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An improved error bound for linear complementarity problems for B-matrices

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Abstract

A new error bound for the linear complementarity problem when the matrix involved is a B-matrix is presented, which improves the corresponding result in (Li *et al.* in Electron. J. Linear Algebra 31(1):476-484, 2016). In addition some sufficient conditions such that the new bound is sharper than that in (García-Esnaola and Peña in Appl. Math. Lett. 22(7):1071-1075, 2009) are provided.

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1 Introduction

Given an $n \times n$ real matrix M and $q \in R^n$, the linear complementarity problem (LCP) is to find a vector $x \in R^n$ satisfying

$$x \geq 0, \qquad Mx + q \geq 0, \qquad (Mx + q)^T x = 0 \tag{1}$$

or to show that no such vector x exists. We denote this problem (1) by LCP(M, q). The LCP(M, q) arises in many applications such as finding Nash equilibrium point of a bimatrix game, the network equilibrium problem, the contact problem and the free boundary problem for journal bearing etc.; for details, see [3–5].

It is well known that the LCP(M, q) has a unique solution for any vector $q \in R^n$ if and only if M is a P-matrix [4]. Here a matrix M is called a P-matrix if all its principal minors are positive. For the LCP(M, q), one of the interesting problems is to estimate

$$\max_{d \in [0,1]^n} \left\| (I - D + DM)^{-1} \right\|_\infty, \tag{2}$$

which can be used to bound the error $\|x - x^*\|_\infty$ [6], that is,

$$\left\| x - x^* \right\|_\infty \leq \max_{d \in [0,1]^n} \left\| (I - D + DM)^{-1} \right\|_\infty \left\| r(x) \right\|_\infty,$$

where x^* is the solution of the LCP(M, q), $r(x) = \min\{x, Mx + q\}$, $D = \operatorname{diag}(d_i)$ with $0 \leq d_i \leq 1$ for each $i \in N$, $d = [d_1, d_2, \ldots, d_n]^T \in [0,1]^n$, and the min operator $r(x)$ denotes the componentwise minimum of two vectors.

When the matrix M for the LCP(M,q) belongs to P-matrices or some subclass of P-matrices, various bounds for (2) were proposed; *e.g.*, see [2, 6–15] and the references therein. Recently, García-Esnaola and Peña in [2] provided an upper bound for (2) when M is a B-matrix as a subclass of P-matrices. Here, a matrix $M = [m_{ij}] \in R^{n,n}$ is called a B-matrix [16] if for each $i \in N = \{1, 2, \dots, n\}$,

$$\sum_{k\in N} m_{ik} > 0, \quad \text{and} \quad \frac{1}{n}\left(\sum_{k\in N} m_{ik}\right) > m_{ij} \quad \text{for any } j \in N \text{ and } j \neq i.$$

Theorem 1 ([2], Theorem 2.2) *Let $M = [m_{ij}] \in R^{n,n}$ be a B-matrix with the form*

$$M = B^+ + C, \tag{3}$$

where

$$B^+ = [b_{ij}] = \begin{bmatrix} m_{11} - r_1^+ & \cdots & m_{1n} - r_1^+ \\ \vdots & & \vdots \\ m_{n1} - r_n^+ & \cdots & m_{nn} - r_n^+ \end{bmatrix}, \qquad C = \begin{bmatrix} r_1^+ & \cdots & r_1^+ \\ \vdots & & \vdots \\ r_n^+ & \cdots & r_n^+ \end{bmatrix}, \tag{4}$$

and $r_i^+ = \max\{0, m_{ij} | j \neq i\}$. Then

$$\max_{d\in[0,1]^n} \left\| (I - D + DM)^{-1} \right\|_\infty \le \frac{n-1}{\min\{\beta, 1\}}, \tag{5}$$

where $\beta = \min_{i\in N}\{\beta_i\}$ and $\beta_i = b_{ii} - \sum_{j\neq i} |b_{ij}|$.

