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A democratic preference aggregation model

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Abstract

We propose a democratic preference aggregation function using trapezoidal fuzzy numbers. Examples are constructed to illustrate the proposed technique to optimize the problem by avoiding paradoxical outcomes without the fear of indecision.

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Background

Democracy is based on the idea that groups make better decisions than individuals. However, majority voting is unable to ensure consistent social positions under all situations. How individual preferences can be aggregated into a collectively preferred alternative is recently studied by Arrow [1] and Sen [2]. In economics and formal epistemology, aggregation of individual preferences on logically related basic propositions into a collective preference is an emerging research area. Decision theory also studies the individual preference aggregation method. Preference aggregation is defined in social choice theory as forming a collective preference in a given set of alternatives. There are two kinds of proposition: the first kind is the premises and the other is the conclusion. Consider, for example, in a city traffic police department, there is a committee of five inspectors to decide to issue a driving license. The decision rule is such that a candidate C will be issued a driving license only if the candidate is physically fit and good in driving. We will say that issuing of a driving license to C is the conclusion (denoted by proposition D, resulting decision) while physically fit (denoted by proposition P) and good in driving (denoted by proposition Q) are the premises. The decision rule can be expressed in the formula $(P \wedge Q) \longleftrightarrow D$. Each inspector of the committee expresses his preference on P, Q, and D such that the rule $(P \wedge Q) \longleftrightarrow D$ is satisfied.

How shall we derive a group decision given the individuals' opinions on premises and conclusion? Assume that each individual expresses preference in the form of a binary decision (Yes/No) on the propositions. If we define the decision as the majority voting, then we may reach an inconsistent position, as shown in Table 1. In the literature on preference aggregation, such a problem is known as discursive dilemma and doctrinal paradox.

Although each inspector expresses a consistent preference, proposition-wise majority voting results in a majority for P and Q but a majority for $\neg D$, which is an inconsistent

Table 1 Discursive dilemma and doctrinal paradox

	P	Q	$(P \wedge Q) \longleftrightarrow D$	D
Inspectors 1 and 5	Yes	Yes	Yes	Yes
Inspectors 2 and 4	Yes	No	Yes	No
Inspector 3	No	Yes	Yes	No
Majority	Yes	Yes	Yes	?

collective result as it violates the rule $(P \wedge Q) \longleftrightarrow D$. The paradox rests with the fact that proposition-wise majority voting can lead a group of rational individuals to endorse an inconsistent collective preference. Such aggregation problems go beyond the specific example: it can be observed in all situations where individual binary decisions need to be combined into a preference of the group.

Condorcet [3] was the first to initiate a research on social choice theory that individual preferences can be aggregated into a collectively preferred alternative. Condorcet made a result that majority voting was a good method for a collectively preferred alternative, but they also found a problem in majority voting method. He suggested a method which is consisted of the pairwise comparison of each alternative. Majority voting decides the winner in each pair, and the collective result is obtained by the combination of all partial results.

Consider, for example, there are three alternatives x, y, and z and five voters V_1, V_2, V_3, V_4, and V_5. Let $\succ_i$ denote voter V_i's preference over $X = \{x, y, z\}$ and $\succ$ denote the collective preference result over X. $V_1 = \{x \succ_1 y, y \succ_1 z\}$, $V_2 = \{y \succ_2 z, z \succ_2 x\}$, $V_3 = \{z \succ_3 x, x \succ_3 y\}$, $V_4 = \{y \succ_2 z, z \succ_2 x\}$, and $V_5 = \{x \succ_1 y, y \succ_1 z\}$ are the five voters' preferences (see Table 2). Transitivity holds for each voter's preference, but unfortunately transitivity does not hold for collective preference 'V_S' (by majority method). This situation is called the Condorcet paradox.

When we combine individual decisions into a collective decision, we may have to face inconsistency in the collective level, like logical consistency and transitivity (in preference aggregation).

Realistic collective decision problems are handled with some extension beyond binary propositional logic into the realm of general multivalued logic (see [4-7]). Duddy and Piggins also used extension in propositional logic in degrees [8]. Beg and Butt [5] proved the (im)possibility theorem in fuzzy logic, similar to those of Arrow [1] and Sen [2]. There cannot exist any preference aggregation procedure that simultaneously satisfies certain consistency conditions [9].

