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To the theory of adaptive signal processing in systems with centrally symmetric receive channels

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Abstract

This paper presents the analytical derivation of joint probability density functions (pdfs) of the maximum likelihood (ML) estimates of a real and complex persymmetric correlation matrices (PCM) of multivariate Gaussian processes. It is oriented at the modifications of the classical Wishart's–Goodman's pdfs adapted to the ML estimates of the data CMs in a wide class of signal processing (SP) problems in systems with centrally symmetric (CS) receive channels. The importance of the derived modified pdfs for such CS systems could be as great as that of the classical Wishart's–Goodman's pdfs for systems with arbitrary receive channels. Some properties of the new obtained joint pdfs are featured.

Keywords: Central symmetry, Correlation matrix, Maximum likelihood estimate, Persymmetry, Probability density function, Wishart–Goodman distribution

1 Introduction

The multivariate statistical analysis of random processes widely uses the Wishart distribution, which describes statistical properties of a maximum likelihood (ML) estimate of the real-valued positively definite correlation matrix (CM) of multivariate Gaussian processes/fields [1–5]. For the ML estimate of the complex-valued Hermitian positively definite general-type data CM, such distribution is derived by Goodman [6–10]. An importance of both these distributions is caused by the fact that the ML CM estimates are widely used in many signal processing contexts: for radar applications [9, 11–17], for "superresolving" direction of arrival (DOA) estimation [7, 12, 18, 19], for multichannel communication systems [10], feature enhanced radar imaging with synthetic aperture radar (SAR) sensors [20, 21], digital beamforming in adaptive array (AA) systems, and fractional SAR modalities [22–25].

The Wishart–Goodman distributions are true for a general type data CM (GCM) of Gaussian processes/fields. These distributions in their classical forms do not take into account CM structure practically possible specificity caused by the peculiarities of signal processing (SP) system. At the same time, prior knowledge about the CM structure specificity could considerably enhance the processing efficiency and analysis precision due to a considerable decrease in the dimensionality of the parameter vectors involved into the adaptation process [13, 26].

The well-known example of such specificity is a persymmetry, i.e., symmetry relative to the secondary diagonal, of symmetric (real-valued) and Hermitian (complex-valued) CMs. Persymmetric CM (PCM) coincides with a result of itself turn relative to the secondary diagonal and thus is completely determined by a set of parameters (CM elements) which quantity is approximately twice less than that for respective general-type CM.

In multichannel (in space or time) signal processing systems, the CM persymmetry could be caused, in particular, by central symmetry (CS) of pairwise-identical receive channels arrangement. Such CS is peculiar to numerous different-purpose space-time signal processing systems (see, for instance, references 1–7 in [27]).

The ML for Hermitian PCM was derived for the first time by Nitzberg in his paper [28]. From the moment of this paper publication, numerous works have been performed to investigate an efficiency of this estimate use

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in CS systems of space-time adaptive signal processing [29–38]. Extensive supplementary list of relevant references is given in [16, 27].

In spite of a diversity of all these works, one common feature is inherent to all of them. Each such work sets an objective to determine statistical characteristics of one or another of function of ML estimate of relevant persymmetric correlation matrix (being usually the Hermitian one) since this function serves as a criterion of respective SP system efficiency. Thus, in [38], such function is "the persymmetric multiband generalized ratio algorithm (PGLR)" and the probability of false alarm and the probability of detection are obtained for this function. Works [28–35] investigate the relative losses, introduced in [9], in signal-to-(interference + noise) ratio (SINR) at the output of CS adaptive detector. These losses are taken relatively to those at the output of optimal detector, and probability density functions (pdfs), mean values, and variances are derived for them. Other criteria and their statistical characteristics under CM persymmetry are considered in [37].

The objectives been set are usually attained by the methods, which have specific distinctions from those been used for an analysis of efficiency of adaptive processing based on ML estimates of GCM. These distinctions are caused by the fact that, under generally concerned conditions, the PCM's initial ML estimate [28] as well as its variants [30, 36, 39] are the sum of two summands. Both these summands have the Wishart (or Wishart–Goodman) distribution with the same matrix of parameters which, however, are not independent [34]. This fact forbids to use directly the methodology, developed in [7, 9, 10], in order to find the statistical characteristics of functions of GCM ML estimates by using Wishart's–Goodman's probability density function (pdf).

For PCM, it is possible to proceed to this methodology after a number of preliminary mathematical transformations [35, 38], which overcome the abovementioned mutual dependence of summands in PCM ML estimate.

Having analyzed these transformations, the author noticed that they solve the problems formulated in [35, 38] as well as "suggest" a way to derive directly a pdf of PCM ML estimates. Such problem, which has not been set and solved in these works, seemed interesting from theoretical and practical considerations. This has stimulated such problem formulation and solution. It was expected that the importance of these distributions for CS systems with PCM of Gaussian inputs should be as great as that of the Wishart's–Goodman's distributions for systems with arbitrary characteristics of receive channels.

The goal of this paper is twofold: (i) to derive closed form analytical expressions for the pdfs of the ML estimates of persymmetric real and complex CMs of Gaussian processes/fields of various natures and (ii) to feature their usefulness in statistical data characterization and operational performance analysis in applications to SP systems that possess a space-time receive channel CS property.

The rest of the paper is organized as follows. Section 2 reviews the persymmetric CMs models. In Sections 3 and 4, we derive the distributions of ML estimates of real and complex persymmetric CMs, respectively, in a closed analytical form. Section 5 exemplifies the usage of the derived distributions in some characteristic applications related to multichannel adaptive SP problems. Conclusion in Section 6 resumes the study.

2 Overview of properties of persymmetric correlation matrices

A. The real $M \times M$ matrix $\mathbf{R} = [r_{i\ell}]_{i,\ell=1}^{M}$ is persymmetric if it coincides with the matrix obtained after rotation of $\mathbf{R}$ with respect to the secondary diagonal, i.e., when the property

$$\mathbf{R} = \mathbf{\Pi}_M \cdot \mathbf{R}^T \cdot \mathbf{\Pi}_M, \quad r_{i\ell} = r_{M+1-\ell,\, M+1-i}, \quad i, \ell \in 1, M, \tag{1}$$

for a real matrix (i.e., matrix composed of real-valued entries) holds.

If the correlation (symmetric) matrix plays a role of $\mathbf{R}$, then the additional equations are true

$$\begin{gathered}\mathbf{R} = \mathbf{\Pi}_M \cdot \mathbf{R}^T \cdot \mathbf{\Pi}_M = \mathbf{\Pi}_M \cdot \mathbf{R} \cdot \mathbf{\Pi}_M = \mathbf{R}^T, \\ r_{i\ell} = r_{M+1-\ell,M+1-i} = r_{M+1-i,M+1-\ell} = r_{\ell i}, \quad i, \ell \in 1, M.\end{gathered} \tag{2}$$

Hereinafter, superscript "^{T}" denotes a vector/matrix transposition;

$$\mathbf{\Pi}_\nu = \sum_{i=1}^{\nu} \mathbf{e}_i \cdot \mathbf{e}_{\nu+1-i}^T = \mathbf{\Pi}_\nu^T, \qquad \mathbf{\Pi}_\nu \cdot \mathbf{\Pi}_\nu^T = \mathbf{I}_\nu, \qquad \mathbf{\Pi}_\nu = \mathbf{\Pi}_\nu^T \tag{3}$$

is the $\nu \times \nu$ orthogonal symmetric permutation matrix with unit entries on its secondary diagonal, and $\mathbf{e}_i$ represents the $i-$ th $(i \in 1, \nu)$ column of the $\nu \times \nu$ unity matrix $\mathbf{I}_\nu$.

For even M $(M = 2 \cdot L)$, matrix (2) allows the following block representation

$$\mathbf{R} = \left[\begin{array}{c|c} \mathbf{R}_{11} & \mathbf{R}_{12} \\ \hline \mathbf{\Pi}_L \cdot \mathbf{R}_{12} \cdot \mathbf{\Pi}_L & \mathbf{\Pi}_L \cdot \mathbf{R}_{11} \cdot \mathbf{\Pi}_L \end{array}\right], \quad \begin{gathered}\mathbf{R}_{11} = \mathbf{R}_{11}^T, \\ \mathbf{R}_{12} = \mathbf{\Pi}_L \cdot \mathbf{R}_{12}^T \cdot \mathbf{\Pi}_L,\end{gathered} \tag{4}$$

where $\mathbf{R}_{11}$ and $\mathbf{R}_{12}$ represent the corresponding $L \times L$ blocks in (4).

