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Blow-up criteria for a two-fluid model of the truncated Euler equations

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Abstract

In this paper, we study a two-fluid model of the truncated Euler equations with partial viscosity. We obtain new blow-up criteria for a solution of the system in terms of vorticity in the homogeneous Besov space $\dot{B}_{\infty,\infty}^{-1}$.

Keywords: blow-up criterion; truncated Euler equations; Besov space

1 Introduction and main results

In this paper, we are concerned with the regularity of the following two-fluid model of the truncated Euler equations with partial viscosity:

$$\begin{cases} u_t - \Delta u + u \cdot \nabla u + \nabla p = \theta e_3, & (x;t) \in \mathbb{R}^3 \times (0;1), \\ \theta_t + u \cdot \nabla \theta = |\nabla u|^2, \\ \nabla \cdot u = 0, \\ u(x,0) = u_0, \qquad \theta(x,0) = \theta_0, \end{cases} \tag{1.1}$$

where $u = (u^1(x,t), u^2(x,t), u^3(x,t))$ denotes the fluid velocity vector field, $p = p(x,t)$ is the scalar pressure, $e_3 = (0,0,1)^T$, while u_0 and θ_0 are given initial velocity and initial temperature, respectively, with $\nabla \cdot u_0 = 0$. For more detailed background of the system, we refer the readers to [1] and the references therein. When $\theta = 0$, the system reduces to the well-known Navier-Stokes equations.

It is well known that the question of global existence or finite time blow-up of smooth solutions for the 3D incompressible Navier-Stokes equations is still unsolved. This challenging problem has attracted significant attention. Therefore, it is interesting to study the blow-up criterion of the solutions for system (1.1). Recently, Sun and Fan [2] first proved the following blow-up criterion for system (1.1):

$$\operatorname{curl} u \in L^1\big(0, T; \dot{B}_{\infty,\infty}^0\big(\mathbb{R}^3\big)\big),$$

$$u \in L^2\big(0, T; \dot{B}_{\infty,\infty}^0\big(\mathbb{R}^3\big)\big).$$

Our purpose in this paper is to establish blow-up criteria of a strong solution for the two-fluid model of the truncated Euler equations in terms of vorticity in the homogeneous Besov space $\dot{B}_{\infty,\infty}^{-1}$. The main difficulty is without viscosity in the second equation for system (1.1).

Now we state our main results as follows.

Theorem 1.1 *Let* $(u_0, \theta_0) \in H^2(\mathbb{R}^3)$, $(u(\cdot, t), \theta(\cdot, t))$ *be a strong solution to system* (1.1) *with the initial data* (u_0, θ_0) *for* $0 \le t < T$. *If the solution* u *satisfies the following condition*:

$$\operatorname{curl} u \in L^2\big(0, T; \dot{B}_{\infty,\infty}^{-1}\big), \tag{1.2}$$

then the solution (u, θ) *can be extended smoothly beyond* $t = T$.

2 Preliminaries and lemmas

Throughout this paper, we introduce some function spaces, notations and important inequalities.

Let $e^{t\triangle}$ denote the heat semigroup defined by

$$e^{t\triangle}f = K_t * f, \quad K_t = (4\pi t)^{-\frac{3}{2}} \exp\left(-\frac{|x|^2}{4t}\right)$$

for $t > 0$ and $x \in \mathbb{R}^3$, where $*$ denotes the convolution of functions defined on $\mathbb{R}^3$.

We now recall the definition of the homogeneous Besov space with negative indices $\dot{B}_{\infty,\infty}^{-\alpha}$ on $\mathbb{R}^3$ and the homogeneous Sobolev space $\dot{H}_q^\alpha$ of exponent $\alpha > 0$. It is known (p.192 of Ref. [3]) that $f \in \mathcal{S}'(\mathbb{R}^3)$ belongs to $\dot{B}_{\infty,\infty}^{-\alpha}$ if and only if $e^{t\triangle}f \in L^\infty$ for all $t > 0$ and $t^{\frac{\alpha}{2}}\|e^{t\triangle}f\|_\infty \in L^\infty(0, \infty; L^\infty)$. The norm of $\dot{B}_{\infty,\infty}^{-\alpha}$ is defined, up to equivalence, by

$$\|f\|_{\dot{B}_{\infty,\infty}^{-\alpha}} = \sup_{t>0}\big(t^{\frac{\alpha}{2}}\|e^{t\Delta}f\|_\infty\big).$$

