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Isogonal deformation of discrete plane curves and discrete Burgers hierarchy

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Abstract

We study deformations of plane curves in the similarity geometry. It is known that continuous deformations of smooth curves are described by the Burgers hierarchy. In this paper, we formulate the discrete deformation of discrete plane curves described by the discrete Burgers hierarchy as isogonal deformations. We also construct explicit formulas for the curve deformations by using the solution of linear diffusion differential/difference equations.

1 Introduction

Integrable deformations of curves play crucial roles in the differential geometry of space/plane curves [30]. Formulating the deformation of curves as the simultaneous system of the Frenet-Serret formula for the Frenet frame of curves and its deformation equation, it naturally gives rise to various integrable systems. This framework can be discretized so that it is consistent with the theory of discrete integrable systems, which is sometimes referred to as the discrete differential geometry [1]. Various deformations of discrete curves have been formulated in this context [6–8, 15–19, 26–29]. The theory of discrete differential geometry of curves is now making progress in explicit constructions of curves, by using the theory of τ functions [22–25].

When we change the geometric structure of space/plane in the framework of Klein geometry, the curve motions are governed by various integrable equations [3–5]. Therefore it may be an interesting and important problem to discretize such deformations of curves consistently with corresponding integrable structures.

In this paper, we consider deformation of the plane curves in the similarity geometry, which is a Klein geometry associated with the linear conformal group. In this setting, it is known that the Burgers hierarchy describes the deformations of similarity curvature of curves. We present discrete deformations of discrete plane curves in the similarity geometry described by the discrete Burgers hierarchy as the isogonal deformations in which each angle of adjacent segments is preserved. The lattice intervals of the hierarchy are generalized to arbitrary functions of corresponding independent variables. Using this formulation, we present explicit formulas of curves for both smooth and discrete cases. We note that the (complex) Burgers equation and its discrete analogue also arise in the curve deformations in complex hyperbola, where the Hamiltonian formulation of the deformation of smooth curves is discussed [15].

In Section 2, we give a brief summary of deformation of smooth plane curves in the similarity geometry, and we see that the Burgers hierarchy naturally arises as the equations for the similarity curvature. We also construct the explicit formula for the family of plane curves corresponding to the shock wave solutions to the Burgers equation. In Section 3, we discretize the whole theory described in Section 2 so that the deformations are governed by the discrete Burgers hierarchy. Formulations of the Burgers and the discrete Burgers hierarchies are discussed in detail in Appendix.

In [21, 31], the deformation theory of plane curves in the similarity geometry can be applied to the construction and generalization of aesthetic curves in CAD. Also, in [9–14] discretizations for the class of nonlinear differential equations describing the motions of plane curves are constructed by using the geometric formulations, resulting in self-adaptive moving mesh discrete model of the original equation. This discretization enables to contruct highly accurate numerical scheme of given equation. The Burgers equation is widely used as the universal model describing one-dimensional nonlinear dissipative system

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after various transformations which are difficult to discretize. It may be possible to construct various useful discrete models by using the result in this paper. We hope that the results in this paper serves as the basis of such industry-based problems.

2 Deformation of smooth curves

Let $\gamma = \gamma(s)$ be a smooth curve in $\mathbb{R}^2$, s be the arc-length, and κ be the curvature of γ. We denote by Sim(2) the similarity transformation group of $\mathbb{R}^2$, that is, $\mathrm{Sim}(2) = \mathrm{CO}(2) \ltimes \mathbb{R}^2$ where CO(2) is the linear conformal group

$$\mathrm{CO}(2) = \left\{A \in \mathrm{GL}\,(2,\mathbb{R}) \,\middle|\, {}^{\mathrm{t}}\!AA = c^2 \,\mathrm{id} \text{ for some constant } c\right\}.$$

The Sim(2)-invariant parameter x is given by the angle function

$$x = \int^s \kappa(s)\, ds, \tag{2.1}$$

and the Sim(2)-invariant curvature u is defined as

$$u = \frac{1}{\kappa^2}\frac{d\kappa}{ds}. \tag{2.2}$$

The x and u are called the *similarity arc-length parameter* and the *similarity curvature*, respectively. If the similarity curvature is constant $u = k_1$, then the inverse of Euclidean curvature is $1/\kappa = -k_1 s + k_2$ for some constant k_2. Thus γ is a log-spiral (if $k_1 \neq 0$) or a circle (if $k_1 = 0, k_2 \neq 0$).

The Sim(2)-invariant frame $\phi = [T, N]$ is given by

$$T = \gamma', \quad N = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} T, \tag{2.3}$$

where the prime means differentiation with respect to the similarity arc-length parameter x. The SO(2)-invariant frame (the Frenet frame) ϕ_{E} given by

$$\phi_{\mathrm{E}} = [T_{\mathrm{E}}, N_{\mathrm{E}}] = \kappa\phi, \quad T_{\mathrm{E}} = \frac{d}{ds}\gamma, \quad N_{\mathrm{E}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} T_{\mathrm{E}},$$

varies according to the Frenet formula

$$\frac{d}{ds}\phi_{\mathrm{E}} = \phi_{\mathrm{E}} \begin{bmatrix} 0 & -\kappa \\ \kappa & 0 \end{bmatrix}.$$

Therefore, by using (2.1) and (2.2), we have

$$\phi' = \phi \begin{bmatrix} -u & -1 \\ 1 & -u \end{bmatrix}. \tag{2.4}$$

We denote by $\gamma(x,t)$ a deformation of a curve $\gamma(x)$. We use the dot to indicate differentiation with respect to time t. Writing $\dot\gamma$ as the linear combination of T and N as

$$\dot\gamma = f(x,t)T + g(x,t)N,$$

we have by using (2.3) that

$$\dot\phi = \phi \begin{bmatrix} f' - fu - g & -g' + gu - f \\ g' - gu + f & f' - fu - g \end{bmatrix}. \tag{2.5}$$

The compatibility condition of the linear system (2.4) and (2.5) is given by

$$g' - gu + f = a, \tag{2.6}$$

$$\dot u + \left(f' - fu - g\right)' = 0, \tag{2.7}$$

for some function $a = a(t)$. Especially, choosing $f = a - u$, $g = -1$ and denoting $t = t_2$, we have

$$\frac{\partial\phi}{\partial t_2} = \phi \begin{bmatrix} -u' + u^2 + 1 - au & -a \\ a & -u' + u^2 + 1 - au \end{bmatrix}, \tag{2.8}$$

$$\frac{\partial u}{\partial t_2} = u'' - 2uu' + au'. \tag{2.9}$$

Equation (2.9) is called the *Burgers equation*, which is linearized to

$$\frac{\partial}{\partial t_2}q = \left(\frac{\partial^2}{\partial x^2} + 1 + a\frac{\partial}{\partial x}\right)q, \tag{2.10}$$

via the Cole-Hopf transformation [20]

$$u = -\left(\log q\right)'. \tag{2.11}$$

Further, the Burgers hierarchy naturally arises as follows [3–5]. Substituting (2.6) into (2.7), we have that

$$\dot u = \left(\Omega^2 + 1\right)g' + au', \tag{2.12}$$

where $\Omega = \partial_x - u - u'\partial_x^{-1}$ is the recursion operator of the Burgers hierarchy (see Appendix A). Here, ∂_x^{-1} is the formal integration operator with respect to x, and in the following, the integration constant should be chosen to be 0. In view of this, we introduce an infinite number of time variables $t = (t_2, t_3, t_4, \ldots)$, and choose $g' = \Omega^{i-3}u'$ ($i \geq 3$). Then the higher flow with respect to the new time variable t_i is given by

$$\frac{\partial}{\partial t_i}\phi = \phi \begin{bmatrix} -\partial_x^{-1}\left(\Omega^{i-1} + \Omega^{i-3} + a\right)u' & -a \\ a & -\partial_x^{-1}\left(\Omega^{i-1} + \Omega^{i-3} + a\right)u' \end{bmatrix}. \tag{2.13}$$

The compatibility condition between (2.4) and (2.13) is the i-th Burgers equation

$$\frac{\partial}{\partial t_i}u = \left(\Omega^{i-1} + \Omega^{i-3} + a\right)u', \tag{2.14}$$

which is linearized to

$$\frac{\partial}{\partial t_i}q = \left(\frac{\partial^i}{\partial x^i} + \frac{\partial^{i-2}}{\partial x^{i-2}} + a\frac{\partial}{\partial x}\right)q, \tag{2.15}$$

via the Cole-Hopf transformation (2.11). Note that the case of $i = 2$ of (2.15) recovers (2.10).