Let us introduce the $2 \cdot L \times 2 \cdot L = M \times M$ matrix

$$\mathbf{S}_M = [s_{i\ell}]_{i,\ell=1}^{2\cdot L} = \frac{1}{\sqrt{2}}(\mathbf{I}_M + \mathbf{J}_M \cdot \mathbf{\Pi}_M) = \frac{1}{\sqrt{2}}\begin{bmatrix} \mathbf{I}_L & \mathbf{\Pi}_L \\ -\mathbf{\Pi}_L & \mathbf{I}_L \end{bmatrix}, \ \mathbf{J}_M = \begin{bmatrix} \mathbf{I}_L & 0 \\ 0 & -\mathbf{I}_L \end{bmatrix} \tag{5}$$

that possesses the following properties (easily verifiable via simple algebraic manipulations)

$$\mathbf{S}_M \cdot \mathbf{S}_M^T = \mathbf{I}_M, \quad \mathbf{S}_M \cdot \mathbf{\Pi}_M = \mathbf{J}_M \cdot \mathbf{S}_M, \quad \mathbf{J}_M \cdot \mathbf{\Pi}_M = -\mathbf{\Pi}_M \cdot \mathbf{J}_M. \tag{6}$$

Using (6), matrix $\mathbf{R}$ (4) can be transformed into the following block-diagonal form

$$\mathbf{R}_M = \mathbf{S}_M \cdot \mathbf{R} \cdot \mathbf{S}_M^T = \begin{bmatrix} \mathbf{R}_{\sum} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Pi}_L \cdot \mathbf{R}_\Delta \cdot \mathbf{\Pi}_L \end{bmatrix}, \quad \begin{matrix} \mathbf{R}_{\sum} = \mathbf{R}_{11} + \mathbf{R}_{12} \cdot \mathbf{\Pi}_L, \\ \mathbf{R}_\Delta = \mathbf{R}_{11} - \mathbf{R}_{12} \cdot \mathbf{\Pi}_L, \end{matrix} \tag{7}$$

with the determinant

$$\det \mathbf{R}_M = |\mathbf{R}_M| = \left|\mathbf{R}_{\sum}\right| \cdot |\mathbf{\Pi}_L \cdot \mathbf{R}_\Delta \cdot \mathbf{\Pi}_L| = \left|\mathbf{R}_{\sum}\right| \cdot |\mathbf{R}_\Delta| = |\mathbf{R}| \tag{8}$$

that coincides with the determinant of the initial matrix

$$\mathbf{R} = \mathbf{S}_M^T \cdot \mathbf{R}_M \cdot \mathbf{S}_M \tag{9}$$

due to orthogonality (6) of matrix $\mathbf{S}_L$ defined in (5).

B. The complex $M \times M$ matrix $\mathbf{C} = [c_{i\ell}]_{i,\ell=1}^M = \mathbf{C}' + j \cdot \mathbf{C}''$ is persymmetric if the following equalities

$$\mathbf{C} = \mathbf{\Pi}_M \cdot \mathbf{C}^T \cdot \mathbf{\Pi}_M, \quad \mathbf{C}' = \mathbf{\Pi}_M \cdot \mathbf{C}'^T \cdot \mathbf{\Pi}_M, \quad \mathbf{C}'' = \mathbf{\Pi}_M \cdot \mathbf{C}''^T \cdot \mathbf{\Pi}_M \tag{10}$$

hold.

If matrix C is associated with a correlation (Hermitian) matrix, the following additional equalities are true

$$\begin{aligned} &\mathbf{C} = \mathbf{\Pi}_M \cdot \mathbf{C}^T \cdot \mathbf{\Pi}_M = \mathbf{\Pi}_M \cdot \mathbf{C}^\sim \cdot \mathbf{\Pi}_M = \mathbf{C}^*, \quad \mathbf{C}^T = \mathbf{C}^\sim, \\ &\mathbf{C}' = \mathbf{\Pi}_M \cdot \mathbf{C}'^T \cdot \mathbf{\Pi}_M = \mathbf{\Pi}_M \cdot \mathbf{C}' \cdot \mathbf{\Pi}_M = \mathbf{C}'^T, \\ &\mathbf{C}'' = \mathbf{\Pi}_M \cdot \mathbf{C}''^T \cdot \mathbf{\Pi}_M = -\mathbf{\Pi}_M \cdot \mathbf{C}'' \cdot \mathbf{\Pi}_M = -\mathbf{C}''^T. \end{aligned} \tag{11}$$

Here, superscripts (˜) and (*) define complex conjugation and Hermitian conjugation (complex conjugation and transposition), respectively.

Let us introduce the unitary $M \times M$ matrix [31, 35, 38]

$$\mathbf{T} = [t_{i,\ell}]_{i,\ell=1}^M = \frac{1}{\sqrt{2}}(\mathbf{I}_M - j \cdot \mathbf{\Pi}_M) \tag{12}$$

that obviously satisfy the properties

$$\mathbf{T} = \mathbf{T}^T = \mathbf{\Pi}_M \cdot \mathbf{T} \cdot \mathbf{\Pi}_M = -j \cdot \mathbf{T}^\sim \cdot \mathbf{\Pi}_M = -j \cdot \mathbf{\Pi}_M \cdot \mathbf{T}^*, \mathbf{T} \cdot \mathbf{T}^* = \mathbf{I}_M. \tag{13}$$

Using (13), matrix (11) can be next transformed into the real symmetric $M \times M$ matrix

$$\mathbf{C}_r = \mathbf{T} \cdot \mathbf{C} \cdot \mathbf{T}^* = \mathbf{C}' + \mathbf{C}''^T \cdot \mathbf{\Pi}_M = \mathbf{C}_r^T = \mathbf{C}' + \mathbf{\Pi}_M \cdot \mathbf{C}'' \tag{14}$$

with the determinant

$$|\mathbf{C}_r| = \left|\mathbf{C}' + \mathbf{C}''^T \cdot \mathbf{\Pi}_M\right| = |\mathbf{C}' + \mathbf{\Pi}_M \cdot \mathbf{C}''| = |\mathbf{C}|, \tag{15}$$

which coincides with that of the initial matrix

$$\mathbf{C} = \mathbf{T}^* \cdot \mathbf{C}_r \cdot \mathbf{T} \tag{16}$$

due to the unitary model (13) of matrix $\mathbf{T}$ defined by (12).

3 pdf of ML estimate of real persymmetric CM

A. Let $M-$ variate random real (i.e., composed of real-valued entries) Gaussian (normal) vectors $\mathbf{y}_i = \left[y_\ell^{(i)}\right]_{\ell=1}^M$ of the $K-$ variate sample $\mathbf{Y} = [\mathbf{y}_i]_{i=1}^K$ be mutually independent and have zero means and identical non-negative definite $M \times M$ CMs $\mathbf{R}$, i.e.,

$$\begin{aligned} &\mathbf{Y} = [\mathbf{y}_i]_{i=1}^K, \quad \mathbf{y}_i = N(0, \mathbf{R}), \quad \overline{\mathbf{y}}_i = 0, \\ &\overline{\mathbf{y}_i \cdot \mathbf{y}_\ell^*} = \mathbf{R} \cdot \delta(i-\ell), \quad i, \ell \in 1, K, \end{aligned} \tag{17}$$

where $\delta(x)$ is the Kroneker symbol, and overbar defines the statistical averaging operator.

The joint pdf $p(\mathbf{Y})$ of elements of sample $\mathbf{Y}$ in this case is given by [3, 5, 12]

$$p(\mathbf{Y}) = (2\pi)^{-K \cdot M/2} \cdot |\mathbf{R}|^{-K/2} \cdot \exp\left\{-\frac{1}{2} \cdot tr\left(\mathbf{R}^{-1} \cdot \mathbf{A}_r\right)\right\}, \tag{18}$$

where $tr(\mathbf{\Phi})$ is the trace (sum of diagonal elements) of a matrix $\mathbf{\Phi}$, and

$$\mathbf{A}_r = \{a_{i\ell}\}_{i,\ell=1}^M = \mathbf{Y} \cdot \mathbf{Y}^T = \sum_{i=1}^K \mathbf{y}_i \cdot \mathbf{y}_i^T = \mathbf{A}_r^T = K \cdot \hat{\mathbf{R}}, \tag{19}$$

represents the $M \times M$ sample (random) CM.

Under conditions (17) and (18), matrix $\hat{\mathbf{R}} = K^{-1} \cdot \mathbf{A}_r$ represents an ML estimate of the real-valued general form CM $\mathbf{R}$ [3, 11–13], and matrix $\mathbf{A}_r$ has the Wishart distribution $W_M^{(\mathbf{R})}(\mathbf{A}_r, K, \mathbf{R})$ with $K - M$ degrees of freedom and parameter matrix $\mathbf{R}$ [2–5, 40]. Then,

$$p\left(\mathbf{A}_r\right) = W_M^{(\mathbf{R})}\left(\mathbf{A}_r, K, \mathbf{R}\right) = \frac{F_M^{(\mathbf{R})}\left(\mathbf{A}_r, K, \mathbf{R}\right)}{f_M^{(\mathbf{R})}\left(K, \mathbf{R}\right)}, \quad (20a)$$

where

$$F_M^{(\mathbf{R})}\left(\mathbf{A}_r, K, \mathbf{R}\right) = \left|\mathbf{A}_r\right|^{\frac{K-M-1}{2}} \cdot exp\left\{-\frac{1}{2}\cdot tr\left(\mathbf{R}^{-1}\cdot\mathbf{A}_r\right)\right\}, \quad (20b)$$

$$f_M^{(\mathbf{R})}\left(K, \mathbf{R}\right) = 2^{K\cdot M/2}\cdot\pi^{M\cdot(M-1)/4}\cdot\left|\mathbf{R}\right|^{K/2}\cdot\prod_{i=1}^{M}\Gamma\left(\frac{K+1-i}{2}\right), \quad K\geq M, \quad (20c)$$

and $\Gamma(x)$ is the gamma function, which for an integer $x = n \geq 1$ is equal to $(n-1)!$.