It is not difficult to see that the bound (5) will be inaccurate when the matrix M has very small value of $\min_{i\in N}\{b_{ii} - \sum_{j\neq i} |b_{ij}|\}$; for details, see [17, 18]. To conquer this problem, Li *et al.*, in [1] gave the following bound for (2) when M is a B-matrix, which improves those provided by Li and Li in [17, 18].

Theorem 2 ([1], Theorem 2.4) *Let $M = [m_{ij}] \in R^{n,n}$ be a B-matrix with the form $M = B^+ + C$, where $B^+ = [b_{ij}]$ is the matrix of* (4). *Then*

$$\max_{d\in[0,1]^n} \left\| (I - D + DM)^{-1} \right\|_\infty \le \sum_{i=1}^{n} \frac{n-1}{\min\{\bar{\beta}_i, 1\}} \prod_{j=1}^{i-1} \frac{b_{jj}}{\bar{\beta}_j}, \tag{6}$$

where $\bar{\beta}_i = b_{ii} - \sum_{j=i+1}^{n} |b_{ij}| l_i(B^+)$ with $l_k(B^+) = \max_{k\le i\le n}\{\frac{1}{|b_{ii}|}\sum_{j=k, j\neq i}^{n} |b_{ij}|\}$, and $\prod_{j=1}^{i-1} \frac{b_{jj}}{\bar{\beta}_j} = 1$ if $i = 1$.

In this paper, we further improve error bounds on the LCP(M,q) when M belongs to B-matrices. The rest of this paper is organized as follows: In Section 2 we present a new error bound for (2), and then prove that this bound is better than those in Theorems 1 and 2. In Section 3, some numerical examples are given to illustrate our theoretical results obtained.

2 Main result

In this section, an upper bound for (2) is provided when M is a B-matrix. Firstly, some definitions, notation and lemmas which will be used later are given as follows.

A matrix $A = [a_{ij}] \in C^{n,n}$ is called a strictly diagonally dominant (*SDD*) matrix if $|a_{ii}| > \sum_{j\neq i}^{n} |a_{ij}|$ for all $i = 1, 2, \ldots, n$. A matrix $A = [a_{ij}] \in R^{n,n}$ is called a nonsingular M-matrix if its inverse is nonnegative and all its off-diagonal entries are nonpositive [3]. In [16] it was proved that a B-matrix has positive diagonal elements, and a real matrix A is a B-matrix if and only if it can be written in the form (3) with B^+ being a *SDD* matrix. Given a matrix $A = [a_{ij}] \in C^{n,n}$, let

$$\begin{aligned} w_{ij}(A) &= \frac{|a_{ij}|}{|a_{ii}| - \sum_{k=j+1,\, k\neq i}^{n} |a_{ik}|}, \quad i \neq j, \\ w_i(A) &= \max_{j\neq i}\{w_{ij}(A)\}, \\ m_{ij}(A) &= \frac{|a_{ij}| + \sum_{k=j+1,\, k\neq i}^{n} |a_{ik}| w_k(A)}{|a_{ii}|}, \quad i \neq j. \end{aligned} \tag{7}$$

Lemma 1 ([19], Theorem 14) *Let $A = [a_{ij}]$ be an $n \times n$ row strictly diagonally dominant M-matrix. Then*

$$\|A^{-1}\|_\infty \leq \sum_{i=1}^{n} \left(\frac{1}{a_{ii} - \sum_{k=i+1}^{n} |a_{ik}| m_{ki}(A)} \prod_{j=1}^{i-1} \frac{1}{1 - u_j(A) l_j(A)} \right),$$

where $u_i(A) = \frac{1}{|a_{ii}|} \sum_{j=i+1}^{n} |a_{ij}|$, $l_k(A) = \max_{k\leq i\leq n}\{\frac{1}{|a_{ii}|}\sum_{j=k,\, j\neq i}^{n} |a_{ij}|\}$, $\prod_{j=1}^{i-1} \frac{1}{1-u_j(A)l_j(A)} = 1$ if $i = 1$, and $m_{ki}(A)$ is defined as in (7).