It is possible to express the postion vector γ in terms of q as follows. The inverse of Euclidean curvature satisfies

$1/\kappa = cq$ for some function $c = c(t)$, because the similarity curvature is logarithmic differentiation of κ, that is, u satisfies that

$$u = \frac{1}{\kappa^2}\frac{\partial\kappa}{\partial s} = \frac{\kappa'}{\kappa^2}\frac{\partial x}{\partial s} = \frac{\kappa'}{\kappa} = \left(\log\kappa\right)'.$$

Since the similarity arclength parameter x is the angle function, we have

$$T_{\mathrm{E}} = \begin{bmatrix}\cos(x+x_0)\\ \sin(x+x_0)\end{bmatrix},$$

where we have incorporated the ambiguity of the angle function $x_0 = x_0(t)$ explicitly. Hence

$$\begin{aligned}\gamma &= \int^x T dx = \int^x \frac{1}{\kappa}\, T_{\mathrm{E}}\, dx\\ &= \int^x c(t)q(x,t)\begin{bmatrix}\cos(x+x_0(t))\\ \sin(x+x_0(t))\end{bmatrix} dx.\end{aligned} \tag{2.16}$$

We determine c and x_0 by the deformation equation (2.13). By differentiating T by t_i (here $\dot{}$ denotes ∂_{t_i}), we have by substituting (2.16) into (2.13),

$$\begin{aligned}\dot T &= (\dot c q + c\dot q)\begin{bmatrix}\cos(x+x_0)\\ \sin(x+x_0)\end{bmatrix} + cq\dot{x_0}\begin{bmatrix}-\sin(x+x_0)\\ \cos(x+x_0)\end{bmatrix}\\ &= \left(\frac{\dot c}{c}+\frac{\dot q}{q}\right)T + \dot{x_0}N.\end{aligned}$$

Note that from (2.14) and (2.11) we have $-\partial_x^{-1}\left(\Omega^{i-1} + \Omega^{i-3} + a\right)u' = \dot q/q$. Similarly for the case of $i = 2$ we also have $-u' + u^2 + 1 - au = \dot q/q$ from (2.10) and (2.11). Then from (2.13) we obtain

$$\dot T = \frac{\dot q}{q}T + aN,$$

which implies $c(t) = c(\text{const.})$ and $x_0 = A(t)$ where $\dot A(t) = a(t)$. Therefore we obtain:

Proposition 2.1. *Let $\gamma = \gamma(x,t)$, $t = (t_2, t_3, t_4, \ldots)$ be a position vector of the plane curve in the similarity geometry satisfying (2.4), (2.8) and (2.13). Then γ admits the representation formula*

$$\gamma = \int^x cq(x,t)\begin{bmatrix}\cos\theta(x,t)\\ \sin\theta(x,t)\end{bmatrix} dx, \quad \theta(x,t) = x + A(t), \tag{2.17}$$

where

$$\frac{\partial A(t)}{\partial t_i} = a(t), \quad i = 2,3,4,\ldots,$$

c is a constant, and $q(x,t)$ satisfies (2.15).

For a shock wave solution to the Burgers hierarchy, we can explictly construct the position vector. For a positive integer M,

$$q(x,t) = e^{t_2} + \sum_{k=1}^{M}\exp\left(\lambda_k x + \sum_{i=2}^{\infty}\left({\lambda_k}^{i} + {\lambda_k}^{i-2} + a\lambda_k\right)t_i + \xi_k\right),$$

solves the linear Eq. (2.15), where $\lambda_1, \xi_1, \ldots, \lambda_M, \xi_M$ are parameters. Then (2.17) gives

$$\begin{aligned}\gamma(x,t) &= \int^x c\, q(x,t)\begin{bmatrix}\cos\theta(x,t)\\ \sin\theta(x,t)\end{bmatrix} dx\\ &= c\sum_{k=0}^{M}\frac{\exp\left(\lambda_k x + \sum_{i=2}^{\infty}(\lambda_k^i + \lambda_k^{i-2} + a)t_i + \xi_k\right)}{1+{\lambda_k}^2}\\ &\quad\times\begin{bmatrix}\lambda_k\cos\theta + \sin\theta\\ \lambda_k\sin\theta - \cos\theta\end{bmatrix},\end{aligned} \tag{2.18}$$

where $\lambda_0 = \xi_0 = 0$. Figures 1 and 2 illustrate motion of plane curves corresponding to M-shock wave solutions ($M = 1, 2$, respectively) of the Burgers Eq. (2.9) with $t_i = 0\ (i \geq 3)$.

Remark 2.2. *The parameter a originally arises as an integration constant in (2.6), and play a role of rotation in the deformation of smooth curves as seen in Proposition 2.1. This parameter can be formally absorbed by a suitable linear transformation of independent variables (see, for example, (2.14) and (2.15)). In the discrete case, however, such manipulation is not applicable since the chain rule does not work effectively. Actually the similar parameter appears in a non-trivial manner in the deformation of discrete curves as shown in Section 3.*

3 Isogonal deformation of discrete curves

In this section, we consider the discrete deformation of discrete plane curves under the similarity geometry, which naturally gives rise to the discrete Burgers equation and its hierarchy. For the definition and fundamental properties of the discrete Burgers hierarchy, the readers may refer to Appendix B.

3.1 Discrete curve

For a map $\gamma : \mathbb{Z} \to \mathbb{R}^2$, $n \mapsto \gamma_n$, if any consecutive three points $\gamma_{n+1}, \gamma_n, \gamma_{n-1}$ are not colinear, we call γ a *discrete plane curve*. For a discrete plane curve γ, we denote by q_n the distance between the adjacent vertices

$$q_n = \left|\gamma_{n+1} - \gamma_n\right|.$$

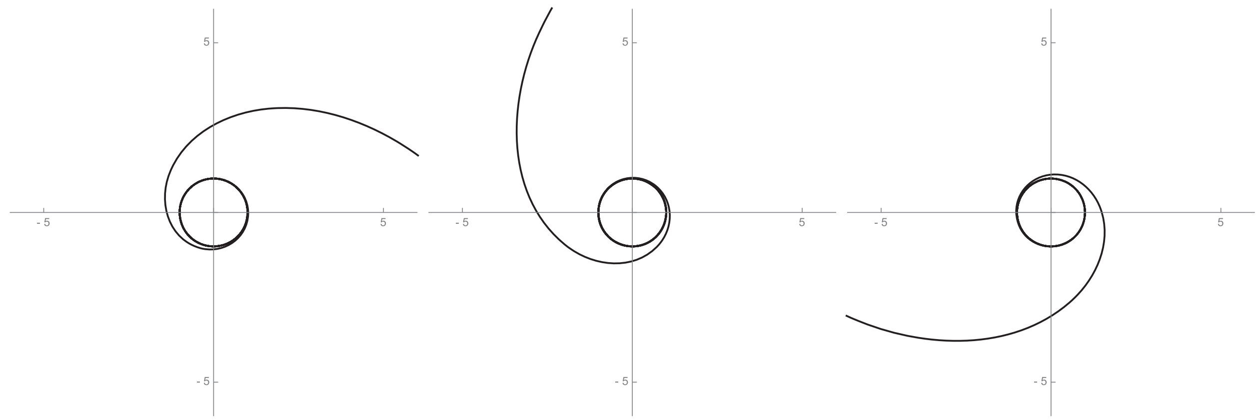

Fig. 1 Motion of plane curves $e^{-t_2}\gamma\,(x,t)$ corresponding to a 1-shock wave solution of the Burgers equation (2.9). Parameters are $c=1$, $a=0$, $\lambda_1=-1$, $\xi_1=0$ and $t_2=-8$ (*left*), 0 (*middle*), 8 (*right*)

We introduce κ_n as the angle between the two vectors $\gamma_n-\gamma_{n-1}$, $\gamma_{n+1}-\gamma_n$. More precisely, we define $\kappa:\mathbb{Z}\to(0,2\pi)$ by

$$\frac{\gamma_{n+1}-\gamma_n}{q_n}=R\left(\kappa_n\right)\frac{\gamma_n-\gamma_{n-1}}{q_{n-1}},$$

where R is the rotation matrix

$$R(x)=\begin{bmatrix}\cos x & -\sin x\\ \sin x & \cos x\end{bmatrix}.$$

Moreover, we put

$$T_n=\gamma_{n+1}-\gamma_n,\quad N_n=R\left(\frac{\pi}{2}\right)T_n,$$

and introduce the map $\phi\colon\mathbb{Z}\to\mathrm{CO}(2)$ by

$$\phi_n=[T_n,N_n]=q_n\left[\,\frac{\gamma_{n+1}-\gamma_n}{q_n},\ R\left(\frac{\pi}{2}\right)\frac{\gamma_{n+1}-\gamma_n}{q_n}\,\right].$$

We call the map ϕ the *similarity Frenet frame* of the discrete plane curve γ.

Proposition 3.1. *The similairity Frenet frame ϕ satisfies the linear difference equation*

$$\phi_{n+1}=\phi_nX_n,\quad X_n=\frac{q_{n+1}}{q_n}R\left(\kappa_{n+1}\right). \tag{3.1}$$

Proof. Since T satisfies

$$\frac{1}{q_{n+1}}T_{n+1}=R\left(\kappa_{n+1}\right)\frac{1}{q_n}T_n,$$

we have

$$\phi_{n+1}=[T_{n+1},N_{n+1}]=\frac{q_{n+1}}{q_n}R\left(\kappa_{n+1}\right)[T_n,N_n]=X_n\phi_n.$$

Since the rotation matrix $R\left(\kappa_{n+1}\right)$ and the matrix ϕ_n commute with each other, the statement is proved. □

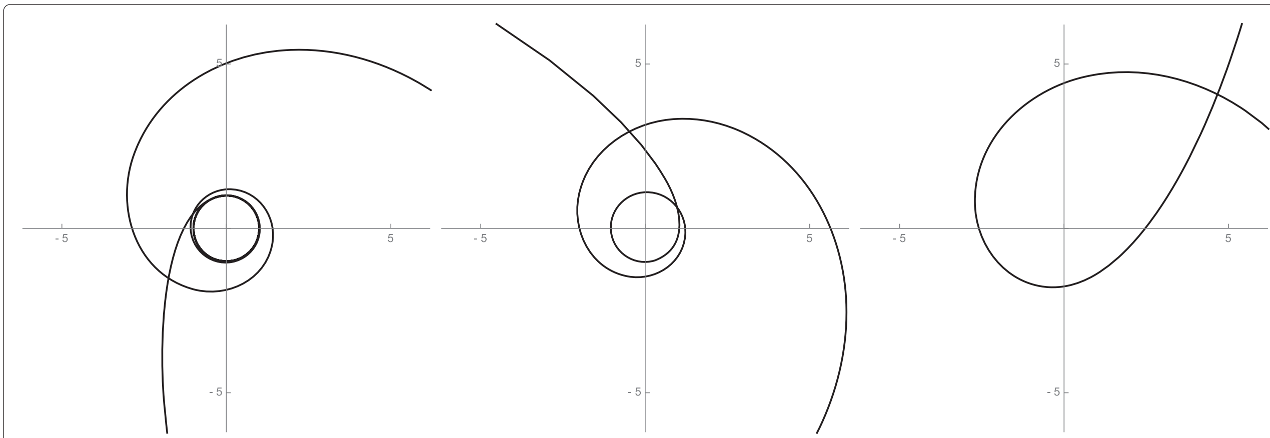

Fig. 2 Motion of plane curves $e^{-t_2}\gamma\,(x,t)$ corresponding to a 2-shock wave solution of the Burgers equation (2.9). Parameters are $c=1$, $a=\pi/4$, $\lambda_1=-1/2$, $\lambda_2=4$, $\xi_1=\xi_2=0$ and $t_2=-12$ (*left*), -2 (*middle*), $-1/10$ (*right*)