Here, the distribution of a random matrix is specified via a joint distribution of random elements that compose such a matrix [3, 9]. Thus, (20) presents an "economical" definition of the pdf $p\left(\mathbf{A}_r\right) = p\left(a_{i\ell}\right)$, $i \in 1, M$, $\ell \in i, M$, as a function of $M\cdot(M+1)/2$ scalar variables, which are completely specified by the real-valued diagonal and above-diagonal elements of the symmetric matrix $\mathbf{A}$ defined by (19).

If CM $\mathbf{R}$ is persymmetric, then under the condition (17), its ML estimate may be written as[1]

$$\hat{\mathbf{R}}_p = \frac{1}{K}\cdot\mathbf{A}_{rp}, \quad \mathbf{A}_{rp} = \frac{1}{2}\cdot\left(\mathbf{Y}\cdot\mathbf{Y}^T + \mathbf{\Pi}_M\cdot\mathbf{Y}\cdot\mathbf{Y}^T\cdot\mathbf{\Pi}_M\right). \quad (21)$$

The problem at hand is to derive the closed-form expression for the pdf of that matrix $\mathbf{A}_{rp}$.

B. The matrix defined by (21) is a sum of two symmetric matrices each being a result of a permutation of another one with respect to its secondary diagonal; thus, it is also symmetric and persymmetric at the same time that follows directly from definition (2). For even $M = 2\cdot L$ (at this stage, we restrict ourselves by that assumption for the sake of simplicity), matrix (21) is defined by $L\cdot(L+1)$ random parameters—its elements $a_{i\ell}$, $i \in 1, L$; $\ell \in i, M+1-i$.

A comparison of (21) with (19) reveals that the first term of matrix $\mathbf{A}_{rp}$ has Wishart distribution (20) with the parameter matrix $\mathbf{R}/2$, and the second term has the same Wishart distribution under the conditions (2). Vectors $\mathbf{\Pi}_M\cdot\mathbf{y}_i$, $i \in 1, K$ of the "inverted" sample $\mathbf{\Pi}_M\cdot\mathbf{Y}$ in (21), possess the same properties (17) as the initial vectors $\mathbf{y}_i$. If these terms are mutually independent, then their sum has the Wishart distribution analogous to (20) with $2\cdot K - M$ degrees of freedom and the parameter matrix $\mathbf{R}/2$ [3–5]. However, for the terms of matrix $\mathbf{A}_{rp}$ defined by (21), this condition is not valid; therefore, its distribution should be different [34].

C. In order to find the desired pdf, we next partition the initial $2\cdot L\times K$ matrix sample $\mathbf{Y} = [\mathbf{y}_i]_{i=1}^{K}$, $\mathbf{y}_i = \left[y_\ell^{(i)}\right]_{\ell=1}^{2L}$ into the $L\times K$ "upper" $\mathbf{Y}_\mathrm{U}$ and "lower" $\mathbf{Y}_\mathrm{L}$ blocks, so that

$$\mathbf{Y} = \begin{bmatrix}\mathbf{Y}_\mathrm{U}\\ \mathbf{Y}_\mathrm{L}\end{bmatrix}, \quad \begin{array}{ll}\mathbf{Y}_\mathrm{U} = [\mathbf{y}_{\mathrm{U}i}]_{i=1}^{K}, & \mathbf{y}_{\mathrm{U}i} = \left[y_\ell^{(i)}\right]_{\ell=1}^{L},\\ \mathbf{Y}_\mathrm{L} = [\mathbf{y}_{\mathrm{L}i}]_{i=1}^{K}, & \mathbf{y}_{\mathrm{L}i} = \left[y_\ell^{(i)}\right]_{\ell=L+1}^{2L}.\end{array} \quad (22)$$

Let us introduce a linear transform of (22) performed with matrix $\mathbf{S}_M$ defined by (5), i.e.,

$$\mathbf{V} = [\mathbf{v}_i]_{i=1}^{K} = \mathbf{S}_M\cdot\mathbf{Y} = \begin{bmatrix}\mathbf{V}_\Sigma\\ \mathbf{V}_\Delta\end{bmatrix},$$
$$\mathbf{V}_\Sigma = \left[\mathbf{v}_{\Sigma i}\right]_{i=1}^{K} = \frac{1}{\sqrt{2}}\cdot\left(\mathbf{Y}_\mathrm{U} + \mathbf{\Pi}_L\cdot\mathbf{Y}_\mathrm{L}\right), \quad \mathbf{V}_\Delta = [\mathbf{v}_{\Delta i}]_{i=1}^{K} = \frac{1}{\sqrt{2}}\cdot\left(\mathbf{Y}_\mathrm{L} - \mathbf{\Pi}_L\cdot\mathbf{Y}_\mathrm{U}\right), \quad (23)$$

which allows rewrite (21), taking into account (6), as follows

$$\mathbf{A}_{rp} = \frac{1}{2}\cdot\left(\mathbf{S}_M^T\cdot\mathbf{V}\cdot\mathbf{V}^T\cdot\mathbf{S}_M + \mathbf{\Pi}_M\cdot\mathbf{S}_M^T\cdot\mathbf{V}\cdot\mathbf{V}^T\cdot\mathbf{S}_M\cdot\mathbf{\Pi}_M\right)$$
$$= \frac{1}{2}\cdot\mathbf{S}_M^T\cdot\left(\mathbf{V}\cdot\mathbf{V}^T + \mathbf{J}_M\cdot\mathbf{V}\cdot\mathbf{V}^T\cdot\mathbf{J}_M\right)\cdot\mathbf{S}_M.$$

It is easy to deduce that taking into account the properties (5) of matrix $\mathbf{J}_M$, the addends embraced in the above formula have identical $L\times L$ diagonal blocks and opposite in signs $L\times L$ off-diagonal blocks. Therefore, one can rewrite

$$\mathbf{A}_{rp} = \mathbf{S}_M^T\cdot\mathbf{B}_\mathbf{V}\cdot\mathbf{S}_M,$$
$$\mathbf{B}_\mathbf{V} = [b_{i\ell}]_{i,\ell=1}^{2L} = \begin{bmatrix}\mathbf{B}_\Sigma & \mathbf{0}\\ \mathbf{0} & \mathbf{B}_\Delta\end{bmatrix} = \mathbf{S}_M\cdot\mathbf{A}_{rp}\cdot\mathbf{S}_M^T, \quad (24a)$$

$$\left|\mathbf{B}_\mathbf{V}\right| = \left|\mathbf{B}_\Sigma\right|\cdot\left|\mathbf{B}_\Delta\right| = \left|\mathbf{A}_{rp}\right|, \quad (24b)$$

where the $L\times L$ diagonal blocks $\mathbf{B}_\Sigma$ and $\mathbf{B}_\Delta$ are expressed as follows:

$$\mathbf{B}_\Sigma = \left[b_{i\ell}^{(\Sigma)}\right]_{i,\ell=1}^{L} = \mathbf{V}_\Sigma\cdot\mathbf{V}_\Sigma T, \quad \mathbf{B}_\Delta = \left[b_{i\ell}^{(\Delta)}\right]_{i,\ell=1}^{L} = \mathbf{V}_\Delta\cdot\mathbf{V}_\Delta^T. \quad (25)$$

Taking into account the interrelations (24), the problem at hand is transformed now into the problem of derivation of the pdf of the auxiliary matrix $\mathbf{B}_\mathbf{V}$ (24).

D. First, note that due to orthogonality of matrix $\mathbf{S}_M$, the Jacobian of the transform, $\mathbf{Y} = \mathbf{S}_M^T\cdot\mathbf{V}$, is equal to unity; hence, the pdf $p(\mathbf{V})$ of the transformed sample $\mathbf{V}$ (23) under conditions (18) becomes

$$p(\mathbf{V}) = (2\pi)^{-K\cdot L}\cdot|\mathbf{R}|^{-K/2}\cdot exp\left\{-\frac{1}{2}tr\left(\mathbf{R}^{-1}\cdot\mathbf{S}_M^T\cdot\mathbf{V}\cdot\mathbf{V}^T\cdot\mathbf{S}_M\right)\right\}.$$

Using the property of the matrix product trace, $tr(\mathbf{A}\cdot\mathbf{B}) = tr(\mathbf{B}\cdot\mathbf{A})$, and taking into account (7)–(9), the latter formula can be rewritten as follows:

$$p(\mathbf{V}) = (2\pi)^{-K\cdot L}\cdot|\mathbf{R}_\Sigma|^{-K/2}\cdot|\mathbf{\Pi}_L\cdot\mathbf{R}_\Delta\cdot\mathbf{\Pi}_L|^{-K/2} \times\exp\left\{-\frac{1}{2}\cdot tr\left(\mathbf{R}_M^{-1}\cdot\mathbf{V}\cdot\mathbf{V}^T\right)\right\}.$$

