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A subclass of analytic functions defined by the Dziok-Raina operator

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Abstract

The main object of the present paper is to introduce a subclass of analytic functions using the Dziok-Raina operator associated with the quasi hypergeometric functions. This class generalizes some well-known classes of starlike and convex functions. The integral means inequalities and the $p-\gamma$-neighborhood of this class are considered. Further, some results concerning the nth-Cesaro means of quasi hypergeometric functions for the class above mentioned are considered.

MSC: 30C45; 30C55

Keywords: quasi hypergeometric functions; Dziok-Raina operator; Cesaro means; integral means

1 Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{U}=\{z:|z|<1\}$ on the complex plane C. Let $\mathcal{S}^*(\delta)$, $\mathcal{C}(\delta)$ denote the subclasses of $\mathcal{A}$ consisting of functions, which are starlike of order δ and convex of order δ, respectively. If f and g are analytic in $\mathcal{U}$, we say that f is subordinate to g in $\mathcal{U}$, written $f\prec g$, if and only if there exists the Schwarz function w, analytic in $\mathcal{U}$ with $w(0)=0$ and $|w(z)|\leq 1$ in $\mathcal{U}$ such that $f(z)=g(w(z))$ $(z\in\mathcal{U})$. The convolution (or Hadamard product) $f*g$ of two functions f,g with series expansions $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ and $g(z)=z+\sum_{n=2}^{\infty}b_nz^n$ is defined by

$$(f*g)(z)=z+\sum_{n=2}^{\infty}a_nb_nz^n.$$

A quasi hypergeometric series is a power series in one complex variable z. Let r, s be nonnegative integers and consider the series

$$\phi_s^r\left(\begin{matrix}\{a_1',b_1'\},\dots,\{a_r',b_r'\}\\ \{a_1,b_1\},\dots,\{a_s,b_s\}\end{matrix}\middle| z\right)=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{r}\Gamma(a_i'+b_i'n)z^n}{\prod_{j=1}^{s}\Gamma(a_j+b_jn)n!},\qquad z\in\mathbb{C},$$

where $a'_1,\dots,a'_r,a_1,\dots,a_s$ are complex numbers and $b'_1,\dots,b'_r,b_1,\dots,b_s$ are positive numbers, which have the relation

$$b'_1+\cdots+b'_r=b_1+\cdots+b_s+1.$$

This function is a general one in the single variable case. In [1], the author showed that the above series is convergent for $|z|<c$ where c denotes the constant

$$c=b_1^{\prime-b'_1}\cdots b_r^{\prime-b'_r}b_1^{-b_1}\cdots b_s^{-b_s}.$$

The function ϕ_s^r satisfies the differential equation

$$\frac{d}{dz}\phi=\prod_{j=1}^{s}P_{b_j}(a_j,b_j)\prod_{i=1}^{r}P_{b'_i}\bigl(a'_i+b'_i,-b'_i\bigr)\phi,$$

where P defines a fractional derivative operator of order b as the following:

$$P_\sigma(a,b)f(z)=\frac{1}{\Gamma(b)}\int_0^1 t^{a-1}(1-t)^{b-1}f\bigl(t^\sigma z\bigr)\,dt,\quad \sigma>0.$$

For more details on this operator, see [2]. For $b'_i=b_j$ and $r=s+1$, then the function ϕ_s^r reduces to the hypergeometric function of higher order

$$\phi_s^{s+1}\left(\left(\begin{matrix}a'_1,\dots,a'_{s+1}\\a_1,\dots,a_s\end{matrix}\right)\middle|\,z\right),$$

and the above differential equation reduces to ordinary differential equation

$$\prod_{j=1}^{s}\left(a_j+z\frac{d}{dz}\right)\frac{z}{dz}\phi=\prod_{i=1}^{s+1}\left(a'_i+z\frac{d}{dz}\right)\phi.$$

Quasi hypergeometric functions are known as Fox-Wright functions and they appeared as an extension of a generalized hypergeometric functions. Recently, these functions have been given considerable attention by theoretical physicists. Indeed, those functions play an important role in conformal field theory and fractional exclusion statistics such as the quasi-algebraic functions and the partition functions. For a mathematical background for these functions, see [1, 3].