We introduce now the homogeneous Sobolev space $\dot{H}_q^\alpha(\mathbb{R}^3)$, which is defined as the set of functions $f \in L^r(\mathbb{R}^3)$, $\frac{1}{r} = \frac{1}{q} - \frac{\alpha}{3}$ such that $(-\Delta)^{\frac{\alpha}{2}}f \in L^q(\mathbb{R}^3)$. This space is endowed with the norm

$$\|f\|_{\dot{H}_q^\alpha} = \big\|(-\Delta)^{\frac{\alpha}{2}}f\big\|_{L^q},$$

and when $q = 2$, we just let $\dot{H}_2^\alpha(\mathbb{R}^3) = \dot{H}^\alpha(\mathbb{R}^3)$.

Lemma 2.1 [4] *Let* $1 < p < q < \infty$ *and* $s = \alpha(\frac{q}{p} - 1) > 0$. *Then there exists a constant depending only on* α, p *and* q *such that the estimate*

$$\|f\|_{L^q} \le C\big\|(-\Delta)^{\frac{s}{2}}f\big\|_{L^p}^{\frac{p}{q}}\|f\|_{\dot{B}_{\infty,\infty}^{-\alpha}}^{1-\frac{p}{q}} \tag{2.1}$$

holds for all $f \in \dot{H}_p^\alpha(\mathbb{R}^3) \cap \dot{B}_{\infty,\infty}^{-\alpha}(\mathbb{R}^3)$.

In particular, for $s = 1$, $p = 2$ and $q = 4$, we get $\alpha = 1$ and

$$\|f\|_{L^4} \le C\|f\|_{\dot{H}^1}^{\frac{1}{2}}\|f\|_{\dot{B}_{\infty,\infty}^{-1}}^{\frac{1}{2}}. \tag{2.2}$$

Lemma 2.2 [5] *For any function* $f \in W^{1,s}(\mathbb{R}^3)$ ($s \ge 1$), *and* $r \ge 1$, *we have*

$$\|f\|_{L^\gamma} \le C\|f\|_{L^2}^{1-\alpha}\|\nabla f\|_{L^2}^\alpha, \tag{2.3}$$

where

$$\alpha = \frac{\frac{1}{r} - \frac{1}{\gamma}}{\frac{1}{3} - \frac{1}{s} - \frac{1}{r}}$$

and C is a constant independent of f.

3 Proof of the main results

In this section, we prove Theorem 1.1.

Proof of Theorem 1.1 First, we multiply the both sides of the first equation of (1.1) by u and the second equation of (1.1) by θ, respectively; after integration by parts over $\mathbb{R}^3$, we get

$$\frac{1}{2}\frac{d}{dt}\int u^2\,dx + \int |\nabla u|^2\,dx \le \int |u\theta|\,dx, \tag{3.1}$$

$$\frac{1}{2}\frac{d}{dt}\int \theta^2\,dx = \int \theta|\nabla u|^2\,dx. \tag{3.2}$$

Next, we multiply the both sides of the first equation of (1.1) by $(-\Delta u)$; after integration by parts over $\mathbb{R}^3$, we get

$$\frac{1}{2}\frac{d}{dt}\int |\nabla u|^2\,dx + \int |\Delta u|^2\,dx \le \int u \cdot \nabla u \Delta u\,dx + \int |\Delta u|\theta\,dx. \tag{3.3}$$

Using the Young inequality and adding (3.2) to (3.3), we obtain

$$\frac{d}{dt}\left(\int |\nabla u|^2\,dx + \int \theta^2\,dx\right) + \int |\Delta u|^2\,dx$$
$$\le \int u \cdot \nabla u \Delta u\,dx + \int \theta|\nabla u|^2\,dx + \int \theta^2\,dx. \tag{3.4}$$

It is easy to see that

$$\int u \cdot \nabla u \Delta u\,dx = -\sum_{i,k}\int \partial_k u \cdot \partial_i u \cdot \partial_k u\,dx$$
$$\le \|\nabla u\|_{L^2}\|\nabla u\nabla u\|_{L^2}$$
$$\le \|\nabla u\|_{L^2}\|\nabla u\|_{L^4}^2$$
$$\le \|\nabla u\|_{L^2}\|\nabla u\|_{\dot{B}_{\infty,\infty}^{-1}}\|\Delta u\|_{L^2}$$
$$\le C\|\nabla u\|_{L^2}^2\|\nabla \times u\|_{\dot{B}_{\infty,\infty}^{-1}}^2 + \frac{1}{2}\|\Delta u\|_{L^2}^2, \tag{3.5}$$

where we use the Hölder inequality, the Young inequality and (2.2).