Lemma 2 ([17], Lemma 3) *Let $\gamma > 0$ and $\eta \geq 0$. Then, for any $x \in [0,1]$,*

$$\frac{1}{1 - x + \gamma x} \leq \frac{1}{\min\{\gamma, 1\}}$$

and

$$\frac{\eta x}{1 - x + \gamma x} \leq \frac{\eta}{\gamma}.$$

Lemma 3 ([18], Lemma 5) *Let $A = [a_{ij}]$ with $a_{ii} > \sum_{j=i+1}^{n} |a_{ij}|$ for each $i \in N$. Then, for any $x_i \in [0,1]$,*

$$\frac{1 - x_i + a_{ii} x_i}{1 - x_i + a_{ii} x_i - \sum_{j=i+1}^{n} |a_{ij}| x_i} \leq \frac{a_{ii}}{a_{ii} - \sum_{j=i+1}^{n} |a_{ij}|}.$$

Lemmas 2 and 3 will be used in the proofs of the following lemma and Theorem 3.

Lemma 4 *Let $M = [m_{ij}] \in R^{n,n}$ be a B-matrix with the form $M = B^+ + C$, where $B^+ = [b_{ij}]$ is the matrix of* (4). *And let $B_D^+ = I - D + DB^+ = [\tilde{b}_{ij}]$ where $D = \operatorname{diag}(d_i)$ with $0 \leq d_i \leq 1$. Then*

$$w_i(B_D^+) \leq \max_{j\neq i} \left\{ \frac{|b_{ij}|}{b_{ii} - \sum_{k=j+1,\, k\neq i}^{n} |b_{ik}|} \right\}$$

and

$$m_{ij}(B_D^+) \le v_{ij}(B^+) < 1,$$

where $w_i(B_D^+)$, $m_{ij}(B_D^+)$ *are defined as in* (7), *and*

$$v_{ij}(B^+) = \frac{1}{b_{ii}}\left(|b_{ij}| + \sum_{\substack{k=j+1,\\ k\neq i}}^{n}\left(|b_{ik}| \cdot \max_{h\neq k}\left\{\frac{|b_{kh}|}{b_{kk} - \sum_{\substack{l=h+1,\\ l\neq k}}^{n} |b_{kl}|}\right\}\right)\right).$$

Proof Note that

$$[B_D^+]_{ij} = \tilde{b}_{ij} = \begin{cases} 1 - d_i + d_i b_{ij}, & i = j, \\ d_i b_{ij}, & i \neq j. \end{cases}$$

Since B^+ is *SDD*, $b_{ii} - \sum_{\substack{k=j+1,\\ k\neq i}}^{n} |b_{ik}| > |b_{ij}|$ for each $i \neq j$. Hence, by Lemma 2 and (7), it follows that

$$\begin{aligned} w_i(B_D^+) &= \max_{j\neq i}\{w_{ij}(B_D^+)\} = \max_{j\neq i}\left\{\frac{|b_{ij}|d_i}{1 - d_i + b_{ii}d_i - \sum_{\substack{k=j+1,\\ k\neq i}}^{n} |b_{ik}|d_i}\right\} \\ &\le \max_{j\neq i}\left\{\frac{|b_{ij}|}{b_{ii} - \sum_{\substack{k=j+1,\\ k\neq i}}^{n} |b_{ik}|}\right\} < 1. \end{aligned} \tag{8}$$