Now for $z\in\mathcal{U}$, and $r\le s+1$, let

$$\psi\left(\begin{matrix}\{a'_1,b'_1\},\dots,\{a'_r,b'_r\}\\\{a_1,b_1\},\dots,\{a_s,b_s\}\end{matrix}\middle|\,z\right)=z\phi_s^r\left(\begin{matrix}\{a'_1,b'_1\},\dots,\{a'_r,b'_r\}\\\{a_1,b_1\},\dots,\{a_s,b_s\}\end{matrix}\middle|\,z\right)$$

and ψ is of the form

$$\psi(z)=z+\sum_{n=2}^{\infty}\frac{\prod_{i=1}^{r}\Gamma(a'_i+b'_j(n-1))z^n}{\prod_{j=1}^{s}\Gamma(a_j+b_j(n-1))(n-1)!}.\tag{1.2}$$

We recall the Dziok-Raina linear operator [4] as follows:

For $f \in \mathcal{A}$, the operator $\mathcal{M}_s^r[a_i'; a_j; b_i'; b_j]$ $(i = 1,\dots,r,\ j = 1,\dots,s)$ is defined by the Hadamard product

$$\mathcal{M}_s^r\left(\begin{array}{c}\{a_1',b_1'\},\dots,\{a_r',b_r'\}\\ \{a_1,b_1\},\dots,\{a_s,b_s\}\end{array}\middle| z\right)f(z) = \psi\left(\begin{array}{c}\{a_1',b_1'\},\dots,\{a_r',b_r'\}\\ \{a_1,b_1\},\dots,\{a_s,b_s\}\end{array}\middle| z\right) * f(z).$$

For a function of the form (1.1) and function ψ of the form (1.2), we derive

$$\mathcal{M}_s^r\left(\begin{array}{c}\{a_1',b_1'\},\dots,\{a_r',b_r'\}\\ \{a_1,b_1\},\dots,\{a_s,b_s\}\end{array}\middle| z\right)f(z) = z + \sum_{n=2}^{\infty} \Upsilon_n a_n z^n, \tag{1.3}$$

where, for convenience,

$$\Upsilon_n = \frac{\prod_{i=1}^{r}\Gamma(a_i' + b_i'(n-1))}{\prod_{j=1}^{s}\Gamma(a_j + b_j(n-1))(n-1)!}.$$

For the sake of simplicity, we write

$$\mathcal{M}_s^r\left(\begin{array}{c}\{a_1',b_1'\},\dots,\{a_r',b_r'\}\\ \{a_1,b_1\},\dots,\{a_s,b_s\}\end{array}\middle| z\right)f(z) = \mathcal{M}_s^r\big[a_i'; a_j; b_i'; b_j\big]f(z).$$

It should be remarked that the linear operator (1.3) is a generalization of many operators considered earlier. For $b_i' = 1$ $(i = 1,\dots,r)$ and $b_j = 1$ $(j = 1,\dots,s)$, $r = s+1$, we obtain the Dziok-Srivastava linear operator [5]. This includes (as its special cases) various other linear operators, for example, the ones introduced and studied by Ruscheweyh [6], Carlson-Shaffer [7] and Bernardi-Livingston operators [8–10]. Also, many interesting subclasses of analytic functions associated with the operator (1.3) and one may refer to [11, 12].

Lemma 1.1 [4] *For $f \in \mathcal{A}$, we have the following*:

(i) $\mathcal{M}_0^1[0; -; 1; -]f(z) = f(z)$,
(ii) $\mathcal{M}_0^1[2; -; 1; -]f(z) = zf'(z)$.

Now using $\mathcal{M}_s^r[a_i'; a_j; b_i'; b_j]f$, we define the following subclass of analytic functions.