Often, human preference is ambiguous and vague and cannot be estimated with an exact numeric value, so the binary values are not suitable to model real-world situations. Bellman and Zadeh [10] used the concept of fuzzy set theory to handle imprecision (or uncertainty) to solve the ambiguity and vagueness in human preference. Fuzzy logic

Table 2 Condorcet paradox

	$x \succ y$	$y \succ x$	$y \succ z$	$z \succ y$	$x \succ z$	$z \succ x$
V_1 and V_5	1	0	1	0	1	0
V_2 and V_4	0	1	1	0	0	1
V_3	1	0	0	1	0	1
V_S	1	0	1	0	0	1

can help in aggregation procedures to make the collective preference set democratic in nature. Trapezoidal fuzzy numbers are the best way to model preferences. Recently, a lot of work on trapezoidal fuzzy numbers has been done by Li et al. [11-13]. In this paper, trapezoidal fuzzy numbers are used for the truth value of the proposition. The rest of the paper is structured as follows: In the 'Methods' section, some basic relations and operations for trapezoidal fuzzy numbers are given and problems are reformulated in the setting of trapezoidal fuzzy numbers. The 'Results' section illustrates how the paradoxes are resolved in trapezoidal fuzzy numbers to find democratic fuzzy aggregation function. The 'Discussion' section presents the discussion in utility maximization. Conclusion of the whole work is given in the last section.

Methods

In some problems, propositions are vague and thus can have truth values between true and false. Beg and Butt [6] tried to use fuzzy logic for the solution of doctrinal and Condercet paradoxes. We can use trapezoidal fuzzy numbers to model the vagueness of human knowledge.

Definition 2.1. *A function 'A' given by*

$$A(x) = \begin{cases} 0 & \textit{if} \;\; x < x_1 \text{or } x > x_4, \\ \frac{x-x_1}{x_2-x_1} & \textit{if} \;\; x_1 \le x < x_2, \\ 1 & \textit{if} \;\; x_2 \le x < x_3, \\ \frac{x-x_4}{x_3-x_4} & \textit{if} \;\; x_3 \le x \le x_4, \end{cases}$$

where $x_1 < x_2 < x_3 < x_4$, *is called a trapezoidal fuzzy number. Symbolically, A is denoted by* (x_1, x_2, x_3, x_4) (*see p. 330 in* [14]).

Let $D[0, 1]$ denote the set of all trapezoidal fuzzy numbers such that $0 \le x_1$ and $x_4 \le 1$. We assume that individual preferences take values on $D[0, 1]$.

Definition 2.2. ([15]) *We define an operator* Δ *on trapezoidal fuzzy numbers, which is given by*

$$A\Delta B = (0 \vee (x_1 + y_1 - 1), 0 \vee (x_2 + y_2 - 1), 0 \vee (x_3 + y_3 - 1), 0 \vee (x_4 + y_4 - 1)),$$

where $A = (x_1, x_2, x_3, x_4)$, $B = (y_1, y_2, y_3, y_4)$, *and* $A, B \in D[0, 1]$. *Also,* $A = B$ if $x_1 = y_1$, $x_2 = y_2$, $x_3 = y_3$, *and* $x_4 = y_4$.

Definition 2.3. *The constant 'k' multiple of trapezoidal fuzzy number A is defined as* $kA = k(x_1, x_2, x_3, x_4) = (k \times x_1, k \times x_2, k \times x_3, k \times x_4)$ *where* $A = (x_1, x_2, x_3, x_4)$, $A \in D[0, 1]$, *and* $k \in [0, 1]$.

Definition 2.4. ([15]) *The relation* $\preceq$ *in* $D[0, 1]$ *is introduced as follows: Let* $A = (x_1, x_2, x_3, x_4)$, $B = (y_1, y_2, y_3, y_4)$, *and* $A, B \in D[0, 1]$:

If $x_4 < y_4$, then $A \prec B$;
If $x_4 = y_4$ and

(i) $x_3 < y_3$, then $A \prec B$;

(ii) $x_3 = y_3$ and

(a) $x_2 < y_2$, then $A \prec B$;
(b) $x_2 = y_2$ and $x_1 < y_1$, then $A \prec B$.