Furthermore, it follows from (7), (8) and Lemma 2 that for each $i \neq j$ ($j < i \le n$)

$$\begin{aligned} m_{ij}(B_D^+) &= \frac{|b_{ij}| \cdot d_i + \sum_{\substack{k=j+1,\\ k\neq i}}^{n} |b_{ik}| \cdot d_i \cdot w_k(B_D^+)}{1 - d_i + b_{ii} \cdot d_i} \\ &\le \frac{1}{b_{ii}} \cdot \left(|b_{ij}| + \sum_{\substack{k=j+1,\\ k\neq i}}^{n} |b_{ik}| \cdot w_k(B_D^+)\right) \\ &\le \frac{1}{b_{ii}}\left(|b_{ij}| + \sum_{\substack{k=j+1,\\ k\neq i}}^{n}\left(|b_{ik}| \cdot \max_{h\neq k}\left\{\frac{|b_{kh}|}{b_{kk} - \sum_{\substack{l=h+1,\\ l\neq k}}^{n} |b_{kl}|}\right\}\right)\right) \\ &= v_{ij}(B^+) \\ &< \frac{1}{b_{ii}}\left(|b_{ij}| + \sum_{\substack{k=j+1,\\ k\neq i}}^{n} |b_{ik}|\right) < 1. \end{aligned}$$

The proof is completed. □

By Lemmas 1, 2, 3 and 4, we give the following bound for (2) when M is a B-matrix.

Theorem 3 *Let* $M = [m_{ij}] \in R^{n,n}$ *be a B-matrix with the form* $M = B^+ + C$, *where* $B^+ = [b_{ij}]$ *is the matrix of* (4). *Then*

$$\max_{d\in[0,1]^n} \left\|(I - D + DM)^{-1}\right\|_\infty \le \sum_{i=1}^{n} \frac{n-1}{\min\{\widehat{\beta}_i, 1\}} \prod_{j=1}^{i-1} \frac{b_{jj}}{\bar{\beta}_j}, \tag{9}$$

where $\widehat{\beta_i} = b_{ii} - \sum_{k=i+1}^{n} |b_{ik}| \cdot v_{ki}(B^+)$ *with* $v_{ki}(B^+)$ *is defined in Lemma* 4, $\bar{\beta}_i$ *is defined in Theorem* 2, *and* $\prod_{j=1}^{i-1} \frac{b_{jj}}{\bar{\beta}_j} = 1$ *if* $i = 1$.

Proof Let $M_D = I - D + DM$. Then

$$M_D = I - D + DM = I - D + D\bigl(B^+ + C\bigr) = B_D^+ + C_D,$$

where $B_D^+ = I - D + DB^+ = [\tilde{b}_{ij}]$ and $C_D = DC$. Similarly to the proof of Theorem 2.2 in [2], we find that B_D^+ is an *SDD M*-matrix with positive diagonal elements and that

$$\bigl\|M_D^{-1}\bigr\|_\infty \le \bigl\|\bigl(I + \bigl(B_D^+\bigr)^{-1} C_D\bigr)^{-1}\bigr\|_\infty \bigl\|\bigl(B_D^+\bigr)^{-1}\bigr\|_\infty \le (n-1)\bigl\|\bigl(B_D^+\bigr)^{-1}\bigr\|_\infty. \tag{10}$$

Next, we give an upper bound for $\|(B_D^+)^{-1}\|_\infty$. By Lemma 1, we have

$$\bigl\|\bigl(B_D^+\bigr)^{-1}\bigr\|_\infty \le \sum_{i=1}^{n} \left(\frac{1}{1 - d_i + b_{ii}d_i - \sum_{k=i+1}^{n} |b_{ik}| \cdot d_i \cdot m_{ki}(B_D^+)} \prod_{j=1}^{i-1} \frac{1}{1 - u_j(B_D^+) l_j(B_D^+)} \right), \tag{11}$$

where

$$u_j\bigl(B_D^+\bigr) = \frac{\sum_{k=j+1}^{n} |b_{jk}| d_j}{1 - d_j + b_{jj}d_j}, \qquad l_k\bigl(B_D^+\bigr) = \max_{k \le i \le n} \left\{ \frac{\sum_{j=k, j \ne i}^{n} |b_{ij}| d_i}{1 - d_i + b_{ii}d_i} \right\},$$

and

$$m_{ki}\bigl(B_D^+\bigr) = \frac{|b_{ki}| \cdot d_k + \sum_{l=i+1, l \ne k}^{n} |b_{kl}| \cdot d_k \cdot w_l(B_D^+)}{1 - d_k + b_{kk} \cdot d_k}$$

with $w_l(B_D^+) = \max_{h \ne l} \{ \frac{|b_{lh}| d_l}{1 - d_l + b_{ll}d_l - \sum_{s=h+1, s \ne l}^{n} |b_{ls}| d_l} \}$.