Definition 1.1 Given $\alpha \in (0,1]$ and functions

$$\Phi(z) = z + \sum_{n=2}^{\infty}\lambda_n z^n, \qquad \Psi(z) = z + \sum_{n=2}^{\infty}\mu_n z^n,$$

analytic in $\mathcal{U}$ such that $\lambda_n \geq 0$, $\mu_n \geq 0$, $\lambda_n \geq \mu_n$, $n \geq 2$, we say that $f \in \mathcal{A}$ is in $\mathcal{M}_s^r[a_i', a_j, b_i', b_j, \Phi, \Psi, \alpha]$ if $f(z) * \Psi(z) \neq 0$ and

$$\left|\frac{\mathcal{M}_s^r[a_i'; a_j; b_i'; b_j](f * \Phi)(z)}{\mathcal{M}_s^r[a_i'; a_j; b_i'; b_j](f * \Psi)(z)} - 1\right| < \alpha,$$

where $\mathcal{M}_s^r[a_i'; a_j; b_i'; b_j]f(z)$ is given by (1.3). We further let

$$\mathcal{M}\mathcal{T}_s^r\big[a_i'; a_j; b_i'; b_j; \Phi, \Psi, \alpha\big] = \mathcal{M}_s^r\big[a_i'; a_j; b_i'; b_j, \Phi, \Psi, \alpha\big] \cap \mathcal{T},$$

where

$$\mathcal{T} = \left\{ f \in \mathcal{A} : f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, z \in \mathcal{U} \right\}, \tag{1.4}$$

a subclass of $\mathcal{A}$ being introduced and studied by Silverman [13].

By suitable choices of the values r, s, a'_i, a_j, b'_i, b_j, Φ, Ψ, and α, we obtain various subclasses. As illustrations, we present some examples.

Example 1.1 For $r = 1$, $s = 0$, we have

$$\begin{aligned}\mathcal{M}\mathcal{T}_0^1(0,-,1,-,\Phi,\Psi,\alpha) &= \mathcal{D}_{\mathcal{T}}(\Phi,\Psi,\alpha) \\ &= \left\{ f \in \mathcal{T} : \left| \frac{(f * \Phi)(z)}{(f * \Psi)(z)} - 1 \right| < \alpha \right\}.\end{aligned}$$

If $\alpha = \frac{1-\delta}{2(1-\nu)}$, then we have the class

$$\mathcal{M}\mathcal{T}_0^1\left(0,-,1,-,\Phi,\Psi,\frac{1-\delta}{2(1-\nu)}\right) = \left\{ f \in \mathcal{T} : \left| \frac{(f * \Phi)(z)}{(f * \Psi)(z)} - 1 \right| < \frac{1-\delta}{2(1-\nu)} \right\}.$$

This class was studied by Frasin [14], Frasin and Darus [15, 16].

Example 1.2 For $r = 1$, $s = 0$, $\alpha = 1 - \delta$, we obtain

$$\mathcal{M}\mathcal{T}_0^1(0,-,1,-,\Phi,\Psi,1-\delta) = \mathcal{D}_{\mathcal{T}}(\Phi,\Psi,\delta) = \left| \frac{(f * \Phi)(z)}{(f * \Psi)(z)} - 1 \right| < 1 - \delta,$$

where $\mathcal{D}_{\mathcal{T}}(\Phi,\Psi,\delta)$ was studied by Juneja *et al.* [7]. In particular, for $r = 1$, $s = 0$, $\Phi(z) = \frac{z}{(1-z)^2}$, $\Psi(z) = \frac{z}{1-z}$, $\alpha = 1 - \delta$

$$\mathcal{M}\mathcal{T}_0^1\left(0,-,1,-,\frac{z}{(1-z)^2},\frac{z}{1-z},\alpha = 1-\delta\right) = \mathcal{S}^*_{\mathcal{T}}(\delta) = \left\{ f \in \mathcal{T} : \left| \frac{zf'(z)}{f(z)} \right| < 1 - \delta \right\}$$