Next, taking into account the explained above properties (7) and (23)–(25), we obtain

$$\begin{aligned} tr\left(\mathbf{R}_M^{-1}\cdot\mathbf{V}\cdot\mathbf{V}^T\right) &= tr\left(\mathbf{R}_\Sigma^{-1}\cdot\mathbf{V}_\Sigma\cdot\mathbf{V}_\Sigma^T\right) \\ &\quad + tr\left(\mathbf{\Pi}_L\cdot\mathbf{R}_\Delta^{-1}\cdot\mathbf{\Pi}_L\cdot\mathbf{V}_\Delta\cdot\mathbf{V}_\Delta^T\right) \\ &= tr\left(\mathbf{R}_M^{-1}\cdot\mathbf{B}_\mathbf{V}\right), \end{aligned} \tag{26a}$$

that yield

$$p(\mathbf{V}) = p\left(\mathbf{V}_\Sigma\right)\cdot p(\mathbf{V}_\Delta) \tag{26b}$$

where

$$p\left(\mathbf{V}_\Sigma\right) = (2\pi)^{-K\cdot L/2}\cdot\left|\mathbf{R}_\Sigma\right|^{-K/2}\cdot\exp\left\{-\frac{1}{2}tr\left(\mathbf{R}_\Sigma^{-1}\cdot\mathbf{V}_\Sigma\cdot\mathbf{V}_\Sigma^T\right)\right\}, \tag{27a}$$

$$p(\mathbf{V}_\Delta) = (2\pi)^{-K\cdot L/2}|\mathbf{\Pi}_L\cdot\mathbf{R}_\Delta\cdot\mathbf{\Pi}_L|^{-K/2}\cdot\exp\left\{-\frac{1}{2}tr\left(\mathbf{\Pi}_L\cdot\mathbf{R}_\Delta^{-1}\cdot\mathbf{\Pi}_L\cdot\mathbf{V}_\Delta\cdot\mathbf{V}_\Delta^T\right)\right\}. \tag{27b}$$

The properties (17) and (4) of the CM blocks admit the following representations

$$\begin{aligned} &\overline{\mathbf{y}_{Ui}\cdot\mathbf{y}_{Ui}{}^T} = \mathbf{R}_{11}, \quad \overline{\mathbf{y}_{Li}\cdot\mathbf{y}_{Ui}{}^T} = \mathbf{\Pi}_L\cdot\mathbf{R}_{12}\cdot\mathbf{\Pi}_L, \\ &\overline{\mathbf{y}_{Ui}\cdot\mathbf{y}_{Li}{}^T} = \mathbf{R}_{12}, \quad \overline{\mathbf{y}_{Li}\cdot\mathbf{y}_{Li}{}^T} = \mathbf{\Pi}_L\cdot\mathbf{R}_{11}\cdot\mathbf{\Pi}_L, \quad i\in\overline{1,K}, \end{aligned}$$

and using the definitions (23), (22), and (7), it is easy to deduce that matrices $\mathbf{R}_\Sigma$ and $\mathbf{\Pi}_L\cdot\mathbf{R}_\Delta\cdot\mathbf{\Pi}_L$, which specify the entries in the corresponding expressions (27), can be expressed as follows,

$$\mathbf{R}_\Sigma = \overline{\mathbf{v}_{\Sigma i}\cdot\mathbf{v}_{\Sigma i}^T}, \quad \mathbf{\Pi}_L\cdot\mathbf{R}_\Delta\cdot\mathbf{\Pi}_L = \overline{\mathbf{v}_{\Delta i}\cdot\mathbf{v}_{\Delta i}^T}, \quad i\in\overline{1,K}. \tag{28}$$

In doing so, it is easy to observe that $K-$ variate "summ", $\mathbf{V}_\Sigma$ and "difference", $\mathbf{V}_\Delta$; samples (23) of the random $L = M/2-$ variate vectors $\mathbf{v}_{\Sigma i}$ and $\mathbf{v}_{\Delta i}$ $(i\in\overline{1,K})$ have normal distributions (27), and matrices $\mathbf{B}_\Sigma$ and $\mathbf{B}_\Delta$ formed via (25) have Wishart distributions with $K-M$ degrees of freedom and the parameter matrices $\mathbf{R}_\Sigma$ and $\mathbf{\Pi}_L\cdot\mathbf{R}_\Delta\cdot\mathbf{\Pi}_L$, respectively, i.e.,

$$p\left(\mathbf{B}_\Sigma\right) = W_L^{(\mathbf{R})}\left(\mathbf{B}_\Sigma, K, \mathbf{R}_\Sigma\right), \tag{29a}$$

$$p(\mathbf{B}_\Delta) = W_L^{(\mathbf{R})}(\mathbf{B}_\Delta, K, \mathbf{\Pi}_L\cdot\mathbf{R}_\Delta\cdot\mathbf{\Pi}_L). \tag{29b}$$

On the other hand, due to the mutual independence of samples $\mathbf{V}_\Sigma$ and $\mathbf{V}_\Delta$ that follow from (26), matrices $\mathbf{B}_\Sigma$ and $\mathbf{B}_\Delta$ defined by (25) are also mutually independent, and their joint density is, therefore, $p(\mathbf{B}_\Sigma, \mathbf{B}_\Delta) = p(\mathbf{B}_\Sigma)\cdot p(\mathbf{B}_\Delta)$. Multiplying these densities (29a) and (29b) and taking into account (26), (24), and (8), we obtain the density $p\,(\mathbf{B}_V)$ of matrix $\mathbf{B}_V$ defined by (24),

$$p(\mathbf{B}_\mathbf{V}) = \frac{|\mathbf{B}_\mathbf{V}|^{(K-L-1)/2}\cdot\exp\left\{-\frac{1}{2}tr\left(\mathbf{R}_M^{-1}\cdot\mathbf{B}_\mathbf{v}\right)\right\}}{2^{K\cdot L}\cdot\pi^{L\cdot(L-1)/2}\cdot|\mathbf{R}_M|^{K/2}\cdot\prod_{i-1}^{L}\Gamma^2\left(\frac{K+1-i}{2}\right)}. \tag{30}$$

Each of two symmetric $L\times L$ matrices $\mathbf{B}_\Sigma$ and $\mathbf{B}_\Delta$ in the arguments of $p(\mathbf{B}_\Sigma, \mathbf{B}_\Delta)$ are defined by $L\cdot(L+1)/2$ parameters, so the number of such parameters in matrix $\mathbf{B}_V$ (24) is equal to $L\cdot(L+1)$ that exactly coincides with the number of parameters that determine matrix $\mathbf{A}_{rp}$ (21). Therefore, to obtain the desired pdf $p\,(\mathbf{A}_{rp})$ using (30), it is enough to define the Jacobian of the transform (24) that relates $\mathbf{B}_\mathbf{V}$ and $\mathbf{A}_{rp}$.

Using (5) and (24), it is easy to deduce that

$$b_{i\ell} = a_{i\ell} + a_{i,\,2\cdot i+1-\ell}, \quad b_{2\cdot L+1-\ell,\,2\cdot L+1-i} = a_{i\ell} - a_{i,\,2\cdot L+1-\ell}, \quad i\in\overline{1,\,L}; \quad \ell\in\overline{i,L}.$$

Then, the Jacobian matrix of the transform (24) can be written as $\mathbf{I}_{L\cdot(L+1)/2}\otimes\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$ where $\otimes$ defines the Kroneker product, and hence, that Jacobian is equal to $2^{L\cdot(L+1)/2}$.

Replacing in (30) matrix $\mathbf{B}_V$ by its representation (24a) and taking into account (24b), (9), and (8), we obtain

$$p\left(\mathbf{A}_{rp}\right) = \frac{\left|\mathbf{A}_{rp}\right|^{(K-L-1)/2}\cdot\exp\left\{-\frac{1}{2}tr\left(\mathbf{R}^{-1}\cdot\mathbf{A}_{rp}\right)\right\}}{2^{(2\cdot K-L-1)\cdot L/2}\cdot\pi^{L\cdot(L-1)/2}\cdot|\mathbf{R}|^{K/2}\cdot\prod_{i-1}^{L}\Gamma^2\left(\frac{K+1-i}{2}\right)}. \tag{31}$$

The latter formula describes the desired pdf of the real symmetric and persymmetric random matrix $\mathbf{A}_{rp}$ of the ML estimate $\hat{\mathbf{R}}_p$ (21) of the real and also persymmetric CM $\mathbf{R}$ of an even order $M = 2\cdot L$ defined above in (2) and (17). This formula has the same form as the Wishart distribution (20) of the matrix $\mathbf{A}_r$ (19) from the ML estimate $\hat{\mathbf{R}} = K^{-1}\cdot\mathbf{A}_r$ of GCM. However, for formula (31), the reduced number of parameters that determine PCM has resulted in increased on $L = M/2$ number of degrees of freedom. That is why this formula could be considered as modified Wishart distribution of the ML estimate $\hat{\mathbf{R}}_p$ (21) of the real PCM $\mathbf{R}$ of an even order $M = 2\cdot L$.