By Lemmas 2 and 4, we can easily see that, for each $i \in N$,

$$\begin{aligned} \frac{1}{1 - d_i + b_{ii}d_i - \sum_{k=i+1}^{n} |b_{ik}| \cdot d_i \cdot m_{ki}(B_D^+)} &\le \frac{1}{\min\{b_{ii} - \sum_{k=i+1}^{n} |b_{ik}| \cdot m_{ki}(B_D^+), 1\}} \\ &\le \frac{1}{\min\{b_{ii} - \sum_{k=i+1}^{n} |b_{ik}| \cdot v_{ki}(B^+), 1\}} \\ &= \frac{1}{\min\{\widehat{\beta_i}, 1\}}, \end{aligned} \tag{12}$$

and that, for each $k \in N$,

$$l_k\bigl(B_D^+\bigr) = \max_{k \le i \le n} \left\{ \frac{\sum_{j=k, j \ne i}^{n} |b_{ij}| d_i}{1 - d_i + b_{ii}d_i} \right\} \le \max_{k \le i \le n} \left\{ \frac{1}{b_{ii}} \sum_{j=k, j \ne i}^{n} |b_{ij}| \right\} = l_k\bigl(B^+\bigr) < 1. \tag{13}$$

Furthermore, according to Lemma 3 and (13), it follows that, for each $j \in N$,

$$\frac{1}{1 - u_j(B_D^+) l_j(B_D^+)} = \frac{1 - d_j + b_{jj}d_j}{1 - d_j + b_{jj}d_j - \sum_{k=j+1}^{n} |b_{jk}| \cdot d_j \cdot l_j(B_D^+)} \le \frac{b_{jj}}{\bar{\beta}_j}. \tag{14}$$

By (11), (12) and (14), we have

$$\left\| \left(B_D^+\right)^{-1} \right\|_\infty \le \frac{1}{\min\{\widehat{\beta}_1, 1\}} + \sum_{i=2}^{n} \left(\frac{1}{\min\{\widehat{\beta}_i, 1\}} \prod_{j=1}^{i-1} \frac{b_{jj}}{\bar{\beta}_j} \right). \tag{15}$$

The conclusion follows from (10) and (15). □

The comparisons of the bounds in Theorems 2 and 3 are established as follows.

Theorem 4 *Let $M = [m_{ij}] \in R^{n,n}$ be a B-matrix with the form $M = B^+ + C$, where $B^+ = [b_{ij}]$ is the matrix of* (4). *Let $\bar{\beta}_i$ and $\widehat{\beta}_i$ be defined in Theorems* 2 *and* 3, *respectively. Then*

$$\sum_{i=1}^{n} \frac{n-1}{\min\{\widehat{\beta}_i, 1\}} \prod_{j=1}^{i-1} \frac{b_{jj}}{\bar{\beta}_j} \le \sum_{i=1}^{n} \frac{n-1}{\min\{\bar{\beta}_i, 1\}} \prod_{j=1}^{i-1} \frac{b_{jj}}{\bar{\beta}_j}.$$