Obviously, $D[0,1]$ is an ordered set.

Definition 2.5. ([15]) *In* $D[0,1]$,

$$\min(A,B) = (\min(x_1,y_1), \min(x_2,y_2), \min(x_3,y_3), \min(x_4,y_4))$$

and

$$\max(A,B) = (\max(x_1,y_1), \max(x_2,y_2), \max(x_3,y_3), \max(x_4,y_4))$$

are called min *and* max *operators for trapezoidal fuzzy numbers, respectively.*

Definition 2.6. ([15]) *A fuzzy implication* $\Longrightarrow$ *for trapezoidal fuzzy numbers is a map*

$$\Longrightarrow: D[0,1] \times D[0,1] \to D[0,1]$$

satisfying

(i) $[(0,0,0,0) \Longrightarrow (0,0,0,0)] = (1,1,1,1)$,
(ii) $[(0,0,0,0) \Longrightarrow (1,1,1,1)] = (1,1,1,1)$,
(iii) $[(1,1,1,1) \Longrightarrow (0,0,0,0)] = (0,0,0,0)$,
(iv) $[(1,1,1,1) \Longrightarrow (0,0,0,0)] = (1,1,1,1)$.

An example of a fuzzy implication for trapezoidal fuzzy numbers A and B is

$$A \Longrightarrow B = \begin{cases} (1-\max(|x_1-y_1|, |x_2-y_2|, |x_3-y_3|, |x_4-y_4|), \\ 1-\max(|x_1-y_1|, |x_2-y_2|, |x_3-y_3|, |x_4-y_4|), \\ 1-\min(|x_1-y_1|, |x_2-y_2|, |x_3-y_3|, |x_4-y_4|), & ; \text{if } A \succ B \\ 1-\min(|x_1-y_1|, |x_2-y_2|, |x_3-y_3|, |x_4-y_4|)) \\ (1,1,1,1) & ; \text{if } A \preceq B. \end{cases}$$

Definition 2.7. *The weighted average operator 'avg' of trapezoidal fuzzy numbers is defined as follows: Let* $A_1 = (x_{11}, x_{12}, x_{13}, x_{14})$, $A_2 = (x_{21}, x_{22}, x_{23}, x_{24})$, ..., $A_n = (x_{n1}, x_{n2}, x_{n3}, x_{n4})$, *and* $A_1, A_2, \ldots, A_n \in D[0,1]$:

$$\operatorname{avg}(A_1, A_2, \ldots, A_n) = \left(\frac{w_1 \times x_{11} + w_2 \times x_{21} + \cdots + w_n \times x_{n1}}{w_1 + w_2 + \cdots + w_n}, \frac{w_1 \times x_{12} + w_2 \times x_{22} + \cdots + w_n \times x_{n2}}{w_1 + w_2 + \cdots + w_n}, \frac{w_1 \times x_{13} + w_2 \times x_{23} + \cdots + w_n \times x_{n3}}{w_1 + w_2 + \cdots + w_n}, \frac{w_1 \times x_{14} + w_2 \times x_{24} + \cdots + w_n \times x_{n4}}{w_1 + w_2 + \cdots + w_n} \right),$$

where w_i *is the weight assigned to truth value* A_i.

Note that if $w_i = 1$ for all i, then $\operatorname{avg}(A_1, A_2, \ldots, A_n)$ is denoted by $\operatorname{average}(A_1, A_2, \ldots, A_n)$.

Example 2.8. Here we reformulate our problem in a trapezoidal fuzzy number, which is illustrated and summarized in Table 1. Table 3 is a clear illustration of Table 1 for trapezoidal fuzzy numbers and also the average of inspectors' decision for each proposition. In this case, we get at least one solution of doctrinal paradox for trapezoidal fuzzy numbers.

