(i) $\|Tu\| \le \|u\|$, $u \in P \cap \partial\Omega_1$ *and* $\|Tu\| \ge \|u\|$, $u \in P \cap \partial\Omega_2$, *or*

(ii) $\|Tu\| \ge \|u\|$, $u \in P \cap \partial\Omega_1$ *and* $\|Tu\| \le \|u\|$, $u \in P \cap \partial\Omega_2$.

Then T has a fixed point in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

3 Main results

Theorem 3.1 *Suppose that* (H_1), (H_2) *and* (H_3) *hold. In addition, assume that the following conditions hold*:

(H_4) $\lim_{u\to 0^+} \sup \max_{0\le t\le 1} \frac{g(t,u)}{u} = 0$;

(H_5) $\lim_{u\to +\infty} \inf \min_{a\le t\le b} \frac{f(t,u)}{u} = +\infty$.

Then problem (1.1) *has at least one positive solution for λ small enough, and problem* (1.1) *has no positive solution for λ large enough.*

Proof For $\lambda > 0$ small enough, let

$$N = \frac{5}{6}\left[\int_0^1 s(1-s)q(s)\,ds + \frac{\alpha}{6(1-\alpha\eta)}\int_0^1 K(\eta,s)q(s)\,ds\right]^{-1}. \tag{3.1}$$

From (H_4), there exists a constant $R_1 > 0$ such that $g(t,u) \le Nu$ for $(t,u) \in [0,1] \times (0,R_1]$.

Let $\Omega_1 = \{u \in K : \|u\| < R_1\}$, $0 < \lambda \le (1-\alpha\eta)R_1$. For any $u \in K \cap \partial\Omega_1$, we get

$$\begin{aligned}
Au(t) &= \int_0^1 G(t,s)f\big(s,u(s)\big)\,ds + \frac{\alpha t^3}{6(1-\alpha\eta)}\int_0^1 K(\eta,s)f\big(s,u(s)\big)\,ds + \frac{\lambda t^3}{6(1-\alpha\eta)} \\
&\le \int_0^1 s(1-s)f\big(s,u(s)\big)\,ds + \frac{\alpha}{6(1-\alpha\eta)}\int_0^1 K(\eta,s)f\big(s,u(s)\big)\,ds + \frac{\lambda}{6(1-\alpha\eta)} \\
&\le \int_0^1 s(1-s)q(s)g\big(s,u(s)\big)\,ds + \frac{\alpha}{6(1-\alpha\eta)}\int_0^1 K(\eta,s)q(s)g\big(s,u(s)\big)\,ds + \frac{\lambda}{6(1-\alpha\eta)} \\
&\le N\int_0^1 s(1-s)q(s)u(s)\,ds + \frac{N\alpha}{6(1-\alpha\eta)}\int_0^1 K(\eta,s)q(s)u(s)\,ds + \frac{R_1(1-\alpha\eta)}{6(1-\alpha\eta)} \\
&\le N\|u\|\int_0^1 s(1-s)q(s)\,ds + \frac{N\alpha\|u\|}{6(1-\alpha\eta)}\int_0^1 K(\eta,s)q(s)\,ds + \frac{R_1}{6} \\
&= \frac{5}{6}R_1 + \frac{1}{6}R_1 = R_1 = \|u\|.
\end{aligned}$$

Therefore

$$\|Au\| \le \|u\|, \quad u \in K \cap \partial\Omega_1.$$

On the other hand, let

$$M = \left[\int_a^b \frac{1}{6}a^3 s(1-s)\,ds + \frac{\alpha a^3}{6(1-\alpha\eta)}\int_a^b K(\eta,s)\,ds\right]^{-1}. \tag{3.2}$$

From (H_5), there exists $R > 0$ such that $f(t,u) \geq 48Mu$ for $(t,u) \in [a,b] \times [R,+\infty)$. Let $R_2 > \frac{6R}{a^3} > R_1$, and let $\Omega_2 = \{u \in K : \|u\| < R_2\}$. For any $u \in K \cap \partial\Omega_2$ and $t \in [a,b]$, we know that $u(t) \geq \frac{1}{6}t^3\|u\| = \frac{1}{6}a^3 R_2 > R$ and