Proof Note that

$$\bar{\beta}_i = b_{ii} - \sum_{j=i+1}^{n} |b_{ij}| l_i\left(B^+\right), \qquad \widehat{\beta}_i = b_{ii} - \sum_{k=i+1}^{n} |b_{ik}| v_{ki}\left(B^+\right),$$

and B^+ is a *SDD* matrix, it follows that for each $i \neq j$ $(j < i \le n)$

$$\begin{aligned} v_{ij}\left(B^+\right) &= \frac{1}{b_{ii}} \left(|b_{ij}| + \sum_{\substack{k=j+1, \\ k \neq i}}^{n} \left(|b_{ik}| \cdot \max_{h \neq k} \left\{ \frac{|b_{kh}|}{b_{kk} - \sum_{\substack{l=h+1, \\ l \neq k}}^{n} |b_{kl}|} \right\} \right) \right) \\ &< \frac{1}{b_{ii}} \sum_{\substack{k=j, \\ k \neq i}}^{n} |b_{ik}| \\ &\le \max_{j \le i \le n} \left\{ \frac{1}{b_{ii}} \sum_{\substack{k=j, \\ k \neq i}}^{n} |b_{ik}| \right\} = l_j\left(B^+\right). \end{aligned}$$

Hence, for each $i \in N$

$$\widehat{\beta}_i = b_{ii} - \sum_{k=i+1}^{n} |b_{ik}| v_{ki}\left(B^+\right) > b_{ii} - \sum_{k=i+1}^{n} |b_{ik}| l_i\left(B^+\right) = \bar{\beta}_i,$$

which implies that

$$\frac{1}{\min\{\widehat{\beta}_i, 1\}} \le \frac{1}{\min\{\bar{\beta}_i, 1\}}.$$

This completes the proof. □

Remark here that, when $\bar{\beta}_i < 1$ for all $i \in N$, then

$$\frac{1}{\min\{\widehat{\beta}_i, 1\}} < \frac{1}{\min\{\bar{\beta}_i, 1\}},$$

which yields

$$\sum_{i=1}^{n} \frac{n-1}{\min\{\widehat{\beta}_i, 1\}} \prod_{j=1}^{i-1} \frac{b_{jj}}{\bar{\beta}_j} < \sum_{i=1}^{n} \frac{n-1}{\min\{\bar{\beta}_i, 1\}} \prod_{j=1}^{i-1} \frac{b_{jj}}{\bar{\beta}_j}.$$

Next it is proved that the bound (9) given in Theorem 3 can improve the bound (5) in Theorem 1 (Theorem 2.2 in [2]) in some cases.

Theorem 5 *Let $M = [m_{ij}] \in R^{n,n}$ be a B-matrix with the form $M = B^+ + C$, where $B^+ = [b_{ij}]$ is the matrix of* (4). *Let β, $\bar{\beta}_i$ and $\widehat{\beta}_i$ be defined in Theorems* 1, 2 *and* 3, *respectively, and let $\alpha = 1 + \sum_{i=2}^{n} \prod_{j=1}^{i-1} \frac{b_{jj}}{\bar{\beta}_j}$ and $\widehat{\beta} = \min_{i\in N}\{\widehat{\beta}_i\}$. If one of the following conditions holds*:

(i) $\widehat{\beta} > 1$ *and* $\alpha < \frac{1}{\beta}$;

(ii) $\widehat{\beta} < 1$ *and* $\alpha\beta < \widehat{\beta}$,

then

$$\sum_{i=1}^{n} \frac{n-1}{\min\{\widehat{\beta}_i, 1\}} \prod_{j=1}^{i-1} \frac{b_{jj}}{\bar{\beta}_j} < \frac{n-1}{\min\{\beta, 1\}}.$$

Proof When $\widehat{\beta} > 1$ and $\alpha < \frac{1}{\beta}$, we can easily get

$$\sum_{i=1}^{n} \frac{n-1}{\min\{\widehat{\beta}_i, 1\}} \prod_{j=1}^{i-1} \frac{b_{jj}}{\bar{\beta}_j} < \frac{n-1}{\min\{\widehat{\beta}, 1\}} \sum_{i=1}^{n} \prod_{j=1}^{i-1} \frac{b_{jj}}{\bar{\beta}_j} = (n-1)\alpha < \frac{n-1}{\beta} \le \frac{n-1}{\min\{\beta, 1\}}.$$

Similarly, for $\widehat{\beta} < 1$ and $\alpha\beta < \widehat{\beta}$, the conclusion can be proved directly. □

3 Numerical examples

Two examples are given to show that the bound in Theorem 3 is sharper than those in Theorems 1 and 2.