One important property of majority selection: min((0.45, 0.48, 0.54, 0.59), (0.15, 0.19, 0.27, 0.31), (0.75, 0.79, 0.85, 0.89)) $\preceq$ $(\theta_1(x_1), \theta_1(x_2), \theta_1(x_3), \theta_1(x_4))$ $\preceq$ max((0.45, 0.48, 0.54, 0.59), (0.15, 0.19, 0.27, 0.31), (0.75, 0.79, 0.85, 0.89)), min((0.61, 0.65, 0.71, 0.75), (0.61, 0.65, 0.71, 0.75), (0.05, 0.09, 0.16, 0.22)) $\preceq$ $(\theta_2(x_1), \theta_2(x_2), \theta_2(x_3), \theta_2(x_4))$ $\preceq$ max((0.61, 0.65, 0.71, 0.75), (0.61, 0.65, 0.71, 0.75), (0.05, 0.09, 0.16, 0.22)), and min((0.45, 0.48, 0.54, 0.59), (0.15, 0.19, 0.27, 0.31), (0.05, 0.09, 0.16, 0.22)) $\preceq$ $(\theta_3(x_1), \theta_3(x_2), \theta_3(x_3), \theta_3(x_4))$ $\preceq$ max((0.45, 0.48, 0.54, 0.59), (0.15, 0.19, 0.27, 0.31), (0.05, 0.09, 0.16, 0.22)) is in Table 3.

Simply, (0.15, 0.19, 0.27, 0.31) $\preceq (\theta_1(x_1), \theta_1(x_2), \theta_1(x_3), \theta_1(x_4)) \preceq$ (0.75, 0.79, 0.85, 0.89), (0.05, 0.09, 0.16, 0.22) $\preceq (\theta_2(x_1), \theta_2(x_2), \theta_2(x_3), \theta_2(x_4)) \preceq$ (0.61, 0.65, 0.71, 0.75) and (0.05, 0.09, 0.16, 0.22) $\preceq (\theta_3(x_1), \theta_3(x_2), \theta_3(x_3), \theta_3(x_4)) \preceq$ (0.45, 0.48, 0.54, 0.59) such that $0.15 \leq \theta_1(x_1) \leq 0.75$, $0.19 \leq \theta_1(x_2) \leq 0.79$, $0.27 \leq \theta_1(x_3) \leq 0.85$, $0.31 \leq \theta_1(x_4) \leq 0.89$, $0.05 \leq \theta_2(x_1) \leq 0.61$, $0.09 \leq \theta_2(x_2) \leq 0.65$, $0.16 \leq \theta_2(x_3) \leq 0.71$, $0.22 \leq \theta_2(x_4) \leq 0.75$, $0.05 \leq \theta_3(x_1) \leq 0.45$, $0.09 \leq \theta_3(x_2) \leq 0.48$, $0.16 \leq \theta_3(x_3) \leq 0.54$, and $0.22 \leq \theta_3(x_4) \leq 0.59$, but the values of $\theta_i(x_j)$ will satisfy the condition for trapezoidal fuzzy numbers of $D[0, 1]$.

At the same time, let $\pi(P)$ denote the degree of truth of the proposition P, and the fuzzy integrity constraint is $\{\pi(P)\Delta\pi(Q) \Longrightarrow \pi(R)\}$. Suppose that the committee members are rational and they never violate the fuzzy integrity constraints (IC) (see List [16]). Now by using the above given Δ operator and fuzzy implication $\Longrightarrow$, the IC can be translated as $\pi(R) \geq \max(0, \pi(P) + \pi(Q) - 1)$: Here $\pi(R) = f_i(\pi(P), \pi(Q))$ is a particular rule of inference for individual i (see [17]).

Results

We assume that decision makers are rational and they have freedom to fully express their opinions on a proposition with which they do not agree or disagree. So any number from $D[0, 1]$ that best representstheir opinions can be opted. A finite set of n individuals and a finite set X of propositions over which individuals have to make their preferences are called an *agenda*. A preference set A_i for an individual i is an n-tuple containing the degree of truth for each proposition. Let $A_i = (h_{i1}, h_{i2}, \ldots, h_{i|X|})$, where $|X|$ denotes the