3.2 Isogonal deformation

3.2.1 General settings

We next consider the deformation of the curves. We write the deformed curve as $\overline{\gamma}$, and we also express the data associated with $\overline{\gamma}$ by putting $\overline{}$. For instance, we define the function $\overline{\kappa}: \mathbb{Z} \to (0, 2\pi)$ by

$$\frac{\overline{\gamma}_{n+1} - \overline{\gamma}_n}{\overline{q}_n} = R\left(\overline{\kappa}_n\right) \frac{\overline{\gamma}_n - \overline{\gamma}_{n-1}}{\overline{q}_{n-1}}, \quad \overline{q}_n = \left|\overline{\gamma}_{n+1} - \overline{\gamma}_n\right|. \quad (3.2)$$

Lemma 3.2. *The necessary and sufficient condition for the deformation $\gamma \mapsto \overline{\gamma}$ being isogonal, namely, $\overline{\kappa} = \kappa$, is that there exist a positive-valued function H and a constant a satisfying*

$$\overline{T}_n = H_n \phi_n \begin{bmatrix} \cos\ a \\ \sin\ a \end{bmatrix}. \quad (3.3)$$

Proof. Since both $\overline{T}$ and T are planar vectors, it is obvious that there exist a positive-valued function H and an angle a such that

$$\overline{T}_n = H_n R\left(a_n\right) T_n = H_n \phi_n \begin{bmatrix} \cos\ a_n \\ \sin\ a_n \end{bmatrix}.$$

Therefore the equality $\overline{\kappa} = \kappa$ holds if and only if the angle a is independent of n. □

Proposition 3.3. *We fix $\delta \in \mathbb{R}_{>0}$, $a, f_0, g_0 \in \mathbb{R}$ and a positive-valued function H. We introduce the functions f, g by the recursion relation*

$$\begin{bmatrix} f_{n+1} \\ g_{n+1} \end{bmatrix} = \frac{1}{\delta} \frac{q_n}{q_{n+1}} R\left(-\kappa_{n+1}\right) \begin{bmatrix} 1 + \delta f_n - H_n \cos a \\ \delta g_n - H_n \sin a \end{bmatrix}, \quad (3.4)$$

and define the deformation $\gamma \mapsto \overline{\gamma}$ by

$$\overline{\gamma}_n = \gamma_n - \delta\left(f_n T_n + g_n N_n\right). \quad (3.5)$$

Then we have the following:

(1) The deformation is isogonal. Namely, for the angle $\overline{\kappa}_n$ defined by (3.2), we have $\overline{\kappa}_n = \kappa_n$.
(2) The similarity Frenet frame $\overline{\phi}$ of the discrete curve $\overline{\gamma}$ can be expresed in terms the frame ϕ of γ as

$$\overline{\phi}_n = \phi_n Y_n, \quad Y_n = H_n R(a).$$

Proof. We compute the difference of $\overline{\gamma}$ by using (3.5), (3.1) and (3.4)

$$\begin{aligned}
\overline{T}_n = \overline{\gamma}_{n+1} - \overline{\gamma}_n = {} & \gamma_{n+1} - \delta\left(f_{n+1} T_{n+1} + g_{n+1} N_{n+1}\right) \\
& - \gamma_n + \delta\left(f_n T_n + g_n N_n\right) \\
= {} & \left\{1 - \delta \frac{q_{n+1}}{q_n}\left(f_{n+1} \cos \kappa_{n+1}\right.\right. \\
& \left.\left. - g_{n+1} \sin \kappa_{n+1}\right) + \delta f_n\right\} T_n \\
& + \delta\left\{-\frac{q_{n+1}}{q_n}\left(f_{n+1} \sin \kappa_{n+1}\right.\right. \\
& \left.\left. + g_{n+1} \cos \kappa_{n+1}\right) + g_n\right\} N_n \\
= {} & H_n\left(\cos a\, T_n + \sin a\, N_n\right).
\end{aligned}$$

Then we have (3.3), which means $\overline{\kappa} = \kappa$. The frame of $\overline{\gamma}$ satisfies

$$\overline{\phi}_n = \left[\overline{T}_n, R\left(\frac{\pi}{2}\right) \overline{T}_n\right] = H_n \phi_n R(a),$$

which completes the proof. □

Repeating the deformation in Proposition 3.3, we have the sequence of isogonal deformations of discrete plane curves $\gamma^0 = \gamma, \gamma^1 = \overline{\gamma}, \ldots, \gamma^m = \overline{\gamma^{m-1}}, \ldots$. We write

$$\begin{gathered}
q_n^m = \left|\gamma_{n+1}^m - \gamma_n^m\right|, \\
T_n^m = \gamma_{n+1}^m - \gamma_n^m, \quad N_n^m = R\left(\frac{\pi}{2}\right) T_n^m, \\
\frac{\gamma_{n+1}^m - \gamma_n^m}{q_n^m} = R\left(\kappa_n^m\right) \frac{\gamma_n^m - \gamma_{n-1}^m}{q_{n-1}^m}.
\end{gathered}$$

Proposition 3.4. *Let κ be the angles associated with the discrete curve γ^0. For each $m \in \mathbb{Z}$, we fix $\delta_m > 0$, real numbers a_m, f_0^m, g_0^m, and the positive-valued function H^m, and we introduce the functions f^m, g^m by the recursion relation*

$$\begin{bmatrix} f_{n+1}^m \\ g_{n+1}^m \end{bmatrix} = \frac{1}{\delta_m} \frac{q_n^m}{q_{n+1}^m} R\left(-\kappa_{n+1}\right) \begin{bmatrix} 1 + \delta_m f_n^m - H_n^m \cos a_m \\ \delta_m g_n^m - H_n^m \sin a_m \end{bmatrix}. \quad (3.6)$$

Then defining the discrete curves γ^m by

$$\gamma_n^{m+1} = \gamma_n^m - \delta_m\left(f_n^m T_n^m + g_n^m N_n^m\right), \quad (3.7)$$

we have the following:

(1) *For each m, it holds that* $\kappa^m = \kappa^0 = \kappa$.
(2) *The similarity Frenet frames* ϕ^m, ϕ^{m+1} *satisfy the system of linear difference equations*

$$\phi^m_{n+1} = \phi^m_n X^m_n, \quad X^m_n = \frac{q^m_{n+1}}{q^m_n} R\left(\kappa_{n+1}\right), \tag{3.8}$$

$$\phi^{m+1}_n = \phi^m_n Y^m_n, \quad Y^m_n = H^m_n R\left(a_m\right). \tag{3.9}$$

(3) *The compatibility condition of the system of linear difference Eqs. (3.8)–(3.9) is*

$$\frac{q^{m+1}_{n+1}}{q^{m+1}_n}\frac{q^m_n}{q^m_{n+1}} = \frac{H^m_{n+1}}{H^m_n}. \tag{3.10}$$

3.2.2 Discrete Burgers flow

Let us consider a special case where the function κ is a constant $\kappa_n = \epsilon$. For each $m \in \mathbb{Z}$, let δ_m be a positive constant, and we set

$$a_m = 0, \quad f^m_0 = \frac{1}{\epsilon^2}\left(\frac{q^m_{-1}}{q^m_0} - \cos\epsilon\right), \quad g^m_0 = \frac{\sin\epsilon}{\epsilon^2},$$
$$H^m_n = 1 + \frac{\delta_m}{\epsilon^2}\left(\frac{q^m_{n+1}}{q^m_n} - 2\cos\epsilon + \frac{q^m_{n-1}}{q^m_n}\right). \tag{3.11}$$

Then the solution of the difference Eq. (3.6) is given by

$$f^m_n = \frac{1}{\epsilon^2}\left(\frac{q^m_{n-1}}{q^m_n} - \cos\epsilon\right), \quad g^m_n = \frac{\sin\epsilon}{\epsilon^2}.$$

Defining the deformation of the discrete curve by (3.7) by using this solution, the compatibility condition (3.10) yields that the ratio $u^m_n = q^m_{n+1}/q^m_n$ obeys (a variant of) the discrete Burgers equation (see (B.3) with $i = 2$)

$$\frac{u^{m+1}_n}{u^m_n} = \frac{1 + \frac{\delta_m}{\epsilon^2}\left(u^m_{n+1} - 2\cos\epsilon + \frac{1}{u^m_n}\right)}{1 + \frac{\delta_m}{\epsilon^2}\left(u^m_n - 2\cos\epsilon + \frac{1}{u^m_{n-1}}\right)}. \tag{3.12}$$

The length $q^m_n = \left|\gamma^m_{n+1} - \gamma^m_n\right|$ satisfy the linear difference equation

$$\frac{q^{m+1}_n - q^m_n}{\delta_m} = \frac{q^m_{n+1} - 2q^m_n\cos\epsilon + q^m_{n-1}}{\epsilon^2}.$$

Remark 3.5. *The function* H^m_n *defined by (3.11) is not necessarily positive in general. However, it is possible to make it positive by choosing* $\delta_m > 0$ *appropriately as follows. We put*

$$Q_m = \min_n \frac{q^m_{n+1} + q^m_{n-1}}{q^m_n} = \min_n\left(u^m_{n+1} + \frac{1}{u^m_{n-1}}\right).$$