and for $r = 1$, $s = 0$, $\Phi(z) = \frac{z+z^2}{(1-z)^3}$, $\Psi(z) = \frac{z}{(1-z)^2}$, $\alpha = 1 - \delta$, we have

$$\mathcal{M}\mathcal{T}_0^1\left(0,-,1,-,\frac{z+z^2}{(1-z)^3},\frac{z}{(1-z)^2},\alpha = 1-\delta\right) = \mathcal{C}_{\mathcal{T}}(\delta) = \left\{ f \in \mathcal{T} : \left| \frac{zf''(z)}{f'(z)} \right| < 1 - \delta \right\},$$

where $\mathcal{S}^*_{\mathcal{T}}(\delta)$ and $\mathcal{C}_{\mathcal{T}}(\delta)$ is the subclasses of $\mathcal{T}$ that are starlike of order δ and convex of order δ, respectively, which were studied by Silverman [13].

Example 1.3 For $r = 1$, $s = 0$, we get

$$\mathcal{M}\mathcal{T}_0^1(2,-,1,-,\Phi,\Psi,\alpha) = \left| \frac{(f * \Phi)'(z)}{(f * \Psi)'(z)} - 1 \right| < \alpha.$$

Example 1.4 For $r = 1$, $s = 0$, $\Phi(z) = \frac{z}{(1-z)^2}$, $\Psi(z) = \frac{z}{1-z}$, we obtain

$$\mathcal{M}\mathcal{T}_0^1\left(2,-,1,-,\frac{z}{(1-z)^2},\frac{z}{1-z},\alpha\right) = \left\{ f \in \mathcal{T} : \left| \frac{zf''(z)}{f'(z)} \right| < \alpha \right\}.$$

Example 1.5 For $r = 1$, $s = 0$, $\Phi(z) = \frac{z+z^2}{(1-z)^3}$, $\Psi(z) = \frac{z}{(1-z)^2}$, we have

$$\mathcal{M}\mathcal{T}_0^1\left(2,-,1,-,\frac{z+z^2}{(1-z)^3},\frac{z}{(1-z)^2},\alpha\right) = \left\{f \in \mathcal{T} : \left|\frac{z(zf'''(z)+2f''(z))}{zf''(z)+f'(z)} - 1\right| < \alpha\right\}.$$

Theorem 1.2 *Let a function f be defined by* (1.4). *Then* $f \in \mathcal{M}\mathcal{T}_s^r[a_i', a_j, b_i', b_j, \Phi, \Psi, \alpha]$ *if and only if*

$$\sum_{n=2}^{\infty} \frac{[\lambda_n - (1-\alpha)\mu_n]}{\alpha} |\Upsilon_n||a_n| \le 1, \quad \alpha \in (0,1]. \tag{1.5}$$

The result is sharp with the extremal functions

$$f_n(z) = z - \frac{\alpha}{\sigma(\alpha,n)} z^n, \quad n \ge 2,$$

where $\sigma(\alpha,n) = [\lambda_n - (1-\alpha)\mu_n]|\Upsilon_n|$, $\alpha \in (0,1]$.

Proof The above condition is necessary and sufficient for f to be in the class $\mathcal{M}\mathcal{T}_s^r[a_i', a_j, b_i', b_j, \Phi, \Psi, \alpha]$. To prove this theorem, we use similar arguments as given by Darus [17]. □

Remark 1.1 In [18], the author introduced the class $\mathcal{T}\mathcal{W}_\eta(\phi,\varphi;A,B)$, we observe that if $\eta = 0$, $A = \alpha$, $B = 0$ and

$$\phi = \psi\left(\begin{matrix}\{a_1', b_1'\}, \ldots, \{a_r', b_r'\} \\ \{a_1, b_1\}, \ldots, \{a_s, b_s\}\end{matrix}\middle| z\right) * \Phi,$$

$$\varphi = \psi\left(\begin{matrix}\{a_1', b_1'\}, \ldots, \{a_r', b_r'\} \\ \{a_1, b_1\}, \ldots, \{a_s, b_s\}\end{matrix}\middle| z\right) * \Psi.$$

Then Theorem 1.2 can be obtained from Theorem 4 in [18].