4 Distribution density of ML estimate of complex persymmetric CM

A. Let the random complex normal $M-$ variate vectors $\mathbf{y}_i = \left[y_\ell^{(i)}\right]_{\ell=1}^{M} = \mathbf{y}_i' + j\cdot\mathbf{y}_i''$ of the $K-$ variate sample $\mathbf{Y} = [\mathbf{y}_i]_{i=1}^{K}$ be mutually independent and have zero means and identical non-negative definite complex Hermitian $M \times M$ CMs, $\mathbf{C} = [c_{i\ell}]_{i,\ell=1}^{M} = \mathbf{C}' + j\cdot\mathbf{C}''$, i.e.,

$$\mathbf{Y} = \mathbf{Y}' + j\cdot\mathbf{Y}'' = [\mathbf{y}_i]_{i=1}^{K}, \ \ \mathbf{y}_i \sim CN(0, \mathbf{C}), \ \ \overline{\mathbf{y}_i} = 0,$$
$$\overline{\mathbf{y}_i\cdot\mathbf{y}_\ell^*} = \mathbf{C}\cdot\delta(i-\ell), \ \ i, \ell \in 1, K. \tag{32a}$$

The latter means [8, 12] that real $\mathbf{y}_i'$ and imaginary $\mathbf{y}_i''$ parts of vectors $\mathbf{y}_i$ $(i \in 1, K)$ are zero means jointly normal real-valued vectors. Then, the $2\cdot M-$ variate vectors

$$\mathbf{g}_i^T = \left[\mathbf{y}_i'^T, \ \ \mathbf{y}_i''^T\right] \sim N(0, \mathbf{Q}), \ \ \overline{\mathbf{g}}i = 0,$$
$$\overline{\mathbf{g}_i\cdot\mathbf{g}_\ell^T} = \mathbf{Q}\cdot\delta(i-\ell), \ \ i, \ell \in 1, K \tag{32b}$$

are also mutually independent with zero means and identical $2\cdot M \times 2\cdot M$ CMs

$$\mathbf{Q} = \overline{\mathbf{g}_i\cdot\mathbf{g}_i^T} = \begin{bmatrix} \overline{\mathbf{y}_i'\cdot\mathbf{y}_i'^T} & \overline{\mathbf{y}_i'\cdot\mathbf{y}_i''T} \\ \overline{\mathbf{y}_i''\cdot\mathbf{y}_i'^T} & \overline{\mathbf{y}_i''\cdot\mathbf{y}_i''T} \end{bmatrix} = \frac{1}{2}\cdot\begin{bmatrix} \mathbf{C}' & -\mathbf{C}'' \\ \mathbf{C}'' & \mathbf{C}' \end{bmatrix}, \ \ i \in 1, K. \tag{32c}$$

The joint distribution of the sample $\mathbf{Y}$ in that case is given by [8, 12]

$$p(\mathbf{Y}) = \pi^{-K\cdot M}\cdot|\mathbf{C}|^{-K}\cdot\exp\left\{-tr\left(\mathbf{C}^{-1}\cdot\mathbf{A}_c\right)\right\}, \tag{33}$$

where

$$\mathbf{A}_c = [a_{i\ell}]_{i,\ell=1}^{M} = \sum_{i=1}^{K}\mathbf{y}_i\cdot\mathbf{y}_i^* = \mathbf{Y}\cdot\mathbf{Y}^* = \mathbf{A}_c^* = K\cdot\hat{\mathbf{C}}. \tag{34}$$

is the $M \times M$ sample complex CM. Under conditions (32), the matrix $\hat{\mathbf{C}} = K^{-1}\cdot\mathbf{A}_c$ in (34) specifies an ML estimate of the general type complex CM $\mathbf{C}$ [6–12], whereas matrix $\mathbf{A}_c$ is characterized by the complex Wishart distribution, $W_M^{(\mathbf{C})}(\mathbf{A}_c, K, \mathbf{C})$, with $K-M+1$ degrees of freedom and the parameter matrix $\mathbf{C}$ [6, 8, 9], i.e.,

$$p(\mathbf{A}_c) = W_M^{(\mathbf{C})}(\mathbf{A}_c, K, \mathbf{C}) = \frac{\mathbf{F}_M^{(\mathbf{C})}(\mathbf{A}_c, K, \mathbf{C})}{\mathbf{f}_M^{(\mathbf{C})}(K, \mathbf{C})}, \tag{35a}$$

$$\mathbf{F}_M^{(\mathbf{C})}(\mathbf{A}_c, K, \mathbf{C}) = |\mathbf{A}_c|^{K-M}\cdot\exp\left\{-tr\left(\mathbf{C}^{-1}\cdot\mathbf{A}_c\right)\right\}, \tag{35b}$$

$$\mathbf{f}_M^{(\mathbf{C})}(K, \mathbf{C}) = \pi^{M\cdot(M-1)/2}\cdot|\mathbf{C}|^{K}\cdot\prod_{i-1}^{M}\Gamma(K+1-i), \quad K \geq M. \tag{35c}$$

Here, the pdf of the complex matrix $\mathbf{C}$ is treated as the joint distribution of its random real and imaginary parts [6–10]. Thus, (35a) specifies a non-negative function of M^2 parameters

$$p(\mathbf{A}_c) = p\left(a_{11}, a_{22}, ..., a_{MM}, a_{i\ell}', a_{i\ell}''\right), \ \ i \in 1, M-1,$$
$$\ell \in i+1, M.$$

Such parameters are completely defied by the real diagonal elements a_{ii} $(i \in 1, M)$ of the random Hermitian complex matrix $\mathbf{A}_c$ (34) and $M\cdot(M-1)$ real $\left(a_{i\ell}'\right)$ and imaginary $\left(a_{i\ell}''\right)$ parts of its above-diagonal elements $a_{i\ell} = a_{i\ell}' + j\cdot a_{i\ell}''$, $(i \in 1, \ M-1; \ \ell \in i+1, \ M)$.

If CM $\mathbf{C}$ is persymmetric, then under the conditions (32), its ML estimate admits the following representation [28–31, 27–37]:

$$\hat{\mathbf{C}}_p = \frac{1}{K}\cdot\mathbf{A}_{cp},$$
$$\mathbf{A}_{cp} = \frac{1}{2}\left(\mathbf{Y}\cdot\mathbf{Y}^* + \mathbf{\Pi}_M\cdot\mathbf{Y}^{\sim}\cdot\mathbf{Y}^T\cdot\mathbf{\Pi}_M\right) = \mathbf{A}_{cp}^*$$
$$= \mathbf{\Pi}_M\cdot\mathbf{A}_{cp}^{\sim}\cdot\mathbf{\Pi}_M. \tag{36}$$

Thus, the problem at hand now is to find the distribution density of the matrix $\mathbf{A}_{cp}$ in (36).

B. By construction, this matrix $\mathbf{A}_{cp}$ is Hermitian and persymmetric as a sum of two Hermitian matrices each being a result of permutation of another one with respect to the secondary diagonal. Therefore, such $\mathbf{A}_{cp}$ is completely specified by $M\cdot(M+1)/2$ real-valued scalar parameters, among which there are

$$z = \varepsilon\left[\frac{M}{2}\right]\cdot\varepsilon\left[\frac{M+1}{2}\right] = \begin{cases} (L-1)\cdot L, & M = 2\cdot L-1, \\ L^2, & M = 2\cdot L, \end{cases} \tag{37}$$

the matrix imaginary parts $\left(a_{i\ell}''\right)$, and the rest $M\cdot(M+1)/2-z$ real parts $\left(a_{i\ell}'\right)$ of the elements $a_{i\ell}$, $i \in 1, L$; $\ell \in i, M+1-i$ that explicitly specify the whole matrix $\mathbf{A}_{cp}$. In (37), $\varepsilon[x]$ represents the integer part of the embraced variable x.

From a comparison of (36) with (34), it follows that the first addend in matrix $\mathbf{A}_{cp}$ is characterized by the distribution, $W_M^{(\mathbf{C})}(\mathbf{A}_c, K, \mathbf{C}/2)$. The second addend has the same distribution as well, since under the condition (11), vectors $\mathbf{\Pi}_M\cdot\mathbf{y}_i^{\sim}$ $(i \in 1, K)$ of the "reverse" and complex conjugate sample $\mathbf{\Pi}_M\cdot\mathbf{Y}^{\sim}$ possess the same properties as the initial vectors, $\mathbf{y}_i$. The samples $\mathbf{Y}$ and $\mathbf{\Pi}_M\cdot\mathbf{Y}^{\sim}$

are mutually uncorrelated, i.e., $\overline{\mathbf{Y}\cdot\left(\widetilde{\mathbf{\Pi}_M\cdot\mathbf{Y}}\right)^*} = 0$ [8, 12]; however, they are not jointly normal [34]. Absence of mutual correlation does not mean mutual independence that does not allow to represent the joint pdf $p\,(\mathbf{Y}, \mathbf{\Pi}_M \cdot \mathbf{Y}^{\sim})$ via the product $p\,(\mathbf{Y}) \cdot p\,(\mathbf{\Pi}_M \cdot \mathbf{Y}^{\sim})$. In addition, the distribution $W_M^{(\mathbf{C})}\,(\mathbf{A}_c, 2K, \mathbf{C}/2)$ specifies the density of the sum in (36) only under the conditions of mutual independence of the addends.