If $Q_m \geq 2\cos\epsilon$, *then we have for arbitrary n*

$$\frac{q^m_{n+1} + q^m_{n-1}}{q^m_n} - 2\cos\epsilon \geq Q_m - 2\cos\epsilon \geq 0,$$

which gives $H^m_n > 0$. *If* $Q_m < 2\cos\epsilon$, *then choose* δ_m *as*

$$\frac{\epsilon^2}{2\cos\epsilon - Q_m} > \delta_m > 0,$$

then H^m_n *becomes positive. In fact, we have for arbitrary n*

$$H^m_n = 1 + \frac{\delta_m}{\epsilon^2}\left(\frac{q^m_{n+1} + q^m_{n-1}}{q^m_n} - 2\cos\epsilon\right)$$
$$\geq 1 + \frac{\delta_m}{\epsilon^2}\left(Q_m - 2\cos\epsilon\right) > 0.$$

3.2.3 Discrete Burgers flow of higher order

Let us write down the deformation equation corresponding to (2.12). From (3.6) we have that

$$\begin{aligned} u^m_n f^m_{n+1} =& f^m_n\cos\kappa_{n+1} + g^m_n\sin\kappa_{n+1} \\ &+ \frac{\cos\kappa_{n+1} - H^m_n\cos\left(\kappa_{n+1} - a_m\right)}{\delta_m}, \\ u^m_n g^m_{n+1} =& -f^m_n\sin\kappa_{n+1} + g^m_n\cos\kappa_{n+1} \\ &- \frac{\sin\kappa_{n+1} - H^m_n\sin(\kappa_{n+1} - a_m)}{\delta_m}. \end{aligned}$$

We solve the second equation in terms of f^m_n and substitute it into the first equation with $n \mapsto n-1$ so as to obtain that

$$\begin{aligned} &\frac{\sin(\kappa_{n+1} - a_m)}{\sin\kappa_{n+1}}H^m_n + \frac{\sin a_m}{\sin\kappa_n}\frac{1}{u^m_{n-1}}H^m_{n-1} \\ &= 1 + \delta_m\left\{\frac{1}{\sin\kappa_{n+1}}u^m_n g^m_{n+1} - \left(\frac{\cos\kappa_{n+1}}{\sin\kappa_{n+1}} + \frac{\cos\kappa_n}{\sin\kappa_n}\right)g^m_n \right.\\ &\quad \left. + \frac{1}{\sin\kappa_n}\frac{1}{u^m_{n-1}}g^m_{n-1}\right\}. \end{aligned}$$

Then the compatibility condition (3.10) gives

$$\frac{u_n^{m+1}}{u_n^m}\frac{\frac{\sin(\kappa_{n+2}-a_m)}{\sin\kappa_{n+2}}+\frac{\sin a_m}{\sin\kappa_{n+1}}\frac{1}{u_n^{m+1}}}{\frac{\sin(\kappa_{n+1}-a_m)}{\sin\kappa_{n+1}}+\frac{\sin a_m}{\sin\kappa_n}\frac{1}{u_{n-1}^{m+1}}}$$
$$=\frac{1+\delta_m\left\{\frac{1}{\sin\kappa_{n+2}}u_{n+1}^m g_{n+2}^m-\left(\frac{\cos\kappa_{n+2}}{\sin\kappa_{n+2}}+\frac{\cos\kappa_{n+1}}{\sin\kappa_{n+1}}\right)g_{n+1}^m+\frac{1}{\sin\kappa_{n+1}}\frac{1}{u_n^m}g_n^m\right\}}{1+\delta_m\left\{\frac{1}{\sin\kappa_{n+1}}u_n^m g_{n+1}^m-\left(\frac{\cos\kappa_{n+1}}{\sin\kappa_{n+1}}+\frac{\cos\kappa_n}{\sin\kappa_n}\right)g_n^m+\frac{1}{\sin\kappa_n}\frac{1}{u_{n-1}^m}g_{n-1}^m\right\}}, \tag{3.13}$$

or equivalently

$$\frac{u_n^{m+1}}{u_n^m}\frac{\frac{\sin(\kappa_{n+2}-a_m)}{\sin\kappa_{n+2}}+\frac{\sin a_m}{\sin\kappa_{n+1}}\frac{1}{u_n^{m+1}}}{\frac{\sin(\kappa_{n+1}-a_m)}{\sin\kappa_{n+1}}+\frac{\sin a_m}{\sin\kappa_n}\frac{1}{u_{n-1}^{m+1}}}$$
$$=\frac{1+\delta_m\left(\frac{1}{\sin\kappa_{n+2}}u_{n+1}^m e^{\partial_n}-\frac{1}{\sin\kappa_{n+2}}-\frac{1}{\sin\kappa_{n+1}}+\frac{1}{\sin\kappa_{n+1}}\frac{1}{u_n^m}e^{-\partial_n}\right)g_{n+1}^m+\delta_m\left(\tan\frac{\kappa_{n+2}}{2}+\tan\frac{\kappa_{n+1}}{2}\right)g_{n+1}^m}{1+\delta_m\left(\frac{1}{\sin\kappa_{n+1}}u_n^m e^{\partial_n}-\frac{1}{\sin\kappa_{n+1}}-\frac{1}{\sin\kappa_n}+\frac{1}{\sin\kappa_n}\frac{1}{u_{n-1}^m}e^{-\partial_n}\right)g_n^m+\delta_m\left(\tan\frac{\kappa_{n+1}}{2}+\tan\frac{\kappa_n}{2}\right)g_n^m}. \tag{3.14}$$

Equation (3.13) or (3.14) is the general form of the deformation equation of the discrete curves in the framework of the similarity geometry, and is regarded as a discrete counterpart of (2.12).

Theorem 3.6. *For a fixed $m\in\mathbb{Z}$, let $\gamma_n^m\in\mathbb{R}^2$ be a discrete curve, and let $\kappa_n=\angle(\gamma_{n+1}^m-\gamma_n^m,\gamma_n^m-\gamma_{n-1}^m)$, $q_n^m=|\gamma_{n+1}^m-\gamma_n^m|$, $u_n^m=\frac{q_{n+1}^m}{q_n^m}$. For given $\delta_m, a_m, H_0^m\in\mathbb{R}$ and a function $g_n^m\in\mathbb{R}$, we define $H_n^m\in\mathbb{R}$ recursively by*

$$\begin{aligned}&\frac{\sin(\kappa_{n+1}-a_m)}{\sin\kappa_{n+1}}H_n^m+\frac{\sin a_m}{\sin\kappa_n}\frac{1}{u_{n-1}^m}H_{n-1}^m\\&=1+\delta_m\left\{\frac{u_n^m g_{n+1}^m}{\sin\kappa_{n+1}}-\left(\frac{1}{\sin\kappa_{n+1}}+\frac{1}{\sin\kappa_n}\right)g_n^m\right.\\&\qquad\left.+\frac{1}{\sin\kappa_n}\frac{g_{n-1}^m}{u_{n-1}^m}\right\}+\delta_m\left(\tan\frac{\kappa_{n+1}}{2}+\tan\frac{\kappa_n}{2}\right)g_n^m.\end{aligned} \tag{3.15}$$

Then we have:

(1) By choosing δ_m and a_m appropriately, H_n^m becomes positive.
(2) Setting the function f_n^m by

$$\begin{aligned}f_n^m=&-\frac{u_n^m}{\sin\kappa_{n+1}}g_{n+1}^m+\frac{\cos\kappa_{n+1}}{\sin\kappa_{n+1}}g_n^m-\frac{1}{\delta_m}\\&+\frac{\sin(\kappa_{n+1}-a_m)}{\delta_m\sin\kappa_{n+1}}H_n^m,\end{aligned}$$

the condition (3.6) is satisfied. Namely, (3.5) gives a isogonal deformation.
(3) u_n^m satisfies (3.14).

We note that (3.14) yields the discrete Burgers equation and its generalizations to that of higher-order by suitable specialization of g_n^m.

Autonomous case In the case of $\kappa_n=\epsilon=$ const., (3.14) is reduced to

$$\begin{aligned}&\frac{u_n^{m+1}}{u_n^m}\frac{\sin(\epsilon-a_m)+\frac{\sin a_m}{u_n^{m+1}}}{\sin(\epsilon-a_m)+\frac{\sin a_m}{u_{n-1}^{m+1}}}\\&=\frac{1+\frac{\delta_m}{\sin\epsilon}\left(u_{n+1}^m e^{\partial_n}-2+\frac{1}{u_n^m}e^{-\partial_n}\right)g_{n+1}^m+\delta_m\frac{2-2\cos\epsilon}{\sin\epsilon}g_{n+1}^m}{1+\frac{\delta_m}{\sin\epsilon}\left(u_n^m e^{\partial_n}-2+\frac{1}{u_{n-1}^m}e^{-\partial_n}\right)g_n^m+\delta_m\frac{2-2\cos\epsilon}{\sin\epsilon}g_n^m}\end{aligned} \tag{3.16}$$

$$=\frac{1+\delta_m\frac{\epsilon^2}{\sin\epsilon}\Omega_{n+1}^{(2)}g_{n+1}^m+\delta_m\frac{2-2\cos\epsilon}{\sin\epsilon}g_{n+1}^m}{1+\delta_m\frac{\epsilon^2}{\sin\epsilon}\Omega_n^{(2)}g_n^m+\delta_m\frac{2-2\cos\epsilon}{\sin\epsilon}g_n^m}, \tag{3.17}$$

where $\Omega_n^{(2)}$ is the recursion operator of the discrete Burgers hierarchy given in (B.6). Putting $g_n=\frac{\sin\epsilon}{\epsilon^2}$ and $a_m=0$, (3.16) recovers the autonomous discrete Burgers Eq. 3.12. Equation 3.17 is a discrete counterpart of (2.12). Therefore, due to (B.5), by putting g_n^m as

$$g_n^m=\frac{\sin\epsilon}{\epsilon^2}\hat{K}^{(i)}[u_n^m],$$

we obtain a variant of the higher order autonomous discrete Burgers equation

$$\begin{aligned}&\frac{u_n^{m+1}}{u_n^m}\frac{1+\frac{\sin a_m}{\sin(\epsilon-a_m)}\frac{1}{u_n^{m+1}}}{1+\frac{\sin a_m}{\sin(\epsilon-a_m)}\frac{1}{u_{n-1}^{m+1}}}\\&\quad=\frac{1+\delta_m\hat{K}^{(i+2)}[u_{n+1}^m]+\frac{2-2\cos\epsilon}{\epsilon^2}\delta_m\hat{K}^{(i)}[u_{n+1}^m]}{1+\delta_m\hat{K}^{(i+2)}[u_n^m]+\frac{2-2\cos\epsilon}{\epsilon^2}\delta_m\hat{K}^{(i)}[u_n^m]},\end{aligned}$$

which corresponds to (2.14).