2 Integral means inequalities

In [13], Silverman found that function $f_2(z) = z - \frac{z^2}{2}$ is often an extremal for the family $\mathcal{T}$. He applied this function to prove the integral means inequality in [19], that is for all $f \in \mathcal{T}$, $\eta > 0$ and $0 < r < 1$

$$\int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^\eta d\theta \le \int_0^{2\pi} \left|f_2\left(re^{i\theta}\right)\right|^\eta d\theta.$$

In the following theorem, we obtain the integral means inequality for the class $\mathcal{M}\mathcal{T}_s^r[a_i', a_j, b_i', b_j, \Phi, \Psi, \alpha]$. We first state a lemma given by Littlewood [20] as follows.

Lemma 2.1 *If the functions f and g are analytic in $\mathcal{U}$ with $g \prec f$, then for $\eta > 0$ and $0 < r < 1$,*

$$\int_0^{2\pi} \left|g\left(re^{i\theta}\right)\right|^\eta d\theta \le \int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^\eta d\theta.$$

The next theorem is as the following.

Theorem 2.2 *Let* $f \in \mathcal{M}\mathcal{T}_s^r[a_i', a_j, b_i', b_j, \Phi, \Psi, \alpha]$, $\sigma(\alpha, n)$ *be a nondecreasing sequence and* $f_2(z)$ *be defined by*

$$f_2(z) = z - \frac{\alpha}{\sigma(\alpha,2)} z^2,$$

where

$$\sigma(\alpha,2) = \bigl[\lambda_2 - (1-\alpha)\mu_2\bigr]|\Upsilon_2| \tag{2.1}$$

and Υ_2 *is given by*

$$\Upsilon_2 = \frac{\prod_{i=1}^r \Gamma(a_i' + b_i')}{\prod_{j=1}^s \Gamma(a_j + b_j)}.$$

Then for $z = re^{i\theta}$, $0 < r < 1$, *we obtain*

$$\int_0^{2\pi} \bigl|f\bigl(re^{i\theta}\bigr)\bigr|^\eta \, d\theta \le \int_0^{2\pi} \bigl|f_2\bigl(re^{i\theta}\bigr)\bigr|^\eta \, d\theta. \tag{2.2}$$

Proof For a function f of the form (1.4) and $z = re^{i\theta}$, the inequality (2.2) is equivalent to

$$\int_0^{2\pi} \left|1 - \sum_{n=2}^{\infty} |a_n| z^{n-1}\right|^\eta d\theta \le \int_0^{2\pi} \left|1 - \frac{\alpha}{\sigma(\alpha,2)} z\right|^\eta d\theta.$$

By Lemma 2.1, it suffices to show that

$$\sum_{n=2}^{\infty} |a_n| z^{n-1} \prec \frac{\alpha}{\sigma(\alpha,2)} z. \tag{2.3}$$

Setting $\sum_{n=2}^{\infty} |a_n| z^{n-1} = \frac{\alpha}{\sigma(\alpha,2)} w(z)$, we have from (1.5) and (2.3),

$$\bigl|w(z)\bigr| = \left|\sum_{n=2}^{\infty} \frac{\sigma(\alpha,2)}{\alpha} |a_n| z^{n-1}\right| \le |z| \sum_{n=2}^{\infty} \frac{\sigma(\alpha,2)}{\alpha} |a_n| \le |z| < 1.$$

By the definition of subordination, we have (2.3) and this completes the proof. □

In the view of last theorem, we state the next corollaries.