$$\begin{aligned}
(Au)\left(\frac{1}{2}\right) &= \int_0^1 G\left(\frac{1}{2},s\right)f\bigl(s,u(s)\bigr)\,ds \\
&\quad + \frac{1}{8}\cdot\frac{\alpha}{6(1-\alpha\eta)}\int_0^1 K(\eta,s)f\bigl(s,u(s)\bigr)\,ds + \frac{1}{8}\cdot\frac{\lambda}{6(1-\alpha\eta)} \\
&\geq \frac{1}{8}\int_0^1 \frac{1}{6}s(1-s)f\bigl(s,u(s)\bigr)\,ds + \frac{\alpha}{48(1-\alpha\eta)}\int_0^1 K(\eta,s)f\bigl(s,u(s)\bigr)\,ds \\
&\geq \frac{1}{8}\int_a^b \frac{1}{6}s(1-s)f\bigl(s,u(s)\bigr)\,ds + \frac{\alpha}{48(1-\alpha\eta)}\int_a^b K(\eta,s)f\bigl(s,u(s)\bigr)\,ds \\
&\geq M\int_a^b s(1-s)u(s)\,ds + \frac{\alpha M}{1-\alpha\eta}\int_a^b K(\eta,s)u(s)\,ds \\
&\geq M\int_a^b \frac{1}{6}a^3 s(1-s)u(s)\,ds + \frac{\alpha M a^3}{6(1-\alpha\eta)}\int_a^b K(\eta,s)u(s)\,ds \\
&= M\left[\int_a^b \frac{1}{6}a^3 s(1-s)\,ds + \frac{\alpha a^3}{6(1-\alpha\eta)}\int_a^b K(\eta,s)\,ds\right]\|u\| \\
&= \|u\|,
\end{aligned}$$

which implies that

$$\|Au\| \geq \|u\| \quad \text{for } u \in K \cap \partial\Omega_2.$$

By Lemma 2.5 we know that problem (1.1) has at least one positive solution.

For λ large enough, we prove that problem (1.1) has no positive solution. Otherwise, there exists $0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n < \cdots$ with $\lim_{n\to\infty}\lambda_n = +\infty$ such that problem (1.1) has a positive solution $u_n(t)$, then we get

$$\begin{aligned}
u_n\left(\frac{1}{2}\right) &= \int_0^1 G\left(\frac{1}{2},s\right)f\bigl(s,u_n(s)\bigr)\,ds \\
&\quad + \frac{\alpha}{48(1-\alpha\eta)}\int_0^1 K(\eta,s)f\bigl(s,u_n(s)\bigr)\,ds + \frac{\lambda_n}{48(1-\alpha\eta)} \\
&\geq \frac{\lambda_n}{48(1-\alpha\eta)} \to +\infty \quad \text{as } n \to +\infty.
\end{aligned}$$

Hence $\|u_n\| \to +\infty$ as $n \to +\infty$.

Again from (H_5), there exist $R' > 0$ and $M > 0$ such that

$$f(t,u) \geq 96Mu \quad \text{for } (t,u) \in [a,b] \times \bigl[R',+\infty\bigr),$$

where M is defined by (3.2). Let n be large enough. Choose $R_2' > \frac{6a^3}{R'}$ such that $\|u_n\| \geq R_2'$. Thus, we get

$$\begin{aligned}
\|u_n\| &\geq u_n\left(\frac{1}{2}\right) \\
&= \int_0^1 G\left(\frac{1}{2}, s\right) f\big(s, u_n(s)\big)\, ds \\
&\quad + \frac{\alpha}{48(1-\alpha\eta)} \int_0^1 K(\eta, s) f\big(s, u_n(s)\big)\, ds + \frac{\lambda_n}{48(1-\alpha\eta)} \\
&\geq \frac{1}{8} \int_0^1 \frac{1}{6} s(1-s) f\big(s, u_n(s)\big)\, ds + \frac{\alpha}{48(1-\alpha\eta)} \int_0^1 K(\eta, s) f\big(s, u_n(s)\big)\, ds \\
&\geq \frac{1}{8} \int_a^b \frac{1}{6} s(1-s) f\big(s, u_n(s)\big)\, ds + \frac{\alpha}{48(1-\alpha\eta)} \int_a^b K(\eta, s) f\big(s, u_n(s)\big)\, ds \\
&\geq 2M \int_a^b s(1-s) u_n(s)\, ds + \frac{2M\alpha}{1-\alpha\eta} \int_a^b K(\eta, s) u_n(s)\, ds \\
&\geq 2M \left[\int_a^b \frac{1}{6} a^3 s(1-s)\, ds + \frac{\alpha a^3}{6(1-\alpha\eta)} \int_a^b K(\eta, s)\, ds\right] \|u_n\| = 2\|u_n\|,
\end{aligned}$$

which is a contradiction. The proof is complete. □

Remark 3.1 The conclusion of Theorem 3.1 also holds if $\lambda = 0$.