Table 3 Revisit of Table 1 for trapezoidal fuzzy numbers

	P	Q	R	$(P\Delta Q) \Longrightarrow R$
Inspectors 1 and 5	(0.45, 0.48, 0.54, 0.59)	(0.61, 0.65, 0.71, 0.75)	(0.45, 0.48, 0.54, 0.59)	(1, 1, 1, 1)
Inspectors 2 and 4	(0.15, 0.19, 0.27, 0.31)	(0.61, 0.65, 0.71, 0.75)	(0.15, 0.19, 0.27, 0.31)	(1, 1, 1, 1)
Inspector 3	(0.75, 0.79, 0.85, 0.89)	(0.05, 0.09, 0.16, 0.22)	(0.05, 0.09, 0.16, 0.22)	(1, 1, 1, 1)
Average	(0.45, 0.49, 0.55, 0.60)	(0.42, 0.46, 0.53, 0.57)	(0.22, 0.25, 0.32, 0.37)	(1, 1, 1, 1)
Majority	$(\theta_1(x_1), \theta_1(x_2), \theta_1(x_3), \theta_1(x_4))$	$(\theta_2(x_1), \theta_2(x_2), \theta_2(x_3), \theta_2(x_4))$	$(\theta_3(x_1), \theta_3(x_2), \theta_3(x_3), \theta_3(x_4))$	(1, 1, 1, 1)

cardinality of X and $h_{ij} \in D[0,1]$ for $j \in (1, 2, \ldots, |X|)$. A profile is an n-tuple $(A_1, A_2, \ldots, A_n)$ of individual preference sets. Function f is an aggregation that assigns to each profile $(A_1, A_2, \ldots, A_n)$ a collective preference set $(h_1^*, h_2^*, \ldots, h_{|X|}^*) = f(A_1, A_2, \ldots, A_n)$. Here $h_j^* \in D[0,1]$ for $j \in (1, 2, \ldots, |X|)$.

The truth value of some proposition $\varphi \in X$ for the preference set A_i is $A_i(\varphi)$. The truth value of some proposition $\varphi \in X$ for the collective preference set $f(A_1, A_2, \ldots, A_n)$ is $f(A_1, A_2, \ldots, A_n)(\varphi) = f(A_1(\varphi), A_2(\varphi), \ldots, A_n(\varphi))$.

If $f(A_1, A_2, \ldots, A_n) = A_i$ for some $i \in (1, 2, \ldots, n)$ and every $(A_1, A_2, \ldots, A_n)$, then f is 'dictatorship.'

Function f is 'manipulable' if and only if there exist some voter i, proposition φ, and profile $(A_1, A_2, \ldots, A_n)$ such that $A_i(\varphi) \neq f(A_1, A_2, \ldots, A_i, \ldots, A_n)(\varphi)$, but $A_i(\varphi) = f(A_1, A_2, \ldots, A_i^*, \ldots, A_n)(\varphi)$ for some alternate preference set A_i^*.

Function f is 'independent' if and only if for all propositions $\varphi \in X$ there is a function $g_\varphi : D[0,1]^n \to D[0,1]$ such that for all $(A_1, A_2, \ldots, A_n)$, we have $f(A_1, A_2, \ldots, A_n)(\varphi) = g_\varphi(A_1(\varphi), A_2(\varphi), \ldots, A_n(\varphi))$.

Preference aggregation formally investigates how to aggregate a finite number of preference bases into a collective one. This formal framework consists of a propositional language $\aleph$ which is built up from a finite set P of propositional letters standing for atomic propositions. An interpretation is a function from P to $D[0,1]$. Function $\pi(.)$ represents the truth function that maps elements in set P to $D[0,1]$. Let the preference base K_i for the agent i be the following set $(\pi_i(p_1), \pi_i(p_2), \ldots, \pi_i(p_{|P|}))$, where $|P|$ denotes the cardinality of P. A preference set is the set $E = \{K_1, K_2, \ldots, K_n\}$. Given a set of IC in a fuzzy setting, ψ maps E and IC into a new (collective) preference base $\psi_{\mathrm{IC}}(E)$, and this process is called fuzzy aggregation for trapezoidal fuzzy numbers.

Let W denote the set of all interpretations and the distance between interpretations is a real-valued function

$$d : W \times W \to \Re$$

such that for all $w, w', w'' \in W$:

1. $d(w, w') \geq 0$.
2. $d(w, w') = 0$ if and only if $w = w'$.
3. $d(w, w') = d(w', w)$.
4. $d(w, w'') \leq d(w, w') + d(w', w'')$.