Non-autonomous case For the case of generic κ_n, we see that the recursion operator of the non-autonomous discrete Burgers hierarchy appears in the right hand side of (3.14). In fact, we have

$$\frac{1}{\sin\kappa_{n+1}}u_n^m e^{\partial_n}-\frac{1}{\sin\kappa_{n+1}}-\frac{1}{\sin\kappa_n}+\frac{1}{\sin\kappa_n}\frac{1}{u_{n-1}^m}e^{-\partial_n} = \epsilon_n^{(i+2)}\Omega_n^{(2,i+2)},$$

by parametrizing $\sin\kappa_n$ as

$$\sin\kappa_n=\begin{cases}\epsilon_{n-1}^{(i+1)} & i=2l,\\ \epsilon_n^{(i+1)} & i=2l+1,\end{cases}$$

where $\epsilon_n^{(i)}$ and $\Omega_n^{(i)}$ are given in (B.13) and (B.14), respectively. For the simplest case $i=0$, we choose $g_n^m=1$ in (3.14) and have

$$\frac{u_n^{m+1}}{u_n^m}\frac{\frac{\sin(\kappa_{n+2}-a_m)}{\sin\kappa_{n+2}}+\frac{\sin a_m}{\sin\kappa_{n+1}}\frac{1}{u_n^{m+1}}}{\frac{\sin(\kappa_{n+1}-a_m)}{\sin\kappa_{n+1}}+\frac{\sin a_m}{\sin\kappa_n}\frac{1}{u_{n-1}^{m+1}}} = \frac{1+\delta_m\epsilon_{n+1}^{(2)}K_{n+1}^{(2)}[u_{n+1}^m]+\delta_m\left(\tan\frac{\kappa_{n+2}}{2}+\tan\frac{\kappa_{n+1}}{2}\right)}{1+\delta_m\epsilon_n^{(2)}K_n^{(2)}[u_n^m]+\delta_m\left(\tan\frac{\kappa_{n+1}}{2}+\tan\frac{\kappa_n}{2}\right)},$$

which is a non-autonomous discrete analogue of the Burgers equation (2.9). If we set $a_m=0$, we obtain a simpler version of the non-autonomous discrete Burgers equation

$$\frac{u_n^{m+1}}{u_n^m}=\frac{1+\delta_m\epsilon_{n+1}^{(2)}K_{n+1}^{(2)}[u_{n+1}^m]+\delta_m\left(\tan\frac{\kappa_{n+2}}{2}+\tan\frac{\kappa_{n+1}}{2}\right)}{1+\delta_m\epsilon_n^{(2)}K_n^{(2)}[u_n^m]+\delta_m\left(\tan\frac{\kappa_{n+1}}{2}+\tan\frac{\kappa_n}{2}\right)}.$$

For $i>0$, we put $g_n^m=K_n^{(i)}[u_n^m]$ and find that u_n^m satisfies a variant of non-autonomous higher-order discrete Burgers equation

$$\frac{u_n^{m+1}}{u_n^m}\frac{\frac{\sin(\kappa_{n+2}-a_m)}{\sin\kappa_{n+2}}+\frac{\sin a_m}{\sin\kappa_{n+1}}\frac{1}{u_n^{m+1}}}{\frac{\sin(\kappa_{n+1}-a_m)}{\sin\kappa_{n+1}}+\frac{\sin a_m}{\sin\kappa_n}\frac{1}{u_{n-1}^{m+1}}} = \frac{1+\delta_m\epsilon_{n+1}^{(i+2)}K_{n+1}^{(i+2)}[u_{n+1}^m]+\delta_m\left(\tan\frac{\kappa_{n+2}}{2}+\tan\frac{\kappa_{n+1}}{2}\right)K_{n+1}^{(i)}[u_{n+1}^m]}{1+\delta_m\epsilon_n^{(i+2)}K_n^{(i+2)}[u_n^m]+\delta_m\left(\tan\frac{\kappa_{n+1}}{2}+\tan\frac{\kappa_n}{2}\right)K_n^{(i)}[u_n^m]}, \tag{3.18}$$

which is a non-autonomous discrete analogue of the higher-order Burgers Eq. 2.14. Note that $q_n^m=|\gamma_{n+1}^m-\gamma_n^m|$ satisfies the linear equation

$$\frac{1}{\delta_m}\left\{\frac{\sin(\kappa_{n+1}-a_m)}{\sin\kappa_{n+1}}q_n^{m+1}+\frac{\sin a_m}{\sin\kappa_n}q_{n-1}^{m+1}-q_n^m\right\} = \epsilon_n^{(i+2)}L_n^{(i+2)}\left[q_n^m\right]+\left(\tan\frac{\kappa_{n+1}}{2}+\tan\frac{\kappa_n}{2}\right)L_n^{(i)}\left[q_n^m\right].$$

We now prove Theorem 3.6. The statement (2) and (3) are derived immediately by solving (3.6) and using the compatibility condition (3.10). For the statement (1), we have the following as a sufficient condition for the positivity of H_n^m:

Lemma 3.7. *We assume that κ_n satisfies $0<\kappa_n<\pi$ or $-\pi<\kappa_n<0$ for all n. For each m, we choose δ_m and a_m in the following manner:*

$$\begin{cases}0<\delta_m & (U_{min}^m>0)\\ 0<\delta_m<-1/U_{min}^m & (U_{min}^m<0)\end{cases},\qquad \begin{cases}\kappa_{max}-\pi<a_m<0 & (0<\kappa_n<\pi)\\ 0<a_m<\kappa_{min}+\pi & (-\pi<\kappa_n<0)\end{cases}, \tag{3.19}$$

where

$$U_n^m=\frac{u_n^m g_{n+1}^m}{\sin\kappa_{n+1}}-\left(\frac{1}{\sin\kappa_{n+1}}+\frac{1}{\sin\kappa_n}\right)g_n^m+\frac{1}{\sin\kappa_n}\frac{g_{n-1}^m}{u_{n-1}^m} +\left(\tan\frac{\kappa_{n+1}}{2}+\tan\frac{\kappa_n}{2}\right)g_n^m,$$

and

$$U_{max}^m=\max_n U_n^m,\quad U_{min}^m=\min_n U_n^m,\quad \kappa_{min}=\min_n\kappa_n,$$
$$\kappa_{max}=\max_n\kappa_n.$$

Then we have $H_n^m>0$.

Proof. We first write the recursion relation (3.15) as

$$H_n^m=-\alpha_n^m H_{n-1}^m+\beta_n^m\left(1+\delta_m U_n^m\right), \tag{3.20}$$

where

$$\alpha_n^m=\frac{\sin a_m\sin\kappa_{n+1}}{u_{n-1}^m\sin(\kappa_{n+1}-a_m)\sin\kappa_n},$$
$$\beta_n^m=\frac{\sin\kappa_{n+1}}{\sin(\kappa_{n+1}-a_m)}.$$

Then (3.20) can be solved formally as

$$H_n^m=\left(H_0^m+\sum_{\nu=0}^n\beta_\nu^m(1+\delta_m U_\nu^m)\prod_{k=0}^{\nu}(-\alpha_k^m)^{-1}\right) \times\prod_{\mu=0}^n(-\alpha_\mu^m),\quad H_0^m>0.$$

Noticing that $\delta_m, u_n^m>0$, it is sufficient for $H_n^m>0$ that all of the following conditions

$$\frac{\sin\kappa_{n+1}}{\sin(\kappa_{n+1}-a_m)}>0, \tag{3.21}$$
$$1+\delta_m U_n^m>0, \tag{3.22}$$
$$\prod_{\nu=0}^n\left(-\frac{\sin a_m\sin\kappa_{\nu+1}}{\sin(\kappa_{\nu+1}-a_m)\sin\kappa_\nu}\right)>0, \tag{3.23}$$

are satisfied for all n. Then it is easy to see that (3.22) is satisfied by choosing δ_m as (3.19). The conditions (3.21) and (3.22) imply

$$\sin\kappa_n>0,\quad \sin(\kappa_n-a_m)>0,\quad \sin a_m<0\quad \text{for }^\forall n,$$

or

$$\sin\kappa_n<0,\quad \sin(\kappa_n-a_m)<0,\quad \sin a_m>0\quad \text{for }^\forall n, \tag{3.24}$$

from which we have

$$0 < \kappa_n < \pi, \quad \kappa_{\max} - a_m < \pi, \quad -\pi < a_m < 0,$$

or

$$-\pi < \kappa_n < 0, \quad -\pi < \kappa_{\min} - a_m, \quad 0 < a_m < \pi. \tag{3.25}$$

This is equivalent to the second condition in (3.19). □

3.3 Explicit formula

An explicit representation formula for the curve γ_n^m is constructed in a similar manner to the smooth curves.