C. Let us consider the transform of the sample matrix $\mathbf{Y}$ performed by the unitary matrix $\mathbf{T}$ defined in (12), i.e.,

$$\mathbf{V} = \{\mathbf{v}_i\}_{i=1}^{K} = \mathbf{T}\cdot\mathbf{Y} = \mathbf{V}_{\sum} + j\cdot\mathbf{V}_{\Delta}, \tag{38a}$$

$$\mathbf{V}_{\sum} = \left\{\mathbf{v}_{\Sigma i}\right\}_{i=1}^{K} = \frac{1}{\sqrt{2}}\left(\mathbf{Y}' + \mathbf{\Pi}_M\cdot\mathbf{Y}''\right),$$
$$\mathbf{V}_{\Delta} = \{\mathbf{v}_{\Delta i}\}_{i=1}^{K} = \frac{1}{\sqrt{2}}\left(\mathbf{Y}'' - \mathbf{\Pi}_M\cdot\mathbf{Y}'\right), \tag{38b}$$

Using (13) and (38), matrix $\mathbf{A}_{cp}$ defined by (36) admits the following representation

$$\mathbf{A}_{cp} = \frac{1}{2}\cdot\mathbf{T}^*\cdot\left(\mathbf{V}\cdot\mathbf{V}^* + \mathbf{V}^{\sim}\cdot\mathbf{V}^T\right)\cdot\mathbf{T}.$$

Obviously, the addends in braces in this equality are complex conjugate, thus

$$\mathbf{A}_{cp} = \mathbf{T}^*\cdot\mathbf{B}_{\mathbf{V}}\cdot\mathbf{T} \tag{39}$$

where

$$\mathbf{B}_{\mathbf{V}} = [b_{i\ell}]_{i,\ell=1}^{M} = \mathrm{Re}\,(\mathbf{V}\cdot\mathbf{V}^*) = \mathbf{B}_{\sum} + \mathbf{B}_{\Delta} = \mathbf{B}_{\mathbf{V}}^T, \tag{40a}$$

$$\mathbf{B}_{\sum} = \left[b_{i\ell}^{(\sum)}\right]_{i,\ell=1}^{M} = \mathbf{V}_{\sum}\cdot\mathbf{V}_{\sum}^T,\quad \mathbf{B}_{\Delta} = \left[b_{i\ell}^{(\Delta)}\right]_{i,\ell=1}^{M} = \mathbf{V}_{\Delta}\cdot\mathbf{V}_{\Delta}^T. \tag{40b}$$

The representation (39) yields the following equalities

$$\mathbf{B}_{\mathbf{V}} = \mathbf{T}\cdot\mathbf{A}_{cp}\cdot\mathbf{T}^*,\quad |\mathbf{B}_{\mathbf{V}}| = \left|\mathbf{A}_{cp}\right|, \tag{41}$$

that reduce the problem at hand to deriving the pdf of the real symmetric matrix $\mathbf{B}_{\mathbf{V}}$.

D. Note that due to (32a) and (32c), CMs of the "summary" $\mathbf{v}_{\Sigma i}$ and the "difference" $\mathbf{v}_{\Delta i}$ $(i \in \overline{1, K})$ vectors in samples $\mathbf{V}_{\Sigma}$ and $\mathbf{V}_{\Delta}$ (38b) are identical and equal to

$$\overline{\mathbf{v}_{\sum i}\cdot\mathbf{v}_{\sum \ell}^T} = \overline{\mathbf{v}_{\Delta i}\cdot\mathbf{v}_{\Delta\ell}^T} = \mathbf{C}_{\sum}\cdot\delta\,(i-\ell),\quad \mathbf{C}_{\Sigma} = \mathbf{C}_r/2,\quad i,\ell \in \overline{1, K}, \tag{42}$$

whereas (as a consequence of unitary $\mathbf{T}$, and the properties specified by (9) and (33)) the density $p(\mathbf{V})$ of the transformed sample $\mathbf{V}$ (38a) becomes

$$p\,(\mathbf{V}) = (2\cdot\pi)^{-K\cdot M}\left|\mathbf{C}_{\sum}\right|^{-K}\exp\left\{-\frac{1}{2}\cdot tr\left(\mathbf{C}_{\sum}^{-1}\cdot\mathbf{V}\cdot\mathbf{V}^*\right)\right\} \tag{43}$$

where matrix $\mathbf{C}_r$ has been defined in (14).

Taking into account a symmetry of matrices $\mathbf{C}_r$ and $\mathbf{C}_{\Sigma}$, and the expressions (40), it is easy to verify the following equalities

$$\begin{aligned} tr\left(\mathbf{C}_{\sum}^{-1}\cdot\mathbf{V}\cdot\mathbf{V}^*\right) &= tr\left\{\mathbf{C}_{\sum}^{-1}\cdot\mathrm{Re}\,(\mathbf{V}\cdot\mathbf{V}^*)\right\} \\ &= tr\left(\mathbf{C}_{\sum}^{-1}\cdot\mathbf{V}_{\sum}\cdot\mathbf{V}_{\sum}^T\right) \\ &\quad + tr\left(\mathbf{C}_{\sum}^{-1}\cdot\mathbf{V}_{\Delta}\cdot\mathbf{V}_{\Delta}^T\right). \end{aligned} \tag{44}$$

Those (43) can be also re-expressed as

$$p(\mathbf{V}) = p\left(\mathbf{V}_{\sum}, \mathbf{V}_{\Delta}\right) = p\left(\mathbf{V}_{\sum}\right)\cdot p(\mathbf{V}_{\Delta}) \tag{45}$$

where

$$p\left(\mathbf{V}_{\sum}\right) = (2\cdot\pi)^{-K\cdot M/2}\left|\mathbf{C}_{\sum}\right|^{-K/2}\cdot\exp\left\{-\frac{1}{2}\cdot tr\left(\mathbf{C}_{\sum}^{-1}\cdot\mathbf{V}_{\sum}\cdot\mathbf{V}_{\sum}^T\right)\right\}, \tag{46a}$$

$$p(\mathbf{V}_{\Delta}) = (2\cdot\pi)^{-K\cdot M/2}\cdot\left|\mathbf{C}_{\sum}\right|^{-K/2}\cdot\exp\left\{-\frac{1}{2}\cdot tr\left(\mathbf{C}_{\sum}^{-1}\cdot\mathbf{V}_{\Delta}\cdot\mathbf{V}_{\Delta}^T\right)\right\}. \tag{46b}$$

From (18), (20), and (40), it follows now that the pdf(s) of the matrices $\mathbf{B}_{\Sigma}$ and $\mathbf{B}_{\Delta}$ in (40) can be expressed as

$$p\left(\mathbf{B}_{\sum}\right) = W_M^{(\mathbf{R})}\left(\mathbf{B}_{\sum}, K, \mathbf{C}_{\sum}\right),\quad p(\mathbf{B}_{\Delta}) = W_M^{(\mathbf{R})}\left(\mathbf{B}_{\Delta}, K, \mathbf{C}_{\sum}\right). \tag{47a}$$

and because of (45), these matrices $\mathbf{B}_{\Sigma}$ and $\mathbf{B}_{\Delta}$ are mutually independent.

The pdf of the sum (40) is therefore given by

$$\begin{aligned} p(\mathbf{B}_{\mathbf{V}}) &= W_M^{(\mathbf{R})}\left(\mathbf{B}_{\mathbf{V}}, 2K, \mathbf{C}_{\sum}\right) \\ &= \frac{|\mathbf{B}_{\mathbf{V}}|^{(2K-M-1)/2}\cdot\exp\left\{-\frac{1}{2}tr\left(\mathbf{C}_{\sum}^{-1}\cdot\mathbf{B}_{\mathbf{V}}\right)\right\}}{2^{M\cdot K}\cdot\pi^{M\cdot(M-1)/4}\cdot\left|\mathbf{C}_{\sum}\right|^{K}\cdot\prod_{i-1}^{M}\Gamma\left(\frac{2K+1-i}{2}\right)}. \end{aligned} \tag{47b}$$

This formula has been already derived in somewhat different way in [35, 38]. In these works, it has been used in accordance with methodology [7, 9, 10] in order to attain the objectives been set, which, however, did not include a derivation of the pdf of the complex Hermitian persymmetric $M\times M$ matrix $\mathbf{A}_{cp}$ (36). At the same time, it is quite simple to proceed to this pdf form the pdf

(47b) by determination of a Jacobian of the transform (41) which connects the matrices $\mathbf{B_V}$ and $\mathbf{A}_{cp}$.

The symmetric $M \times M$ matrix $\mathbf{B_V}$ is defined by $M \cdot (M+1)/2$ parameters, whose number exactly coincides with the number of parameters that specify the Hermitian persymmetric matrix $\mathbf{A}_{cp}$. For elements $a_{i\ell} = a'_{i\ell} + j \cdot a''_{i\ell}$ of that matrix, the following equalities

$$\alpha_{i\ell} = \alpha_{M_\ell M_i} = \alpha^{\sim}_{M_\ell M_i} = \alpha_{i\ell}^{\sim}, \quad M_k = M+1-k,$$
$$\alpha'_{i\ell} = \alpha'_{M_\ell M_i} = \alpha'_{M_i M_\ell} = \alpha''_{\ell i}, \alpha''_{i\ell} = \alpha''_{M_\ell M_i} = -\alpha''_{M_i M_\ell}$$
$$= -\alpha''_{\ell i}, i, \ell \in 1, M$$

hold.