Corollary 2.1 *Let* $f \in \mathcal{M}\mathcal{T}_o^1[0,-,1,-,\Phi,\Psi,\alpha] = \mathcal{D}\mathcal{T}(\Phi,\Psi,\alpha)$, $\alpha \in (0,1]$ *and* $f_2(z)$ *be defined by*

$$f_2(z) = z - \frac{\alpha}{\lambda_2 - (1-\alpha)\mu_2} z^2.$$

Then for $z = re^{i\theta}$, $0 < r < 1$, *we obtain*

$$\int_0^{2\pi} \bigl|f(z)\bigr|^\eta d\theta \le \int_0^{2\pi} \bigl|f_2(z)\bigr|^\eta \, d\theta. \tag{2.4}$$

Corollary 2.2 *Let* $f \in \mathcal{M}\mathcal{T}_o^1[2,-,1,-,\frac{z}{(1-z)^2},\frac{z}{1-z},\alpha] = \mathcal{S}^*_{\mathcal{T}}(\alpha)$ *and* $f_2(z)$ *be defined by*

$$f_2(z) = z - \frac{\alpha}{2(\alpha+1)}z^2.$$

Then for $z = re^{i\theta}$, $0 < r < 1$, (2.4) *holds true.*

Corollary 2.3 *Let* $f \in \mathcal{M}\mathcal{T}_o^1[2,-,1,-,\frac{z+z^2}{(1-z)^3},\frac{z}{(1-z)^2},\alpha] = \mathcal{C}_{\mathcal{T}}(\alpha)$ *and* $f_2(z)$ *be defined by*

$$f_2(z) = z - \frac{\alpha}{4(\alpha+1)}z^2.$$

Then for $z = re^{i\theta}$, $0 < r < 1$, (2.4) *holds true.*

Remark 2.1 In [21], the author introduced the class $\mathcal{TW}_\eta(\phi,\varphi;A,B)$, we observe that if $\eta = 0$, $A = \alpha$, $B = 0$ and

$$\phi = \psi\left(\begin{matrix}\{a_1',b_1'\},\ldots,\{a_r',b_r'\}\\ \{a_1,b_1\},\ldots,\{a_s,b_s\}\end{matrix}\middle| z\right) * \Phi,$$

$$\varphi = \psi\left(\begin{matrix}\{a_1',b_1'\},\ldots,\{a_r',b_r'\}\\ \{a_1,b_1\},\ldots,\{a_s,b_s\}\end{matrix}\middle| z\right) * \Psi.$$

Then Theorem 2.2 can be obtained from Theorem 7 in [21].

3 Neighborhoods of the class $\mathcal{M}\mathcal{T}_s^r[a_i',a_j,b_i',b_j,\Phi,\Psi,\alpha]$

For f of the form (1.4), and $\gamma \geq 0$, Frasin and Darus [22] investigated the $p-\gamma$-neighborhood of f as the following:

$$M_\gamma^p(f) = \left\{g \in \mathcal{T} : g(z) = z - \sum_{n=2}^{\infty} b_n z^n, \sum_{n=2}^{\infty} n^{p+1}|a_n - b_n| \leq \gamma\right\}, \tag{3.1}$$

where p is a fixed positive integer. In particular, for the identity function $e(z) = z$, we immediately have

$$M_\gamma^p(e) = \left\{g \in \mathcal{T} : g(z) = z - \sum_{n=2}^{\infty} b_n z^n, \sum_{n=2}^{\infty} n^{p+1}|b_n| \leq \gamma\right\}.$$

We note that $M_\gamma^0(f) \equiv N_\gamma(f)$, $M_\gamma^1(f) \equiv M_\gamma(f)$, where $N_\gamma(f)$ is called a γ-neighborhood of f introduced by Ruscheweyh [23] and $M_\gamma(f)$ was defined by Silverman [24].

Now, we investigate $p-\gamma$-neighborhood for functions in the class $\mathcal{M}\mathcal{T}_s^r[a_i',a_j,b_i',b_j,\Phi,\Psi,\alpha]$.