Theorem 3.2 *Suppose that* (H_1), (H_2) *and* (H_3) *hold. In addition, assume that*

(H_6) $\lim_{u\to+\infty} \inf \min_{0\leq t\leq 1} \frac{g(t,u)}{u} = 0$;
(H_7) $\lim_{u\to 0^+} \sup \max_{a\leq t\leq b} \frac{f(t,u)}{u} = +\infty$.

Then problem (1.1) *has at least one positive solution for any* $\lambda \in [0, +\infty)$.

Proof From (H_7), there exist constants $R_1 > 0$ and $M > 0$ such that

$$f(t, u) \geq 48Mu \quad \text{for } (t, u) \in [a, b] \times (0, R_1],$$

where M is defined by (3.2). Let $\Omega_1 = \{u \in K : \|u\| < R_1\}$. For any $u \in K \cap \partial\Omega_1$, we know that

$$\begin{aligned}
(Au)\left(\frac{1}{2}\right) &= \int_0^1 G\left(\frac{1}{2}, s\right) f\big(s, u(s)\big)\, ds \\
&\quad + \frac{\alpha}{48(1-\alpha\eta)} \int_0^1 K(\eta, s) f\big(s, u(s)\big)\, ds + \frac{\lambda}{48(1-\alpha\eta)} \\
&\geq \frac{1}{8} \int_0^1 \frac{1}{6} s(1-s) f\big(s, u(s)\big)\, ds + \frac{\alpha}{48(1-\alpha\eta)} \int_0^1 K(\eta, s) f\big(s, u(s)\big)\, ds \\
&\geq \frac{1}{8} \int_a^b \frac{1}{6} s(1-s) f\big(s, u(s)\big)\, ds + \frac{\alpha}{48(1-\alpha\eta)} \int_a^b K(\eta, s) f\big(s, u(s)\big)\, ds \\
&\geq M \int_a^b s(1-s) u(s)\, ds + \frac{M\alpha}{1-\alpha\eta} \int_a^b K(\eta, s) u(s)\, ds \\
&\geq M \left[\int_a^b \frac{1}{6} a^3 s(1-s)\, ds + \frac{\alpha a^3}{6(1-\alpha\eta)} \int_a^b K(\eta, s)\, ds\right] \|u\| = \|u\|.
\end{aligned}$$

Therefore

$$\|Au\| \geq \|u\| \quad \text{for } u \in K \cap \partial\Omega_1.$$

On the other hand, from (H_6), there exists a constant $R_0 > 0$ such that $g(t,u) \leq Nu$ for $u \geq R_0$, where N is defined by (3.1). Since $g(t,u)$ is continuous on $[0,1] \times [0,+\infty)$, there exists $M^* > 0$ such that $\max_{0\leq t\leq 1} g(t,u) \leq M^*$ for $(t,u) \in [0,1] \times [0,R_0]$. Choose

$$R_2 \geq \max\left\{2R_1, \frac{M^*}{N}, R_0, \frac{\lambda}{1-\alpha\eta}\right\}.$$