   Similarly,

$$\int \theta|\nabla u|^2\,dx \le \|\theta\|_{L^2}\|\nabla u\nabla u\|_{L^2}$$
$$\le \|\theta\|_{L^2}\|\nabla u\|_{L^4}^2$$

$$\leq \|\theta\|_{L^2} \|\nabla u\|_{\dot{B}^{-1}_{\infty,\infty}} \|\Delta u\|_{L^2}$$

$$\leq C\|\theta\|^2_{L^2} \|\nabla \times u\|^2_{\dot{B}^{-1}_{\infty,\infty}} + \frac{1}{2} \|\Delta u\|^2_{L^2}. \tag{3.6}$$

Putting (3.5) and (3.6) into (3.4) and applying the Gronwall inequality yields

$$u \in L^\infty\left(0, T; H^1\right) \cap L^2\left(0, T; H^2\right), \quad \theta \in L^\infty\left(0, T; L^2\right).$$

Multiplying the both sides of the second equation of (1.1) by $|\theta|^{p-2}\theta$ ($2 < p < \infty$), after integration by parts over $\mathbb{R}^3$, we deduce that

$$\frac{1}{p}\frac{d}{dt} \int \theta^p \, dx \leq \int |\theta|^{p-1}|\nabla u|^2 \, dx \leq \|\theta\|^{p-1}_{L^p} \|\nabla u\|^2_{L^{2p}} \leq \|\theta\|^{p-1}_{L^p} \|u\|^2_{H^2},$$

which implies that

$$\theta \in L^\infty\left(0, T; L^p\right) \quad (2 < p < \infty). \tag{3.7}$$

Taking curl on the both sides of the first equation of (1.1), we obtain the following vorticity formulation for the vorticity field $\omega = \operatorname{curl} u$:

$$\partial_t \omega + u \cdot \nabla \omega - \Delta \omega = \operatorname{curl}(\theta e_3) + \omega \cdot \nabla u, \tag{3.8}$$

and then multiplying the both sides of the vorticity equation by $|\omega|^{p-2}\omega$ ($p > 2$), after integration by parts over $\mathbb{R}^3$, we infer that

$$\frac{d}{dt} \int v^2 \, dx + \int |\nabla v|^2 \, dx \leq C \int |\theta| |\nabla| |\omega|^{p-2}\omega| \, dx + C \int \nabla u |\omega|^p \, dx$$

$$\leq C \int |\theta| |v|^{\frac{p-2}{p}} |\nabla v| \, dx + C \int \nabla u |\omega|^{\frac{p}{2}} |\omega|^{\frac{p}{2}} \, dx$$

$$\leq C\|\theta\|_{L^{\frac{4p}{p+2}}} \|v\|^{1-\frac{2}{p}}_{L^4} \|\nabla v\|_{L^2} + C\|\nabla u\|_{L^2} \|v\|^2_{L^4}$$

$$\leq C\|v\|^{1-\frac{2}{p}}_{L^4} \|\nabla v\|_{L^2} + C\|v\|^{\frac{1}{2}}_{L^2} \|\nabla v\|^{\frac{3}{2}}_{L^2}$$

$$\leq C\|v\|^{\frac{1}{2}-\frac{1}{p}}_{L^2} \|\nabla v\|^{\frac{3}{2}-\frac{1}{p}}_{L^2} + C\|v\|^{\frac{1}{2}}_{L^2} \|\nabla v\|^{\frac{3}{2}}_{L^2}$$

$$\leq C\|v\|^2_{L^2} + \frac{1}{2} \|\nabla v\|^2_{L^2}, \tag{3.9}$$

where $v := |\omega|^{\frac{p}{2}}$. Using the Gronwall inequality, we obtain

$$\omega \in L^\infty\left(0, T; L^p\right) \quad (p > 2),$$

which implies that

$$\nabla u \in L^\infty\left(0, T; L^p\right) \quad (p > 2). \tag{3.10}$$

On the other hand, since u is a solution of the Stokes system

$$u_t - \Delta u + \nabla p = -u \cdot \nabla u + \theta e_3 \in L^\infty(0, T; L^p),$$

it follows from the $W^{2,p}$-theory of the Stokes system that

$$u \in L^p(0, T; W^{2,p}) \quad (p > 2). \tag{3.11}$$

So,

$$\nabla u \in L^p(0, T; L^\infty) \quad (p > 2). \tag{3.12}$$

Differentiating the second equation of (1.1) with respect to x_i ($1 \leq i \leq 3$) and multiplying the resulting equation by $\partial_i \theta$, and then integrating by parts over $\mathbb{R}^3$, we get