Next, taking into account (12), we can express the elements of matrix $\mathbf{B_V}$ (41) as follows:

$$b_{i\ell} = \alpha'_{i\ell} - \alpha''_{i,M_\ell}, \quad b_{M_\ell M_i} = \alpha'_{i\ell} + \alpha''_{i,M_\ell},$$

$$b_{i,M_i} = \alpha'_{i,M_i}, i \in 1, M, \ell \in i, M,$$

that allow to compute the Jacobian of the transform (41)

$$\det\begin{bmatrix} \mathbf{I}_z \otimes \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_L \end{bmatrix} = \det\left[\mathbf{I}_z \otimes \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\right] = 2^z, \quad z = \varepsilon\left[\frac{M}{2}\right] \cdot \varepsilon\left[\frac{M+1}{2}\right]. \tag{48}$$

Replacing in (47b) matrix $\mathbf{B_V}$ by its representation (41) and taking into account (48), (42), (16), and (15) yields

$$p(\mathbf{A}_{cp}) = \frac{|\mathbf{A}_{cp}|^{(2K-M-1)/2} \exp\{-tr(\mathbf{C}^{-1} \cdot \mathbf{A}_{cp})\}}{2^z \cdot \pi^{M \cdot (M-1)/4} \cdot |\mathbf{C}|^K \cdot \prod_{i-1}^{M} \Gamma\left(\frac{2K+1-i}{2}\right)}, K \geq L = \varepsilon\left[\frac{M+1}{2}\right]. \tag{49}$$

The latter formula completely defines the desired pdf of the complex Hermitian persymmetric $M \times M$ matrix $\mathbf{A}_{cp}$ present in the ML estimate $\hat{\mathbf{C}}_p$ (36) of the Hermitian PCM $\mathbf{C}$ specified in (11) and (32). This formula has the same form as the Wishart's–Goodman's pdf (35) of the matrix $\mathbf{A}_c$ (34) from the ML estimate $\hat{\mathbf{C}} = K^{-1} \cdot \mathbf{A}_c$ of GCM. However, for formula (49), the reduced number of parameters that determine PCM has resulted in increased on $(M-1)/2$ number of degrees of freedom. That is why this formula could be considered as modified Wishart's–Goodman's distribution of the ML estimate $\hat{\mathbf{C}}_p$ (36) of complex PCM $\mathbf{C}$ of an order M.

Note that in a particular case of $M = 1$, when $L = 1$, $z = 0$, $|\mathbf{C}| = c_{11} = \overline{\left|\mathbf{y}_1^{(i)}\right|^2} = \sigma^2$ and $\mathbf{A}_{cp} = a_{11} = \sum_{i=1}^{K} \left|\mathbf{y}_1^{(i)}\right|^2 = \mathbf{A}_c$, formula (49) is transformed into

$$p(\mathbf{A}_{cp}) = p(\mathbf{A}_c) = p(a_{11}) = \frac{1}{\sigma^2 \cdot (K-1)!} \cdot \left(\frac{a_{11}}{\sigma^2}\right)^{K-1} \cdot \exp\left(-\frac{a_{11}}{\sigma^2}\right), \tag{50}$$

i.e., the pdf (49) turns into the Erlang distribution (50) with the shape and scale parameters, K and σ^2, respectively. Such pdf (50) characterizes, in an explicit statistical sense, the pdf of a sum of K squared magnitudes of independent complex normal random variables with zero means and equal variances σ^2 [39].

5 Exemplifying practical usage of the derived distributions

Distributions (31) and (49) of a persymmetric estimation CM (21) and (36) resemble the pdf(s) (20) and (35) of a general type CM estimates (19) and (34) but with an increased number of the degrees of freedom. In connection with this, the well-known properties of real [1–5] and complex [6–10] Wishart distributions (with relevant modifications) are transferred into the derived here distributions. Here beneath, we feature an importance of those distributions referring to some characteristic SP examples.

A. A non-degenerate transformation

$$\mathbf{B}_{rp} = \mathbf{U} \cdot \mathbf{A}_{rp} \cdot \mathbf{U} \tag{51}$$

of a $2L \times 2L$ real persymmetric matrix $\mathbf{A}_{rp}$ (21) distributed via (31) with the non-random symmetric and persymmetric $2 \cdot L \times 2 \cdot L$ matrix $\mathbf{U} = \mathbf{U}^T = \mathbf{\Pi}_M \cdot \mathbf{U} \cdot \mathbf{\Pi}_M$ gives rise to the random symmetric and persymmetric matrix $\mathbf{B}_{rp}$ (51) with the same distribution but the transformed parametric matrix[2]

$$\mathbf{G} = \mathbf{U} \cdot \mathbf{R} \cdot \mathbf{U}. \tag{52}$$

Indeed, under the conditions (21), such matrix $\mathbf{B}_{rp}$ (51) can be expressed as

$$\mathbf{B}_{rp} = \frac{1}{2}\left(\mathbf{U} \cdot \mathbf{Y} \cdot \mathbf{Y}^T \cdot \mathbf{U} + \mathbf{U} \cdot \mathbf{\Pi}_{2L} \cdot \mathbf{Y} \cdot \mathbf{Y}^T \cdot \mathbf{\Pi}_{2L} \cdot \mathbf{U}\right) = \frac{1}{2}\left(\mathbf{V} \cdot \mathbf{V}^T + \mathbf{\Pi}_{2L} \cdot \mathbf{V} \cdot \mathbf{V}^T \cdot \mathbf{\Pi}_{2L}\right), \tag{53}$$

where $\mathbf{V} = \mathbf{U} \cdot \mathbf{Y} = \{\mathbf{v}_i\}_{\ell=1}^{K}$ is the $K-$ variate sample composed of $2 \cdot L -$ variate random vectors

$$\mathbf{v}_i = N(0, \mathbf{G}), \quad \overline{\mathbf{v}_i} = 0, \quad \overline{\mathbf{v}_i \cdot \mathbf{v}_\ell^T} = \mathbf{G} \cdot \delta(i-\ell), \quad i, \ell \in 1, K. \tag{54}$$

The pdf of matrix $\mathbf{B}_p$ (51) (for a fixed given $K \geq L$) can therefore be expressed as

$$p\left(\mathbf{B}_{rp}\right)=\frac{\left|\mathbf{B}_{rp}\right|^{(K-L-1)/2}\cdot\exp\left\{-\frac{1}{2}tr\left(\mathbf{G}^{-1}\cdot\mathbf{B}_{rp}\right)\right\}}{2^{\left(K-\frac{L+1}{2}\right)\cdot L}\cdot\pi^{\frac{L\cdot(L-1)}{2}}\cdot\left|\mathbf{G}\right|^{\frac{K}{2}}\cdot\prod_{i-1}^{L}\Gamma^{2}\left(\frac{K+1-i}{2}\right)}. \tag{55}$$

This formula permits ease computing of the Jacobian $\left|\frac{\partial(\mathbf{B}_{rp})}{\partial(\mathbf{A}_{rp})}\right|=\frac{p(\mathbf{A}_{rp})}{p(\mathbf{B}_{rp})}$ of the transform (51). Using (31) and taking into account (52), we obtain $\left|\frac{\partial(\mathbf{B}_{rp})}{\partial(\mathbf{A}_{rp})}\right|=|\mathbf{U}|^{L+1}$.

Similarly, the use of the complex matrix (36) (characterized by distribution (49)) in the transform $\mathbf{B}_{cp}=\mathbf{U}\cdot\mathbf{A}_{cp}\cdot\mathbf{U}$ (for a Hermitian and persymmetric $\mathbf{U}$) yields the same distribution $p(\mathbf{B}_{cp})$ (49) for $\mathbf{B}_{cp}$ but with the properly transformed parametric matrix $\mathbf{G}_c=\mathbf{U}\cdot\mathbf{C}\cdot\mathbf{U}$. The Jacobian of the transform is given by $\left|\frac{\partial(\mathbf{B}_{cp})}{\partial(\mathbf{A}_{cp})}\right|=|\mathbf{U}|^{M+1}$.

Note that the computations of the corresponding Jacobians become intractable without knowledge of the analytically closed expressions (31) and (49) for the corresponding pdf(s).

B. In the example, let us compare mean values of relative bias

$$\Delta(\alpha)=\frac{s(\alpha)-\overline{\hat{s}(\alpha)}}{s(\alpha)}=1-\overline{\hat{\nu}(\alpha)},\quad \hat{\nu}(\alpha)=\frac{\hat{s}(\alpha)}{s(\alpha)} \tag{56}$$

between random (estimate) spectral function (SF)

$$\hat{s}(\alpha)=\left(\mathbf{x}^{*}(\alpha)\cdot\hat{\mathbf{C}}^{-1}\cdot\mathbf{x}(\alpha)\right)^{-1} \tag{57a}$$

of Capon method [7, 18, 19] and true value of this SF

$$s(\alpha)=\left(\mathbf{x}^{*}(\alpha)\cdot\mathbf{C}_{p}^{-1}\cdot\mathbf{x}(\alpha)\right)^{-1}. \tag{57b}$$

The latter is inversely proportional to a quadratic form of complex steering vector $\mathbf{x}(\alpha)=\mathbf{x}'(\alpha)+j\cdot\mathbf{x}''(\alpha)$ with matrix $\mathbf{C}_p^{-1}$ being inverse to Hermitian PCM $\mathbf{C}_p$.

As an estimate of this matrix, we will use in (57a) following matrices

$$\hat{\mathbf{C}}=\begin{cases}K^{-1}\cdot\mathbf{A}_c & \text{(a)}\\ K^{-1}\cdot\mathbf{A}_{cp} & \text{(b)}\end{cases} \tag{58}$$

with defining matrices $\mathbf{A}_c$ (34) and $\mathbf{A}_{cp}$ (36) which obey pdfs (35) and (49), respectively. In the latter case, we will also assume that the steering vector $\mathbf{x}(\alpha)$ obeys a condition

$$\mathbf{x}(\alpha)=\mathbf{x}_p(\alpha)=c\cdot\mathbf{\Pi}_M\cdot\widetilde{\mathbf{x}_p}(\alpha),\quad |c|^2=1, \tag{59}$$

which is completely natural for CS receive channels.