Let $\Omega_2 = \{u \in K : \|u\| < R_2\}$. For any $u \in [0,R_2]$, we get

$$f(t,u) \leq q(t)g(t,u) \leq \frac{1}{2}M^* + \frac{1}{2}NR_2.$$

Thus, for any $u \in K \cap \partial\Omega_2$ and $t \in (0,1)$, we know that

$$\begin{aligned}
Au(t) &= \int_0^1 G(t,s)f\bigl(s,u(s)\bigr)\,ds + \frac{\alpha t^3}{6(1-\alpha\eta)}\int_0^1 K(\eta,s)f\bigl(s,u(s)\bigr)\,ds + \frac{\lambda t^3}{6(1-\alpha\eta)} \\
&\leq \int_0^1 s(1-s)f\bigl(s,u(s)\bigr)\,ds + \frac{\alpha}{6(1-\alpha\eta)}\int_0^1 K(\eta,s)f\bigl(s,u(s)\bigr)\,ds + \frac{\lambda}{6(1-\alpha\eta)} \\
&\leq \int_0^1 s(1-s)q(s)g\bigl(s,u(s)\bigr)\,ds + \frac{\alpha}{6(1-\alpha\eta)}\int_0^1 K(\eta,s)q(s)g\bigl(s,u(s)\bigr)\,ds + \frac{\lambda}{6(1-\alpha\eta)} \\
&\leq \int_0^1 s(1-s)q(s)\left(\frac{1}{2}M^* + \frac{1}{2}NR_2\right)ds \\
&\quad + \frac{\alpha}{6(1-\alpha\eta)}\int_0^1 K(\eta,s)q(s)\left(\frac{1}{2}M^* + \frac{1}{2}NR_2\right)ds + \frac{1}{6}R_2 \\
&\leq \frac{1}{2}M^*\left[\int_0^1 s(1-s)q(s)\,ds + \frac{\alpha}{6(1-\alpha\eta)}\int_0^1 K(\eta,s)q(s)\,ds\right] \\
&\quad + \frac{1}{2}NR_2\left[\int_0^1 s(1-s)q(s)\,ds + \frac{\alpha}{6(1-\alpha\eta)}\int_0^1 K(\eta,s)q(s)\,ds\right] + \frac{1}{6}R_2 \\
&\leq \frac{5M^*}{12N} + \frac{5}{12N}NR_2 + \frac{1}{6}R_2 \\
&\leq \frac{5}{12}R_2 + \frac{5}{12}R_2 + \frac{1}{6}R_2 = R_2 = \|u\|.
\end{aligned}$$

Therefore

$$\|Au\| \leq \|u\| \quad \text{for } u \in K \cap \partial\Omega_2.$$

It follows from Lemma 2.5 that problem (1.1) has at least one positive solution. □

4 Examples

Now, we give examples to illustrate the main results in the paper.

Example 4.1 Consider the following boundary value problem:

$$\begin{cases} u^{(4)}(t) + \frac{1}{t^{1+\omega_1+\omega_2+\omega_3}(1-t)^{3+\omega_4}}(u^2+\sqrt{u})\sin^2 2u = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = u''(0) = 0, \\ u''(1) - \frac{1}{3}u''(\frac{1}{2}) = 1. \end{cases} \tag{4.1}$$

Then problem (4.1) has at least one positive solution if $\omega_1 + \omega_2 + \omega_3 < 1$ and $\omega_4 < -1$.

Let

$$q(t) = \frac{1}{t^{1+\omega_1+\omega_2+\omega_3}(1-t)^{3+\omega_4}}, \qquad g(t,u) = 2\bigl(u^2+\sqrt{u}\bigr)\sin^2 2u. \tag{4.2}$$

Take $[a,b] = [\frac{1}{4}, \frac{3}{4}]$. Notice, for any fixed $t \in (0,1)$, that $f(t,x) \leq q(t)g(t,x)$ and $0 < \int_0^1 s(1-s)q(s)\,ds < +\infty$ for $\omega_1 + \omega_2 + \omega_3 < 1$ and $\omega_4 < -1$.

Obviously, conditions $(H_1)\sim(H_3)$ are satisfied.

Now, for any fixed $t \in (0,1)$, (H_4) and (H_5) follow immediately from

$$\lim_{u\to 0^+} \sup \max_{0\leq t\leq 1} \frac{2(u^2+\sqrt{u})\sin^2 2u}{u} = 0,$$

$$\lim_{u\to +\infty} \inf \min_{\frac{1}{4}\leq t\leq \frac{3}{4}} \frac{(u^2+\sqrt{u})\sin^2 2u}{t^{1+\omega_1+\omega_2+\omega_3}(1-t)^{3+\omega_4}u} = +\infty.$$

Thus, the existence of a positive solution follows from Theorem 3.1 if $\omega_1 + \omega_2 + \omega_3 < 1$ and $\omega_4 < -1$.

Example 4.2 Consider the following boundary value problem:

$$\begin{cases} u^{(4)}(t) + \frac{2{,}012|\ln u(t)|+\sin^2 u(t)}{\sqrt{t(1-t)}} = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = u''(0) = 0, \\ u''(1) - \frac{1}{12}u''(\frac{1}{6}) = 2{,}013. \end{cases} \tag{4.3}$$

Then problem (4.3) has at least one positive solution.

Let

$$q(t) = \frac{1}{\sqrt{t(1-t)}}, \qquad g(t,u) = 2{,}013\bigl|\ln u(t)\bigr| + \sin^2 u(t). \tag{4.4}$$

Take $[a,b] = [\frac{1}{4}, \frac{3}{4}]$. Notice, for any fixed $t \in (0,1)$, that $f(t,x) \leq q(t)g(t,x)$ and $0 < \int_0^1 s(1-s)q(s)\,ds < +\infty$.