$$\frac{1}{2} \frac{d}{dt} \int |\nabla \theta|^2 \, dx \leq C \int |\nabla u| |\nabla \theta|^2 \, dx + C \int |\nabla u| |\Delta u| |\nabla \theta| \, dx$$

$$\leq C \|\nabla u\|_{L^\infty} \|\nabla \theta\|_{L^2}^2 + C \|\nabla u\|_{L^\infty} \|\Delta u\|_{L^2} \|\nabla \theta\|_{L^2}.$$

By (3.12) and the Gronwall inequality, we have

$$\theta \in L^\infty(0, T; H^1). \tag{3.13}$$

We multiply the both sides of the equation of (3.8) by $(-\Delta \omega)$; after integration by parts over $\mathbb{R}^3$, we get

$$\frac{1}{2} \frac{d}{dt} \int |\nabla \omega|^2 \, dx + \int |\Delta \omega|^2 \, dx \leq \int |u| |\nabla \omega| |\Delta \omega| \, dx + \int |\Delta \omega| |\nabla \theta| \, dx$$

$$\leq \|u\|_{L^4} \|\Delta \omega\|_{L^2} \|\nabla \omega\|_{L^4} + \|\nabla \theta\|_{L^2} \|\Delta \omega\|_{L^2}$$

$$\leq C \|\Delta \omega\|_{L^2}^{\frac{1}{2}} \|\nabla \omega\|_{L^2}^{\frac{3}{2}} + C \|\Delta \omega\|_{L^2}$$

$$\leq C \|\nabla \omega\|_{L^2}^2 + \frac{1}{2} \|\Delta \omega\|_{L^2}^2 + C,$$

the Gronwall inequality yields

$$u \in L^\infty(0, T; H^2) \cap L^2(0, T; H^3). \tag{3.14}$$

In the following calculations, we use the following commutator estimate and bilinear estimate [6]:

$$\left\| \Lambda^s(fg) - f \Lambda^s g \right\|_{L^p} \leq \left(\|\nabla f\|_{L^{p_1}} \left\| \Lambda^{s-1} g \right\|_{L^{q_1}} + \left\| \Lambda^s f \right\|_{L^{p_2}} \|g\|_{L^{q_2}} \right), \tag{3.15}$$

$$\left\| \Lambda^s(fg) \right\|_{L^p} \leq \left(\|f\|_{L^{p_1}} \left\| \Lambda^s g \right\|_{L^{q_1}} + \left\| \Lambda^s f \right\|_{L^{p_2}} \|g\|_{L^{q_2}} \right), \tag{3.16}$$

with $s > 0$, $\Lambda^s = (-\Delta)^{\frac{s}{2}}$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$. Taking the operation Λ^2 on both sides of the second equation of (1.1), then multiplying them by $\Lambda^2 \theta$, and integrating by parts over

$\mathbb{R}^3$, we have

$$\frac{1}{2}\frac{d}{dt}\int |\Lambda^2\theta|^2\,dx \leq \left|\int \left(\Lambda^2(u\cdot\nabla\theta) - u\nabla\Lambda^2\theta\right)dx\right| + \int \Lambda^2\left(|\nabla u|^2\right)\Lambda^2\theta\,dx$$

$$\leq C\big(\|\nabla u\|_{L^\infty}\big\|\Lambda^2\theta\big\|_{L^2} + \|\|\nabla u\|_{L^\infty}\|\Lambda^2\theta\|_{L^2}u\|_{L^3}\|\nabla\theta\|_{L^6}\big)\big\|\Lambda^2\theta\big\|_{L^2}$$

$$+ C\|\nabla u\|_{L^\infty}\big\|\Lambda^3 u\big\|_{L^2}\big\|\Lambda^2\theta\big\|_{L^2}.$$

Combining (3.12) with (3.14) and using the Gronwall inequality, we obtain

$$\theta \in L^\infty\big(0,T;H^2\big).$$

This completes the proof. $\square$

Authors' contributions
YX provided the model in the paper and some important references [1, 2]. He performed the some analysis work. HM and XF provided the ideas about blow-up criteria in terms of vorticity in the homogeneous Besov space and participated in the design of the study. LM mainly carried out inferences and computations of formulas in the paper. All authors read and approved the final manuscript.

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