As is shown in [7, 10], in the case (a)

$$\hat{\nu}(\alpha)=K^{-1}\cdot d, \tag{60}$$

where d is a random variable with the independent on α Erlang's pdf

$$p_d(x)=((\delta-1)!)^{-1}\cdot x^{\delta-1}\cdot\exp\{-x\},\quad \delta=\delta_g=K-M+1$$

with the shape parameter δ_g and the scale parameter equal to unity [39]. Its mean is $\bar{d}=\delta_g$. Therefore, by virtue of (60), $\overline{\hat{\nu}(\alpha)}=K^{-1}\cdot\bar{d}=1-(M-1)/K$, so that

$$\Delta(\alpha)=\Delta_g=(M-1)/K. \tag{61}$$

In the case (b), let us use the representation (39) for the matrix $\mathbf{A}_{cp}$. Then, taking into account the properties (13) of the matrix $\mathbf{T}$ (12), we will obtain for the SF $\hat{s}(\alpha)$ (57a) the following

$$\hat{s}(\alpha)=K^{-1}\cdot\left(\mathbf{z}^{*}(\alpha)\cdot\mathbf{B}_{\mathbf{V}}^{-1}\cdot\mathbf{z}(\alpha)\right)^{-1},\quad \mathbf{z}(\alpha)=\mathbf{T}\cdot\mathbf{x}(\alpha),$$

where $\mathbf{B_V}$ is the matrix (40) with the pdf (47). Under conditions (59), the vector

$$\mathbf{z}(\alpha)=(1-j)\cdot\mathbf{x}_{\Delta}(\alpha)/\sqrt{2},\quad \mathbf{x}_{\Delta}(\alpha)=\mathbf{x}'(\alpha)-\mathbf{x}''(\alpha),$$

and the latter SF is transformed to the formula

$$\hat{s}(\alpha)=K^{-1}\cdot\left(\mathbf{x}_{\Delta}^{*}(\alpha)\cdot\mathbf{B}_{\mathbf{V}}^{-1}\cdot\mathbf{x}_{\Delta}(\alpha)\right)^{-1},$$

which has a quadratic form of real-valued vector $\mathbf{x}_{\Delta}(\alpha)$ with real-valued symmetric matrix $\mathbf{B_V}$, with pdf (47), in denominator.

Next, using the methodology [7, 9, 10], it is possible to demonstrate that, under considered conditions, quantity $\hat{\nu}(\alpha)$ (56) obeys following equality similar to (60)

$$\hat{\nu}(\alpha)=K^{-1}\cdot d_1, \tag{62}$$

where d_1 is a random variable with the independent on α pdf

$$p_{d_1}(x)=\Gamma\left(\delta_p\right)^{-1}\cdot x^{\delta_p-1}\cdot\exp\{-x\}.\quad \delta_p=K-(M-1)/2.$$

For odd M, this distribution become the Erlang's one [39] with the shape parameter δ_p and the scale parameter equal to unity. That is why mean value $\bar{d}_1=\delta_p$, and, by virtue of (60) and (56),

$$\overline{\hat{\nu}(\alpha)}=K^{-1}\cdot\overline{d_1}=1-(M-1)/(2\cdot K),$$
$$\Delta(\alpha)=\Delta_p=(M-1)/(2\cdot K)=\Delta_g/2.$$

Thereby, under considered conditions, for the same training sample size K, the estimate (58b) reduces as great as twice the bias (56) of SF estimate (57) in contrast to the estimate (58a). This is equivalent to the statement that equal values of bias (56) are provided by the estimate (58b) at the twice smaller training sample size K.

C. Note, that, under conditions (59), the quadratic form in (57a) could be computed in more simple way by using the estimate (58b) instead of (58a). This is easy to follow by rewriting it as

$$\mathbf{x}^*(\alpha)\cdot\hat{\mathbf{C}}^{-1}\cdot\mathbf{x}(\alpha) = \begin{cases} K\cdot q_g(\alpha), & q_g(\alpha)=\mathbf{x}^*(\alpha)\cdot\mathbf{w}_g(\alpha), \quad \mathbf{w}_g(\alpha)=\mathbf{A}_c^{-1}\cdot\mathbf{x}(\alpha), \quad \text{(a)} \\ K\cdot q_p(\alpha), & q_p(\alpha)=\mathbf{x}_p^*(\alpha)\cdot\mathbf{w}_p(\alpha), \quad \mathbf{w}_p(\alpha)=\mathbf{A}_{cp}^{-1}\cdot\mathbf{x}_p(\alpha). \quad \text{(b)} \end{cases} \tag{63}$$

Matrix $\mathbf{A}_{cp}^{-1}$, being inverse to Hermitian persymmetric matrix, is also Hermitian persymmetric one, i.e., $\mathbf{A}_{cp}^{-1}=\mathbf{\Pi}_M\cdot\left(\widetilde{\mathbf{A}_{cp}}\right)^{-1}\cdot\mathbf{\Pi}_M$. That is why, under conditions (59), vector $\mathbf{w}_p(\alpha)=c\cdot\mathbf{\Pi}_M\cdot\widetilde{\mathbf{w}_p}(\alpha)$ and thus it is completely determined by the half ($M/2$) of its components which could be computed at the expense of $\approx M^2/2$ multiplications. Approximately $M/2$ multiplications are enough in order to compute the scalar product $q_p(\alpha)$. Thus, an amount of computations necessary to calculate the quadratic form in (57a) based on (58b) could be twice less than based on (58a).

D. Similar gains in the amount of computations and efficiency (including an efficiency in terms of criteria different from (56)) could be achieved in the systems with CS receive channels also by using other kinds of PCM ML estimates (considered, particularly, in [30–32, 35, 36, 37, 38–19]).

6 Conclusions

The main result of this work is the derivation of the pdfs (37) and (49) of random $M\times M$ real-valued and complex-valued ML estimates (21) and (36) of persymmetric CMs (2) and (11) of multivariate Gaussian processes and fields. Such CMs arise in numerous practical applications, particularly, in the tasks of space-time adaptive signal processing in systems with central symmetry of receive channels [20–32, 27–37, 38, 16, 17]. The derived pdfs have the same form as the classical Wishart's pdf and Wishart's–Goodman's pdf (20) and (35) for GCM ML estimates, but with the number of degrees of freedom being increased approximately on $M/2$. This is due to the approximately half less number of determining parameters (elements) of the PCM. That is why, in the systems with the CS receive channels, the derived distributions appeal to such importance as that of the classical Wishart and Wishart's–Goodman's pdfs in systems with arbitrary receive channels.

Reduced dimensionality of parameters vector for PCM in CS systems increases an efficiency of adaptive signal processing based on ML estimates (21) and (36) as compared with ML estimates (19) and (34). The gain depends on the efficiency criterion and additional conditions which take into account the task specificity. Nevertheless, in the above-considered example as well as in numerous other practically important cases, the adaptive processing based on estimates (21) and (36) under central symmetry of space-time receive channels could provide for some conditions at the expense of K_p training samples, the efficiency (in terms of different criteria) being close to that provided by the estimates (19) and (34) at the expense of $K\approx 2\cdot K_p$ training samples [34, 35, 36, 37].

It is essential that, taking into account the structure specificity of utilized matrices and vectors, it is possible to make signal processing in CS systems based on ML estimates (21) and (36) as well as their variants [30–32, 36, 37] less computationally consuming.

General considerations allow to assume a possibility to obtain similar gain not only in case of Gaussian stochastic processes but also in case of other stochastic processes. The latter, in particular, could be subject to exponential distribution or Weibull one. However, a rigorous proof of this hypothesis is unavailable to the author, since for such non-Gaussian processes, there are currently unknown neither their correlation matrices' ML estimates analogous to (19) and (34) nor these estimates' pdfs analogous to Wishart's (20) pdf and Wishart's–Goodman's (35) one.

Finally, note that statistical characteristics of different functions of ML estimates of Gaussian processes' PCM could be obtained without explicit use of their pdfs (31) and (49) what was illustrated, in particular, in [35, 36, 37, 38, 16, 17]. Nevertheless, the pdfs (31) and (49) are also useful in such case because they allow to perform only one step of transformations similar to (9) and (39) and then to use the methodology [7, 9, 10] in order to find statistical characteristics of functions of GCM ML estimates. Such approach seems to be the easiest one and methodologically reasonable. Namely, this approach is used by the author for comparative analysis of a number of "superresolving" spectral estimation methods in systems with CS receive channels. The results of this analysis are planned to be discussed in a special paper.

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7 Endnotes

[1]For place saving, we omit the derivation of formula (21), as it can be easily obtained according to the procedure employed in [28] when deriving the ML estimate of the Hermitian persymmetric CM.

[2]In application to distribution (35), this property is known as the Goodman theorem [8–10].

Abbreviations

AA: antenna array; CM: correlation matrix; CS: centrally symmetric; GCM: general type correlation matrix; ML: maximum likelihood; PCM: persymmetric correlation matrices; pdf: probability density function; SAR: synthetic aperture radar; SF: spectral function; SINR: signal-to-(interference + noise) ratio; SP: signal processing.

Competing interests
The author declares that he has no competing interests.

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