Proposition 3.8. *Let γ_n^m be a discrete curve satisfying (3.8) and (3.9). Then γ_n^m admits the representation formula*

$$\gamma_n^m = \sum_j^{n-1} q_j^m \begin{bmatrix} \cos\theta_j^m \\ \sin\theta_j^m \end{bmatrix}, \quad \theta_n^m = \sum_\nu^n \kappa_\nu + \sum_\mu^m a_\mu. \tag{3.26}$$

Proof. Since $|\gamma_{n+1}^m - \gamma_n^m|/q_n^m = 1$, there exist a function $\theta_n^m \in [0, 2\pi)$ such that

$$\frac{\gamma_{n+1}^m - \gamma_n^m}{q_n^m} = \begin{bmatrix} \cos\theta_n^m \\ \sin\theta_n^m \end{bmatrix}, \tag{3.27}$$

so that the frame ϕ_n^m is expressed as

$$\phi_n^m = q_n^m R(\theta_n^m). \tag{3.28}$$

Then (3.8) and (3.9) give

$$\theta_{n+1}^m - \theta_n^m - \kappa_n \in 2\pi\mathbb{Z}, \quad \theta_n^{m+1} - \theta_n^m - a_m \in 2\pi\mathbb{Z}, \tag{3.29}$$

and we may assume

$$\theta_{n+1}^m - \theta_n^m - \kappa_n = 0, \quad \theta_n^{m+1} - \theta_n^m - a_m = 0, \tag{3.30}$$

without losing generality, which implies (3.26). □

For the curves constructed from the shock wave solutions of the autonomous discrete Burgers hierarchy, the summation in (3.26) can be computed explicitly. For simplicity, we demonstrate it by taking the case of $i = 2$ with $\kappa_n = \epsilon$ (const.), $\delta_m = \delta$ (const.) in (3.18)

$$\begin{aligned} \frac{u_n^{m+1}}{u_n^m} \frac{\frac{\sin(\epsilon - a_m)}{\sin\epsilon} + \frac{\sin a_m}{\sin\epsilon}\frac{1}{u_n^{m+1}}}{\frac{\sin(\epsilon - a_m)}{\sin\epsilon} + \frac{\sin a_m}{\sin\epsilon}\frac{1}{u_{n-1}^{m+1}}} \\ = \frac{1 + \frac{\delta_m}{\epsilon^2}\left(u_{n+1}^m - 2\cos\epsilon + \frac{1}{u_n^m}\right)}{1 + \frac{\delta_m}{\epsilon^2}\left(u_n^m - 2\cos\epsilon + \frac{1}{u_{n-1}^m}\right)}, \end{aligned} \tag{3.31}$$

which is linearized in terms of q_n^m as

$$\begin{aligned} \frac{\frac{\sin(\epsilon - a_m)}{\sin\epsilon} q_n^{m+1} + \frac{\sin a_m}{\sin\epsilon} q_{n-1}^{m+1} - q_n^m}{\delta} \\ = \frac{q_{n+1}^m - 2q_n^m \cos\epsilon + q_{n-1}^m}{\epsilon^2}. \end{aligned} \tag{3.32}$$

(3.32) admits the solution

$$q_n^m = e^{\mu_0 m} + \sum_{k=1}^M \exp(\lambda_k n + \mu_k m + \xi_k), \tag{3.33}$$

where $M \in \mathbb{N}$, λ_k, ξ_k $(k = 1, \ldots, M)$ are arbitrary constants and

$$\mu_k = \log \frac{1 + \frac{\delta}{\epsilon^2}\left(e^{\lambda_k} - 2\cos\epsilon + e^{-\lambda_k}\right)}{\frac{\sin(\epsilon - a_m)}{\sin\epsilon} + \frac{\sin a_m}{\sin\epsilon} e^{-\lambda_k}}. \tag{3.34}$$

Then, by using the formulas

$$\sum_j^{n-1} c^j \cos(j\epsilon) = \frac{c^{n+1}\cos((n-1)\epsilon) - c^n \cos(n\epsilon)}{c^2 - 2c\cos\epsilon + 1},$$

$$\sum_j^{n-1} c^j \sin(j\epsilon) = \frac{c^{n+1}\sin((n-1)\epsilon) - c^n \sin(n\epsilon)}{c^2 - 2c\cos\epsilon + 1},$$

where c is a constant satisfying $c^j \to 0$ $(j \to -\infty)$, we have from (3.26) that

$$\begin{aligned} \gamma_n^m = \sum_{k=0}^M \frac{\exp(\lambda_k n + \mu_k m + \xi_k)}{e^{2\lambda_k} - 2e^{\lambda_k}\cos\epsilon + 1} \\ \times \begin{bmatrix} e^{\lambda_k}\cos\theta_{n-1}^m - \cos\theta_n^m \\ e^{\lambda_k}\sin\theta_{n-1}^m - \sin\theta_n^m \end{bmatrix}, \end{aligned} \tag{3.35}$$

with $\lambda_0 = \xi_0 = 0$ and $\lambda_k > 0$.

Figures 3, 4 illustrate motion of discrete plane curves corresponding to M-shock wave solutions ($M = 1, 2$, respectively) of the discrete Burgers equation (3.31) with parameters $a = \pi/3$, $\epsilon = \pi/4$, $\delta = 1$, $\xi_1 = 0$.

Appendixes

A Burgers hierarchy

The Burgers hierarchy is the family of nonlinear partial differential equations obtained from the linear partial differential equations

$$\frac{\partial q}{\partial t_i} = \frac{\partial^i q}{\partial x^i}, \quad i = 1, 2, 3, \ldots \tag{A.1}$$

through the Cole-Hopf transformation

$$u = -\frac{\partial}{\partial x}\log q. \tag{A.2}$$

By noticing that

$$q = e^{-\int u\,dx}, \tag{A.3}$$

the nonlinear equations in the hierachy are expressed [2] as

$$\begin{aligned} \frac{\partial u}{\partial t_i} &= K_i[u], \\ K_i[u] &= -\frac{\partial}{\partial x}\left(e^{\int u\,dx}\frac{\partial^i}{\partial x^i}e^{-\int u\,dx}\right). \end{aligned} \tag{A.4}$$

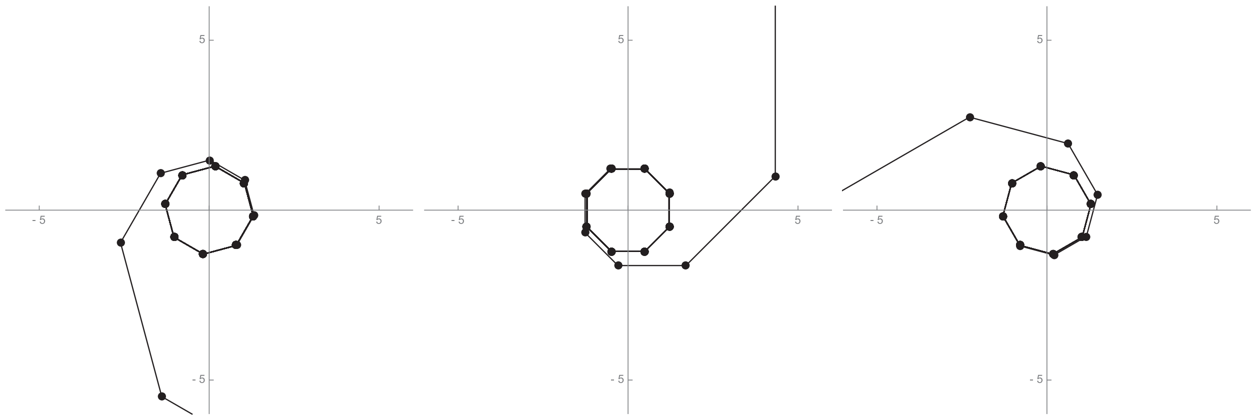

Fig. 3 Motion of discrete plane curves $e^{-\mu_0 m}\gamma_n^m$ corresponding to a 1-shock wave solution of the discrete Burgers Eq. 3.31. Parameters are $\lambda_1 = 1$, and $m = -8$ (*left*), 0 (*middle*), 8 (*right*)

Some of the flows of the hierarchy are given by

$$\begin{aligned} i = 1: &\quad K_1[u] = u', \\ i = 2: &\quad K_2[u] = u'' - 2uu', \\ i = 3: &\quad K_3[u] = u''' - (u')^2 - uu'' + 3u^2u'. \end{aligned}$$

An elementary calculation shows the following relation between $K_i[u]$ and $K_{i-1}[u]$:

$$K_i[u] = \Omega K_{i-1}[u], \quad \Omega = \partial_x - u - u'\partial_x^{-1}. \tag{A.5}$$

Here, Ω is called the recursion operator of the Burgers hierarchy, by which the equations in the hierarchy can be expressed as

$$\frac{\partial u}{\partial t_i} = \Omega^{i-1}K_1[u] = \Omega^{i-1}u', \quad i \geq 2. \tag{A.6}$$

B Discrete Burgers hierachy

B.1 Discrete Burgers hierarchy

Let δ, ϵ be constants. For $i = 0, 1, 2, \ldots$, we consider the family of linear difference equations

$$\frac{q_n^{m+1} - q_n^m}{\delta} = \widehat{L}^{(i)}\left[q_n^m\right], \tag{B.1}$$

where

$$\widehat{L}^{(i)}[\,q_n^m] = \begin{cases} \Delta^i q_n^m & i = 2l, \\ e^{\partial_n/2}\Delta^i q_n^m & i = 2l+1. \end{cases}$$