One possible choice for distance function is the Hamming distance:

$$\begin{aligned} d^*(w, w') &= \frac{1}{4} \sum_{\forall x \in P} |w(x) - w'(x)| \\ &= \frac{1}{4} \sum_{\forall x \in P} (|w(x_1) - w'(x_1)| + |w(x_2) - w'(x_2)| \\ &\qquad + |w(x_3) - w'(x_3)| + |w(x_4) - w'(x_4)|). \end{aligned}$$

Another possible choice for distance function is the square distance:

$$\begin{aligned} d^{**}(w, w') &= \sum_{\forall x \in P} (w(x_1) - w'(x_1))^2 + (w(x_2) - w'(x_2))^2 \\ &\qquad + (w(x_3) - w'(x_3))^2 + (w(x_4) - w'(x_4))^2). \end{aligned}$$

Now let us define a preference aggregation operator in this framework. For any interpretation $w \in W$ and any profile of preference basis $K \in K^n$, the distance between an interpretation and a profile can now be defined as

$$D^d(w, K) = \sum_{\forall i} d(w, K_i).$$

Our objective is to choose w which minimizes this distance and does not violate any IC in the fuzzy setting. To minimize the distance is the same as to minimize a measure of disagreement in the society by bringing the collective preference set close to the individual preference sets as possible. Individual disagreement brings about individual disutility. Thus, we seek to minimize the societal disutility which is assumed to be the sum of individual disutilities. We have many distance and dissimilarity measures in the literature like distance 'd_h' [18], distances 'd_{LR}' and 'd_f' [19,20], distance 'D_{MLR}' [21], etc. In this paper, we have chosen the Hamming distance d^* and square distance d^{**}. Choosing any distance or dissimilarity measure is solely at our discretion provided that it satisfies certain normative principles.

Let w be any arbitrary interpretation. In this case, $w(P) = (\theta_1, \theta_2, \ldots, \theta_{|P|})$, where we have $\theta_i \in D[0, 1]$ for all $1 \leq i \leq |P|$ and $|P|$ denotes the cardinality of P.

We can think a collective preference set $\bar{F}$ which is the weighted average of individual preference sets. We believe that the closer our collective preference set F is to $\bar{F}$, the more legitimate and democratic is our final preference. Now we assert that an optimal fuzzy aggregation function is democratic if the solution is as close as possible to the average of individual preferences. Such a view is held only for the purpose of illustration in our paper, and there could be different possible ways for democratization.

A fuzzy aggregation function is democratic as well as optimal if we make the solution of Table 3 as close as possible to the avg(inspector 1, inspector 2, inspector 3, inspector 4, inspector 5). This could be easily achieved by adding the penalty to the objective function. Here we use the distance function $d(w,\text{avg})$ as penalty. So $d(w,\text{avg})$ is the degree of democracy.

Now d^* is a generalization of the Hamming distance, and we can use it in trapezoidal fuzzy numbers to avoid the doctrinal paradox. We can formulate the fuzzy aggregation as an optimization problem which can be stated as

Minimize

$$D^{d^*}(w, K) = \sum_{\forall i} d^*(w, K_i)$$

and

$$d^*(w, \text{avg})$$

subject to the fuzzy IC.

Here $w(P) = (\theta_1, \theta_2, \ldots, \theta_{|P|})$, $\min\{K_1(j), K_2(j), \ldots, K_n(j)\} \leq \theta_j \leq \max\{K_1(j), K_2(j), \ldots, K_n(j)\}$, and $\theta_j \in D[0, 1]$ for $j \in \{1, 2, 3, \ldots, |P|\}$, where $K_i(j)$ denotes the jth element of the preference base K_i.