Obviously, conditions $(H_1)\sim(H_3)$ are satisfied.

Now, for any fixed $t \in (0,1)$, (H_6) and (H_7) follow immediately from

$$\lim_{u\to 0^+} \sup \max_{\frac{1}{4}\leq t\leq \frac{3}{4}} \frac{2{,}012|\ln u(t)| + \sin^2 u(t)}{\sqrt{t(1-t)}u} = +\infty,$$

$$\lim_{u\to +\infty} \inf \min_{0\leq t\leq 1} \frac{2{,}012|\ln u(t)| + \sin^2 u(t)}{u} = 0.$$

Thus, the existence of a positive solution follows from Theorem 3.2.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the work was realized in collaboration with same responsibility. All authors read and approved the final manuscript.

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References
1. Ma, R, Zhang, J, Fu, S: The method of upper and lower solutions for fourth-order two point boundary value problems. J. Math. Anal. Appl. **215**, 415-422 (1997)
2. Aftabizadeh, AR: Existence and uniqueness theorems for fourth-order boundary value problems. J. Math. Anal. Appl. **116**, 415-426 (1986)
3. Yang, Y: Fourth-order two-point boundary value problems. Proc. Am. Math. Soc. **104**, 175-180 (1988)
4. Gupta, CP: Existence and uniqueness results for the bending of an elastic beam equation at resonance. J. Math. Anal. Appl. **135**, 208-225 (1988)
5. Wei, Z: Positive solutions of singular boundary value problems of fourth-order differential equations. Acta Math. Sin. **42**, 715-722 (1999)
6. Ma, R, Wang, H: On the existence of positive solutions of fourth order ordinary differential equation. Appl. Anal. **59**, 225-231 (1995)
7. Harjani, J, López, B, Sadarangani, K: Fixed point theorems for mixed monotone operators and applications to integral equations. Nonlinear Anal. **74**, 1749-1760 (2011)
8. Yao, Q: Existence and multiplicity of positive solutions to a class of nonlinear cantilever beam equations. J. Syst. Sci. Math. Sci. **1**, 63-69 (2009)
9. Sun, Y: Positive solutions for third-order three-point nonhomogeneous boundary value problems. Appl. Math. Lett. **22**, 45-51 (2009)
10. Zhang, K, Wang, C: The existence of positive solutions of a class of fourth-order singular boundary value problems. Acta Math. Sci., Ser. A **29**, 127-135 (2009)
11. O'Regan, D: Solvability of some fourth (and higher) order singular boundary value problems. J. Math. Anal. Appl. **161**, 78-116 (1991)
12. Schroder, J: Fourth order two-point boundary value problems; estimates by two-sided bounds. Nonlinear Anal. **8**, 107-114 (1984)
13. Guo, L, Sun, J, Zhao, Y: Existence of positive solution for nonlinear third-order three-point boundary value problem. Nonlinear Anal. **68**, 3151-3158 (2008)
14. Bai, Z, Wang, H: On the positive solutions of some nonlinear fourth-order beam equations. J. Math. Anal. Appl. **270**, 357-368 (2002)
15. Graef, JR, Qian, C, Yang, B: A three point boundary value problem for nonlinear fourth-order differential equations. J. Math. Anal. Appl. **287**, 217-233 (2003)
16. Yao, Q: Existence and iteration of *n* symmetric positive solutions for a singular two-point boundary value problem. Comput. Math. Appl. **47**, 1195-1200 (2004)
17. Yao, Q: Local existence of multiple positive solutions to a singular cantilever beam equation. J. Math. Anal. Appl. **363**, 138-154 (2010)
18. Agarwal, RP: On fourth order boundary value problems arising in beam analysis. Differ. Integral Equ. **2**, 91-110 (1989)
19. Ji, C, O'Regan, D, Yan, B, Agarwal, RP: Nonexistence and existence of positive solutions for second order singular three-point boundary value problems with derivative dependent and sign-changing nonlinearities. J. Appl. Math. Comput. **36**, 61-87 (2011)
20. Sun, Y, Liu, L, Zhang, J, Agarwal, RP: Positive solutions of singular three-point boundary value problems for second order differential equations. J. Comput. Appl. Math. **230**, 738-750 (2009)
21. Guo, D, Lakshmikantham, V: Nonlinear Problems in Abstract Cones. Academic Press, New York (1988)
22. Deimling, K: Nonlinear Functional Analysis. Springer, New York (1985)

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