Here Δ is a central-difference operator in n defined as

$$\Delta = \frac{e^{\partial_n/2} - e^{-\partial_n/2}}{\epsilon}.$$

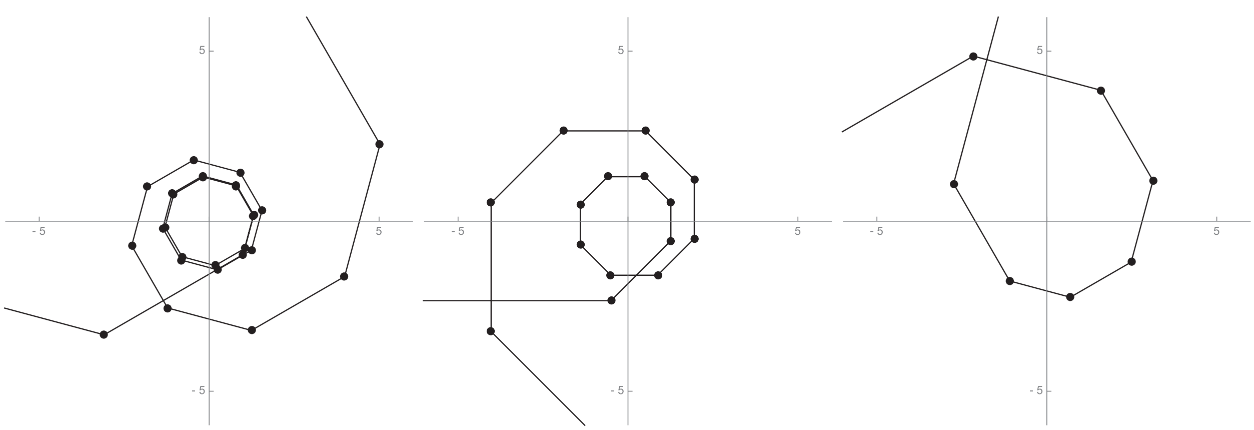

Fig. 4 Motion of discrete plane curves $e^{-\mu_0 m}\gamma_n^m$ corresponding to a 2-shock wave solution of the discrete Burgers Eq. 3.31. Parameters are $\lambda_1 = 1/3$, $\lambda_2 = -3$, $\xi_2 = 0$, and $m = -13$ (*left*), -6 (*middle*), -1 (*right*)

The first few examples of $\widehat{L}^{(i)}\left[q_n^m\right]$ are given by

$$\begin{aligned}
i=0: &\quad \widehat{L}^{(0)}\left[q_n^m\right]=q_n^m,\\
i=1: &\quad \widehat{L}^{(1)}\left[q_n^m\right]=\frac{q_{n+1}^m-q_n^m}{\epsilon},\\
i=2: &\quad \widehat{L}^{(2)}\left[q_n^m\right]=\frac{q_{n+1}^m-2q_n^m+q_{n-1}^m}{\epsilon^2},\\
i=3: &\quad \widehat{L}^{(3)}\left[q_n^m\right]=\frac{q_{n+2}^m-3q_{n+1}^m+3q_n^m-q_{n-1}^m}{\epsilon^3}.
\end{aligned}$$

The *discrete Burgers hierarchy* is a family of nonlinear difference equations obtained from (B.1) by the discrete Cole-Hopf transformation [15]

$$u_n^m=\frac{q_{n+1}^m}{q_n^m}. \tag{B.2}$$

Noticing that

$$q_n^m=\prod_k^{n-1}u_k^m,$$

the i-th order equation in the hierarchy can be written as

$$\frac{u_n^{m+1}}{u_n^m}=\frac{1+\delta\widehat{K}^{(i)}\left[u_{n+1}^m\right]}{1+\delta\widehat{K}^{(i)}\left[u_n^m\right]}, \tag{B.3}$$

where

$$\widehat{K}^{(i)}\left[u_n^m\right]=\frac{1}{q_n^m}\widehat{L}^{(i)}\left[q_n^m\right].$$

For instance, the first few $\widehat{K}^{(i)}$ are given by

$$\begin{aligned}
i=0: &\quad \widehat{K}^{(0)}\left[u_n^m\right]=1,\\
i=1: &\quad \widehat{K}^{(1)}\left[u_n^m\right]=\frac{1}{\epsilon}\left(u_n^m-1\right),\\
i=2: &\quad \widehat{K}^{(2)}\left[u_n^m\right]=\frac{1}{\epsilon^2}\left(u_n^m-2+\frac{1}{u_{n-1}^m}\right),\\
i=3: &\quad \widehat{K}^{(3)}\left[u_n^m\right]=\frac{1}{\epsilon^3}\left(u_{n+1}^m u_n^m-3u_n^m+3-\frac{1}{u_{n-1}^m}\right).
\end{aligned}$$

The discrete Burgers hierarchy admits the *recursion operators* which generate higher order flows from lower ones.

Proposition B.1. *It holds that*

$$\widehat{K}^{(i+1)}\left[u_n^m\right]=\Omega_n\widehat{K}^{(i)}\left[u_n^m\right],$$

where Ω_n is a difference operator defined by

$$\Omega_n=\begin{cases}\Omega_n^{(\mathrm{odd})}=\frac{1}{\epsilon}\left(u_n^m e^{\partial_n}-1\right) & i=2l,\\ \Omega_n^{(\mathrm{even})}=\frac{1}{\epsilon}\left(1-\frac{1}{u_{n-1}^m}e^{-\partial_n}\right) & i=2l+1.\end{cases} \tag{B.4}$$

In particular, we have

$$\widehat{K}^{(i+2)}\left[u_n^m\right]=\Omega_n^{(2)}\widehat{K}^{(i)}\left[u_n^m\right], \tag{B.5}$$

where

$$\begin{aligned}
\Omega_n^{(2)}&=\Omega_n^{(\mathrm{odd})}\Omega_n^{(\mathrm{even})}=\Omega_n^{(\mathrm{even})}\Omega_n^{(\mathrm{odd})}\\
&=\frac{1}{\epsilon^2}\left(u_n^m e^{\partial_n}-2+\frac{1}{u_{n-1}^m}e^{-\partial_n}\right).
\end{aligned} \tag{B.6}$$

Proof. Since we have

$$\begin{aligned}
\widehat{L}^{(2l+1)}\left[q_n^m\right]&=\frac{e^{\partial_n}-1}{\epsilon}\widehat{L}^{(2l)}\left[q_n^m\right],\quad \widehat{L}^{(2l+2)}\left[q_n^m\right]\\
&=\frac{1-e^{-\partial_n}}{\epsilon}\widehat{L}^{(2l+1)}\left[q_n^m\right],
\end{aligned}$$

from (B.1), we obtain

$$\begin{aligned}
q_n^m\widehat{K}^{(2l+1)}\left[q_n^m\right]&=\frac{e^{\partial_n}-1}{\epsilon}\left(q_n^m\widehat{K}^{(2l)}\left[u_n^m\right]\right)\\
&=\frac{1}{\epsilon}\left(q_{n+1}^m e^{\partial_n}-q_n^m\right)\widehat{K}^{(2l)}\left[q_n^m\right],\\
q_n^m\widehat{K}^{(2l+2)}\left[u_n^m\right]&=\frac{1-e^{-\partial_n}}{\epsilon}\left(q_n^m\widehat{K}^{(2l+1)}\left[u_n^m\right]\right)\\
&=\frac{1}{\epsilon}\left(q_n^m-q_{n-1}^m e^{-\partial_n}\right)\widehat{K}^{(2l+1)}\left[u_n^m\right],
\end{aligned}$$

which immediately yields (B.4). The second half of the statement can be verified by a straightforward calculation. □

By using the recursion operators (B.4) and (B.6), $\widehat{K}^{(i)}\left[u_n^m\right]$ can be expressed as

$$\widehat{K}^{(i)}\left[u_n^m\right]=\begin{cases}\left(\Omega_n^{(2)}\right)^l 1 & i=2l,\\ \left(\Omega_n^{(2)}\right)^l\Omega_n^{(\mathrm{odd})}1 & i=2l+1.\end{cases}$$

B.2 Non-autonomous discrete Burgers hierarchy

We formulate the discrete Burgers hierarchy with arbitrary lattice intervals, which we call *non-autonomous* discrete Burgers hierarchy. The hierarchy introduced in the previous section is sometimes referred to as the *autonomous* discrete Burgers hierarchy. We first introduce the *divided difference* $f\left[x_j,x_{j+1},\ldots,x_{j+n}\right]$ of the function $f(x)$ with the base points $x_j, x_{j+1}, \ldots, x_{j+n}$ recursively by $f[x_j] = f(x_j)$ and

$$\text{first order:}\quad f[x_j, x_{j+1}] = \frac{f[x_{j+1}] - f[x_j]}{x_{j+1} - x_j},$$

$$\text{second order:}\quad f[x_j, x_{j+1}, x_{j+2}] = \frac{f[x_{j+1}, x_{j+2}] - f[x_j, x_{j+1}]}{x_{j+2} - x_j},$$

$$\text{third order:}\quad f[x_j, x_{j+1}, x_{j+2}, x_{j+3}] = \frac{f[x_{j+1}, x_{j+2}, x_{j+3}] - f[x_j, x_{j+1}, x_{j+2}]}{x_{j+3} - x_j},$$

$$n\text{-th order:}\quad f[x_j, x_{j+1}, \ldots, x_{j+n}] = \frac{f[x_{j+1}, \ldots, x_{j+n}] - f[x_j, \ldots, x_{j+n-1}]}{x_{j+n} - x_j}.$$

Among the various properties of the divided differences, we here note the following:

(1) *Expansion formula.*

$$f[x_j, x_{j+1}, \ldots, x_{j+n}] = \sum_{k=0}^{n} \frac{f(x_{j+k})}{\prod_{s=0,\, s\neq k}^{n} (x_{j+k} - x_{j+s})}.$$

For example, we have

$$f[x_j, x_{j+1}] = \frac{f(x_{j+1})}{x_{j+1} - x_j} + \frac{f(x_j)}{x_j - x_{j+1}},$$

$$f[x_j, x_{j+1}, x_{j+2}] = \frac{f(x_{j+2})}{(x_{j+2} - x_{j+1})(x_{j+2} - x_j)} + \frac{f(x_{j+1})}{(x_{j+1} - x_{j+2})(x_{j+1} - x_j)} + \frac{f(x_j)}{(x_j - x_{j+2})(x_j - x_{j+1})},$$

$$\begin{aligned} f[x_j, x_{j+1}, x_{j+2}, x_{j+3}] &= \frac{f(x_{j+3})}{(x_{j+3} - x_{j+2})(x_{j+3} - x_{j+1})(x_{j+3} - x_j)} + \frac{f(x_{j+2})}{(x_{j+2} - x_{j+3})(x_{j+2} - x_{j+1})(x_{j+2} - x_j)} \\ &\quad + \frac{f(x_{j+1})}{(x_{j+1} - x_{j+3})(x_{j+1} - x_{j+2})(x_{j+1} - x_j)} + \frac{f(x_j)}{(x_j - x_{j+3})(x_j - x_{j+2})(x_j - x_{j+1})}. \end{aligned}$$

As an immediate consequence, it follows that $f[x_j, x_{j+1}, \ldots, x_{j+n}]$ is invariant with respect to interchanging the base points.