Example 1 Consider the family of B-matrices in [17]:

$$M_k = \begin{bmatrix} 1.5 & 0.5 & 0.4 & 0.5 \\ -0.1 & 1.7 & 0.7 & 0.6 \\ 0.8 & -0.1\frac{k}{k+1} & 1.8 & 0.7 \\ 0 & 0.7 & 0.8 & 1.8 \end{bmatrix},$$

where $k \ge 1$. Then $M_k = B_k^+ + C_k$, where

$$B_k^+ = \begin{bmatrix} 1 & 0 & -0.1 & 0 \\ -0.8 & 1 & 0 & -0.1 \\ 0 & -0.1\frac{k}{k+1} - 0.8 & 1 & -0.1 \\ -0.8 & -0.1 & 0 & 1 \end{bmatrix}.$$

By computations, we have $\beta = \frac{1}{10(k+1)}$, $\bar{\beta}_1 = \bar{\beta}_2 = \frac{90k+91}{100k+100}$, $\bar{\beta}_3 = 0.99$, $\bar{\beta}_4 = 1$, $\hat{\beta}_1 = \frac{820k+828}{900k+900}$, $\hat{\beta}_2 = 0.99$, $\hat{\beta}_3 = 1$ and $\hat{\beta}_4 = 1$. Then it is easy to verify that M_k satisfies the condition (ii) of

Theorem 5. Hence, by Theorem 1 (Theorem 2.2 in [2]), we have

$$\max_{d\in[0,1]^4} \left\|(I-D+DM_k)^{-1}\right\|_\infty \le \frac{4-1}{\min\{\beta,1\}} = 30(k+1).$$

It is obvious that

$$30(k+1) \longrightarrow +\infty, \quad \text{when } k \longrightarrow +\infty.$$

By Theorem 2, we find that, for any $k \ge 1$,

$$\begin{aligned}
&\max_{d\in[0,1]^4} \left\|(I-D+DM_k)^{-1}\right\|_\infty \\
&\quad \le 3\left(\frac{1}{\bar{\beta}_1} + \frac{1}{\bar{\beta}_2}\cdot\frac{1}{\bar{\beta}_1} + \frac{1}{\bar{\beta}_3}\cdot\frac{1}{\bar{\beta}_1\bar{\beta}_2} + \frac{1}{\bar{\beta}_1\bar{\beta}_2\bar{\beta}_3}\right) \\
&\quad = 3\left(\frac{100k+100}{90k+91} + \frac{(100k+100)^2}{(90k+91)^2} + \frac{2(100k+100)^2}{0.99(90k+91)^2}\right) < 14.5193.
\end{aligned}$$

By Theorem 3, we find that, for any $k \ge 1$,

$$\begin{aligned}
&\max_{d\in[0,1]^4} \left\|(I-D+DM_k)^{-1}\right\|_\infty \\
&\quad \le 3\left(\frac{1}{\hat{\beta}_1} + \frac{1}{\hat{\beta}_2}\cdot\frac{1}{\bar{\beta}_1} + \frac{1}{\bar{\beta}_1\bar{\beta}_2} + \frac{1}{\bar{\beta}_1\bar{\beta}_2\bar{\beta}_3}\right) \\
&\quad = 3\left(\frac{900k+900}{820k+828} + \frac{(100k+100)}{0.99(90k+91)} + \frac{1.99(100k+100)^2}{0.99(90k+91)^2}\right) \\
&\quad < 3\left(\frac{100k+100}{90k+91} + \frac{(100k+100)^2}{(90k+91)^2} + \frac{2(100k+100)^2}{0.99(90k+91)^2}\right).
\end{aligned}$$