The above optimization problem helps us to avoid doctrinal paradox and find an optimal fuzzy aggregation function. We say that an aggregation function is optimal if the collective preference set is close to the individual preference set and weighted average of individual preference sets as possible. Finding a collective social choice function in Table 3 now becomes an optimization problem which can have multiple optimal solutions. The problem in Table 3 can be democratized in d^* and d^{**}. The optimal fuzzy aggregation function gives the solution for Table 3 as $(\theta_1, \theta_2, \theta_3) = ((0.45, 0.48, 0.54, 0.59), (0.42, 0.621, 0.621, 0.621), (0.151, 0.249, 0.32, 0.367))$ with minimum $D^{d^*} + d^*(w,\text{average}) = 1.8527$. The optimal fuzzy aggregation function gives the solution for Table 3 as $(\theta_1, \theta_2, \theta_3) = ((0.45, 0.487, 0.553, 0.598), (0.422, 0.462, 0.462, 0.572), (0.217, 0.252, 0.405, 0.405))$ with minimum $D^{d^{**}} + d^{**}(w, \text{average}) = 1.9164$. The fact that there is at least one solution to the problem shows that doctrinal paradox cannot occur in this case.

We like our aggregation procedure to be strategy proof. Impossibility theorems proved by Dietrich and List [22] were similar to the Gibbard-Satterthwaite theorem on strategy-proof aggregation rules. Given these theorems, we do not claim that our distance-based aggregation method is strategy proof. In fact, Dietrich [9] has proved that independence and monotonicity are properties of an aggregator that result in strategy proofness. Since we do not claim that our distance-based aggregator is independent and monotone simultaneously, the strategy proofness of our aggregator is clear. The nature of our objective function in the optimization problem is such that if an individual was to submit an insincere preference (in an attempt to manipulate the collective preference), any deviation of the collective preference set from this insincere preference has a penalty in the objective function. In this situation, there appears to be a partial corrective mechanism whereby our aggregation method is not easily prone to manipulation.

Example 3.1. Here we reformulate our problem in trapezoidal fuzzy number form, which is illustrated and summarized in Table 2. Table 4 is a clear illustration of Table 2 for trapezoidal fuzzy numbers. Also, the average of voters' decision for each proposition is calculated. Now we get at least one solution of the Condorcet paradox for trapezoidal fuzzy numbers. Suppose that the individuals have a preference structure as shown in Table 4.

Now consider Table 4. Assume that there is a small economy with three individuals and three goods x, y, and z. Individual binary relations over $X = \{x, y, z\}$, namely V_1, V_2, V_3, V_4, and V_5, are linear orders. Any optimal fuzzy social preference aggregation function maps the individual preference set into a social preference set that must be a linear order. Accordingly, this becomes an optimization problem which minimizes the sum of the distances of social preference from the individual preferences and weighted

Table 4 Revisit of Table 2 for trapezoidal fuzzy numbers

	$x \succ y$	$y \succ x$	$y \succ z$	$z \succ y$	$x \succ z$	$z \succ x$
V_1 and V_5	(0.45,0.48, 0.54,0.59)	(0.15,0.19, 0.27,0.31)	(0.45,0.48, 0.54,0.59)	(0.05,0.09, 0.16,0.22)	(0.39,0.42, 0.44,0.48)	(0.15,0.19, 0.27,0.31)
V_2 and V_4	(0.75,0.79, 0.85,0.89)	(0.61,0.65, 0.71,0.75)	(0.15,0.19, 0.27,0.31)	(0.05,0.09, 0.16,0.22)	(0.45,0.48, 0.54,0.59)	(0.25,0.32, 0.35,0.39)
V_3	(0.75,0.79, 0.85,0.89)	(0.05,0.09, 0.16,0.22)	(0.25,0.32, 0.35,0.39)	(0.15,0.19, 0.27,0.31)	(0.61,0.65, 0.71,0.75)	(0.15,0.19, 0.27,0.31)
Average	(0.65,0.69, 0.75,0.79)	(0.27,0.31, 0.38,0.43)	(0.28,0.33, 0.39,0.43)	(0.08,0.12, 0.20,0.25)	(0.48,0.52, 0.56,0.61)	(0.18,0.23, 0.30,0.34)
V_S	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6

average of individual preferences using d^* (subject to fuzzy IC of linear order). Here preference aggregation is molded by representing preference ordering as truth values over trapezoidal fuzzy numbers.