(2) *Autonomization and continuous limit.* Putting the lattice interval to be constant, namely, $x_{j+k} = x_j + k\epsilon$, it follows that

$$f[x_j, x_{j+1}, \ldots, x_{j+n}] = \frac{1}{n!}\Delta_{+x}^{n} f(x_j), \quad \Delta_{+x} f(x) = \frac{f(x+\epsilon) - f(x)}{\epsilon}, \tag{B.7}$$

and thus

$$f\left[x_j, x_{j+1}, \ldots, x_{j+n}\right] \longrightarrow \frac{1}{n!}\frac{d^n f(x_j)}{dx^n} \quad (\epsilon \to 0).$$

In order to formulate the non-autonomous discrete Burgers hierarchy, we first introduce the family of linear difference equations for $q_n^m = q(x_n, t_m)$, $\delta_m = t_{m+1} - t_m$:

$$\frac{q_n^{m+1} - q_n^m}{\delta_m} = L_n^{(i)}[\,q_n^m\,], \tag{B.8}$$

where

$$L_n^{(i)}\left[q_n^m\right] = \begin{cases} q[\,x_{n-l}, \ldots, x_{n+l}] & i = 2l, \\ q[\,x_{n-l}, \ldots, x_{n+l+1}] & i = 2l+1. \end{cases}$$

The first few examples of $L_n^{(i)}[\,q_n^m]$ are given by

$$i = 0:\ L_n^{(0)}\left[q_n^m\right] = q_n^m,$$

$$i = 1:\ L_n^{(1)}\left[q_n^m\right] = \frac{q_{n+1}^m}{x_{n+1} - x_n} + \frac{q_n^m}{x_n - x_{n+1}},$$

$$i = 2:\ L_n^{(2)}\left[q_n^m\right] = \frac{q_{n+1}^m}{(x_{n+1} - x_n)(x_{n+1} - x_{n-1})} + \frac{q_n^m}{(x_n - x_{n+1})(x_n - x_{n-1})} + \frac{q_{n-1}^m}{(x_{n-1} - x_{n+1})(x_{n-1} - x_n)},$$

$$\begin{aligned} i = 3:\ L_n^{(3)}\left[q_n^m\right] &= \frac{q_{n+2}^m}{(x_{n+2} - x_{n+1})(x_{n+2} - x_n)(x_{n+2} - x_{n-1})} + \frac{q_{n+1}^m}{(x_{n+1} - x_{n+2})(x_{n+1} - x_n)(x_{n+1} - x_{n-1})} \\ &\quad + \frac{q_n^m}{(x_n - x_{n+2})(x_n - x_{n+1})(x_n - x_{n-1})} + \frac{q_{n-1}^m}{(x_{n-1} - x_{n+2})(x_{n-1} - x_{n+1})(x_{n-1} - x_n)}. \end{aligned}$$

We note that the following recursion relations hold:

$$L_n^{(2l+1)}\left[q_n^m\right] = \frac{q\left[x_{n-l+1}, \ldots, x_{n+l+1}\right] - q\left[x_{n-l}, \ldots, x_{n+l}\right]}{x_{n+l+1} - x_{n-l}} = \frac{L_{n+1}^{(2l)}\left[q_{n+1}^m\right] - L_n^{(2l)}\left[q_n^m\right]}{x_{n+l+1} - x_{n-l}}, \tag{B.9}$$

$$L_n^{(2l+2)}\left[q_n^m\right] = \frac{q\left[x_{n-l}, \ldots, x_{n+l+1}\right] - q\left[x_{n-l-1}, \ldots, x_{n+l}\right]}{x_{n+l+1} - x_{n-l-1}} = \frac{L_n^{(2l+1)}\left[q_n^m\right] - L_{n-1}^{(2l+1)}\left[q_{n-1}^m\right]}{x_{n+l+1} - x_{n-l-1}}. \tag{B.10}$$

The non-autonomous discrete Burgers hierarchy is a family of nonlinear difference equations obtained from

(B.8) by the discrete Cole-Hopf transformation (B.2). The i-th order equation in the hierarchy is given as

$$\frac{u_n^{m+1}}{u_n^m} = \frac{1+\delta_m K_{n+1}^{(i)}\left[u_{n+1}^m\right]}{1+\delta_m K_n^{(i)}\left[u_n^m\right]}, \tag{B.11}$$

where

$$K_n^{(i)}\left[u_n^m\right] = \frac{1}{q_n^m} L_n^{(i)}\left[q_n^m\right].$$

The recursion operator for the non-autonomous discrete Burgers hierarchy is given as follows:

Proposition B.2. *It holds that*

$$K_n^{(i+1)}\left[u_n^m\right] = \Omega_n^{(1,i+1)} K_n^{(i)}\left[u_n^m\right],$$

where $\Omega_n^{(1,i+1)}$ *is a difference operator defined by*

$$\Omega_n^{(1,i+1)} = \begin{cases} \frac{1}{\epsilon_n^{(i+1)}}\left(u_n^m e^{\partial} - 1\right) & i=2l, \\ \frac{1}{\epsilon_n^{(i+1)}}\left(1 - \frac{1}{u_{n-1}^m} e^{-\partial_n}\right) & i = 2l+1, \end{cases} \tag{B.12}$$

and

$$\epsilon_n^{(i+1)} = \begin{cases} x_{n+l+1} - x_{n-l} & i=2l, \\ x_{n+l+1} - x_{n-l-1} & i=2l+1. \end{cases} \tag{B.13}$$

In particular, we have

$$K_n^{(i+2)}\left[u_n^m\right] = \Omega_n^{(2,i+2)} K_n^{(i)}\left[u_n^m\right],$$

where

$$\begin{aligned} &\Omega_n^{(2,i+2)} \\ &= \Omega_n^{(1,i+2)}\Omega_n^{(1,i+1)} \\ &= \begin{cases} \frac{1}{\epsilon_n^{(i+2)}}\left(\frac{u_n^m}{\epsilon_n^{(i+1)}} e^{\partial_n} - \frac{1}{\epsilon_n^{(i+1)}} - \frac{1}{\epsilon_{n-1}^{(i+1)}} + \frac{1}{\epsilon_{n-1}^{(i+1)} u_{n-1}^m} e^{-\partial_n}\right) & i=2l, \\ \frac{1}{\epsilon_n^{(i+2)}}\left(\frac{u_n^m}{\epsilon_{n+1}^{(i+1)}} e^{\partial_n} - \frac{1}{\epsilon_{n+1}^{(i+1)}} - \frac{1}{\epsilon_n^{(i+1)}} + \frac{1}{\epsilon_n^{(i+1)}}\frac{1}{u_{n-1}^m} e^{-\partial_n}\right) & i=2l+1. \end{cases} \end{aligned} \tag{B.14}$$

Proof. The first half of the statement follows from the recursion relation of the divided differences. Indeed, it follows from (B.9) and (B.10) that

$$L_n^{(2l+1)}\left[q_n^m\right] = \frac{1}{\epsilon_n^{(2l+1)}}\left(e^{\partial_n} - 1\right) L_n^{(2l)}\left[q_n^m\right],$$

$$L_n^{(2l+2)}\left[q_n^m\right] = \frac{1}{\epsilon_n^{(2l+2)}}\left(1 - e^{-\partial_n}\right) L_n^{(2l+1)}\left[q_n^m\right],$$

which are equivalent to

$$K_n^{(2l+1)}\left[u_n^m\right] = \frac{1}{\epsilon_n^{(2l+1)}}\left(u_n^m e^{\partial_n} - 1\right) K_n^{(2l)}\left[u_n^m\right],$$

$$K_n^{(2l+2)}\left[u_n^m\right] = \frac{1}{\epsilon_n^{(2l+2)}}\left(1 - \frac{1}{u_{n-1}^m} e^{-\partial_n}\right) K_n^{(2l+1)}\left[u_n^m\right].$$

Thus we have (B.12). The second half is verified by a direct computation. □

Remark B.3. *The non-autonomous discrete Burgers hierarchy (B.11) reduces to the discrete Burgers hierarchy (B.3) by putting (see (B.7))*

$$x_{n+1} - x_n = \epsilon, \quad L_n^{(i)}\left[q_n^m\right] = \frac{1}{i!}\widehat{L}^{(i)}\left[q_n^m\right], \quad K_n^{(i)}\left[u_n^m\right] = \frac{1}{i!}\widehat{K}^{(i)}\left[u_n^m\right], \quad \Omega_n^{(1,i)} = \frac{1}{i}\widehat{\Omega}_n.$$

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