In particular, when $k = 1$,

$$\begin{aligned}
&3\left(\frac{900k+900}{820k+828} + \frac{(100k+100)}{0.99(90k+91)} + \frac{1.99(100k+100)^2}{0.99(90k+91)^2}\right) \approx 13.9878, \\
&3\left(\frac{100k+100}{90k+91} + \frac{(100k+100)^2}{(90k+91)^2} + \frac{2(100k+100)^2}{0.99(90k+91)^2}\right) \approx 14.3775,
\end{aligned}$$

and the bound (5) in Theorem 1 is

$$\frac{4-1}{\min\{\beta,1\}} = 30(k+1) = 60.$$

When $k = 2$,

$$\begin{aligned}
&3\left(\frac{900k+900}{820k+828} + \frac{(100k+100)}{0.99(90k+91)} + \frac{1.99(100k+100)^2}{0.99(90k+91)^2}\right) \approx 14.0265, \\
&3\left(\frac{100k+100}{90k+91} + \frac{(100k+100)^2}{(90k+91)^2} + \frac{2(100k+100)^2}{0.99(90k+91)^2}\right) \approx 14.4246,
\end{aligned}$$

and the bound (5) in Theorem 1 is

$$\frac{4-1}{\min\{\beta,1\}} = 30(k+1) = 90.$$

Example 2 Consider the following family of B-matrices:

$$M_k = \begin{bmatrix} \frac{1}{k} & \frac{-a}{k} \\ 0 & \frac{1}{k} \end{bmatrix},$$

where $\frac{\sqrt{5}-1}{2} < a < 1$ and $\frac{2-a^2}{1+a} < k < 1$. Then $M_k = B_k^+ + C$ with C is the null matrix.

By simple computations, we can get

$$\beta = \frac{1-a}{k}, \qquad \bar{\beta}_1 = \frac{1-a^2}{k}, \qquad \bar{\beta}_2 = \frac{1}{k}, \qquad \hat{\beta}_1 = \frac{1}{k} \quad \text{and} \quad \hat{\beta}_2 = \frac{1}{k}.$$

It is not difficult to verify that M_k satisfies the condition (i) of Theorem 5. Thus, the bound (6) of Theorem 2 (Theorem 2.4 in [1]) is

$$\sum_{i=1}^{2} \frac{2-1}{\min\{\bar{\beta}_i,1\}} \prod_{j=1}^{i-1} \frac{b_{jj}}{\bar{\beta}_j} = \frac{k+1}{1-a^2},$$

which is larger than the bound

$$\frac{1}{\min\{\beta,1\}} = \frac{k}{1-a}$$

given by (5) in Theorem 1 (Theorem 2.2 in [2]). However, by Theorem 3 we can get

$$\max_{d\in[0,1]^2} \left\| (I-D+DM_k)^{-1} \right\|_\infty \leq \frac{2-a^2}{1-a^2},$$

which is smaller than the bound (5) in Theorem 1, *i.e.*,

$$\frac{2-a^2}{1-a^2} < \frac{k}{1-a}.$$

In particular, when $a = \frac{4}{5}$ and $k = \frac{8}{9}$, the bounds in Theorems 1 and 2 are, respectively,

$$\frac{1}{\min\{\beta,1\}} = \frac{k}{1-a} = \frac{360}{81}$$

and

$$\sum_{i=1}^{2} \frac{2-1}{\min\{\bar{\beta}_i,1\}} \prod_{j=1}^{i-1} \frac{b_{jj}}{\bar{\beta}_j} = \frac{k+1}{1-a^2} = \frac{425}{81},$$

while the bound (9) in Theorem 3 is

$$\sum_{i=1}^{2} \frac{2-1}{\min\{\hat{\beta}_i,1\}} \prod_{j=1}^{i-1} \frac{b_{jj}}{\bar{\beta}_j} = \frac{2-a^2}{1-a^2} = \frac{306}{81}.$$

These two examples show that the bound in Theorem 3 is sharper than those in Theorems 1 and 2.

4 Conclusions

In this paper, we give a new error bound for the linear complementarity problem when the matrix involved is a B-matrix, which improves those bounds obtained in [2] and [1]. Numerical examples are given to illustrate the corresponding results.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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