The problem in Table 4 can be democratized in d^* and d^{**}. The optimal fuzzy preference function gives the solution for Table 4 as (θ_1, θ_2, θ_3, θ_4, θ_5, θ_6) = ((0.69, 0.69, 0.75, 0.79), (0.15, 0.19, 0.27, 0.31), (0.25, 0.32, 0.39, 0.42), (0.05, 0.09, 0.16, 0.22), (0.45, 0.48, 0.56, 0.59), (0.17, 0.22, 0.29, 0.31)) with minimum distance $D^{d^*} + d^*(w,\text{average}) = 1.925$. The optimal fuzzy preference function gives the solution for Table 4 as (θ_1, θ_2, θ_3, θ_4, θ_5, θ_6) = ((0.65, 0.69, 0.75, 0.79), (0.27, 0.31, 0.38, 0.43), (0.28, 0.33, 0.39, 0.43), (0.08, 0.12, 0.2, 0.25), (0.48, 0.52, 0.56, 0.61), (0.18, 0.23, 0.3, 0.34)) with minimum distance $D^{d^{**}} + d^{**}(w,\text{average}) = 1.2864$. Note that the uniqueness of the optimal solution is not guaranteed.

Discussion

We have assumed that agents have only 'epistemic' preferences, i.e., they only care about the 'distance' between the collective preference set which is collectively calculated and the individual preference set they personally decide (see [23]). This distance was originally measured by d^* where P is a finite set of propositional letters standing for atomic propositions. We re-scale this distance to form a new distance measure:

$$d(p,q) = \frac{d^*(p,q)}{4 \times |P|}.$$

Now this re-scaling makes $d(p,q) \in [0,1]$. Let p_i be the profile of individual i. Individual i receives fuzzy 'utility' if the collective preference set is collectively accepted, given by the formula $u_i(q) = \eta(d_i(p_i,q)) \in [0,1]$, where $\eta : [0,1] \to [0,1]$ is the strong fuzzy negation (see [14]) that satisfies:

1. $\eta(0) = 1, \eta(1) = 0.$
2. η is nonincreasing.
3. $\eta(\eta(x)) = x.$

For the sake of simplicity, assume that $\eta(x) = 1 - x$. Now $u_i(q) = 1 - d_i(p_i,q)$ implies that $\sum_{\forall i} u_i(q) + \sum_{\forall i} d_i(p_i,q) = n$. Choosing a collective preference set q which minimizes $\sum_{\forall i} d_i(p_i,q)$ is therefore equivalent to choosing q which maximizes $\sum_{\forall i} u_i(q)$, i.e., the sum of individual utilities. As a result, optimization problems in Tables 3 and 4 can be viewed as social utility maximization problems.

Conclusion

Policy makers could be allowed to opt for trapezoidal fuzzy numbers to express their preferences. It not only gives freedom of expression to the decision makers but also provides us with a wider range of fuzzy connectives that can be used according to the nature of the problem. In this paper, a democratic fuzzy aggregation approach for trapezoidal fuzzy numbers is used to find an interpretation having the least distance with the profile of individual preference sets and avg set. The authors believe that the preference aggregation modeling for trapezoidal fuzzy numbers will have tremendously useful applications. Our aggregation procedure is an improvement of Pigozzi's [24] aggregation method (in the context of binary logic) because cases of dictatorship are highly unlikely in our method. Our aggregation procedure did not give a unique solution. A tie-breaking method can be used in the case of our optimal solution as it is not unique. Assume that we have a preference set $\bar{C} = (\text{average}(\pi_i(P_1)), \text{average}(\pi_i(P_2)), \ldots, \text{average}(\pi_i(P_{|P|})))$. Set $\bar{C}$ might violate the fuzzy integrity constraints. Solutions could be narrowed down by picking solutions which are at a minimal distance from $\bar{C}$ with the help of the tie-breaking method. Another useful method is the aggregation function from the optimal solution, which has minimal disagreement with other aggregation functions, using a dissimilarity or distance measure. However, the important question is, how well behaved is our aggregation operator. We want the collective preference set to be responsive to the individual preference set, and we also want the collective preference set to obey rationality constraints. However, it does not ensure collective rationality. Distance-based operators are used to obtain collective rationality results in a situation of tie or indecision. Fuzzy aggregation methods are useful to construct democratic fuzzy social preference aggregation functions as already given in the previous discussions.

Competing interests
Both authors declare that they have no competing interests.

Authors' contributions
IB and TR contributed equally and significantly in this research. Both authors read and approved the final manuscript.

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