Theorem 3.1 *If* $\sigma(\alpha,n)/n^{p+1}$ *is a nondecreasing sequence, then* $\mathcal{M}\mathcal{T}_s^r[a_i',a_j,b_i',b_j,\Phi,\Psi,\alpha] \subset M_\gamma^p(e)$, *where*

$$\gamma = \frac{2^{p+1}\alpha}{\sigma(\alpha,2)}$$

and $\sigma(\alpha,2)$ *is defined as in* (2.1).

Proof It follows from (1.5) that if $f \in \mathcal{M}\mathcal{T}_s^r[a_i', a_j, b_i', b_j, \Phi, \Psi, \alpha]$, then

$$\sum_{n=2}^{\infty} n^{p+1}|a_n| \le \frac{2^{p+1}\alpha}{\sigma(\alpha, 2)}.$$

This gives that $\mathcal{M}\mathcal{T}_s^r[a_i', a_j, b_i', b_j, \Phi, \Psi, \alpha] \subset M_\gamma^p(e)$. □

Corollary 3.1 $\mathcal{D}_\mathcal{T}[\Phi, \Psi, \alpha] \subset M_\gamma^p(e)$, *where*

$$\gamma = \frac{2^{p+1}\alpha}{\lambda_2 - (1-\alpha)\mu_2}.$$

Corollary 3.2 $\mathcal{D}_\mathcal{T}[\Phi, \Psi, \frac{1-\delta}{2(1-\nu)}] \subset M_\gamma^p(e)$, *where*

$$\gamma = \frac{2^{p+1}(1-\delta)}{2(1-\nu)\lambda_2 - (1+\delta-2\nu)\mu_2}.$$

Corollary 3.3 $\mathcal{M}\mathcal{T}_0^1[2, -1, -, \Phi, \Psi, \alpha] \subset M_\gamma^p(e)$, *where*

$$\gamma = \frac{2^{p+1}\alpha}{2[\lambda_2 - (1-\alpha)\mu_2]}.$$

Corollary 3.4 $\mathcal{M}\mathcal{T}_0^1[2, -1, -, \frac{z}{(1-z)^2}, \frac{z}{1-z}, \alpha] \subset M_\gamma^p(e)$, *where*

$$\gamma = \frac{2^{p+1}\alpha}{2(1+\alpha)}.$$

Corollary 3.5 $\mathcal{M}\mathcal{T}_0^1[2, -1, -, \frac{z+z^2}{(1-z)^3}, \frac{z}{(1-z)^2}, \alpha] \subset M_\gamma^p(e)$, *where*

$$\gamma = \frac{2^{p+1}\alpha}{4(1+\alpha)}.$$

4 Cesaro means

In this section, we investigate results on Cesaro means for the function ψ defined by the form (1.2) and for the class $\mathcal{M}\mathcal{T}_s^r[a_i', a_j, b_i', b_j, \Phi, \Psi, \alpha]$. Quasi hypergeometric functions were considered as a generalization to the generalized hypergeometric functions studied by Ruscheweyh [25], which he observed the following results.

Lemma 4.1 *Let* $0 < a \le b$ *if* $b \ge 2$ *or* $a + b \ge 3$ *then the function of the form* $f(z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} z^{n+1}$, $z \in \mathcal{U}$ *is convex.*

Note that (x) *is the Pochhammer symbol defined by*:

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = \left\{\begin{array}{ll} 1, & n = 0; \\ x(x+1)\cdots(x+n-1), & n = 1, 2, \ldots \end{array}\right\}.$$

Lemma 4.2 *Suppose that* $0 < \prod_{i=1}^{r} \Gamma(a_i' + b_i' n) \le \prod_{j=1}^{s} \Gamma(a_j + b_j n)$, *then*

$$\operatorname{Re}\left\{\frac{\psi(z)}{z}\right\} > \frac{1}{2}, \quad \text{for all } z \in \mathcal{U}.$$

Now let us recall the following by defining $\mathcal{S}^*$, $\mathcal{C}$, $\mathcal{QS}^*$ and $\mathcal{QC}$ the subclasses of $\mathcal{A}$ consisting of functions that are starlike in $\mathcal{U}$ and convex in $\mathcal{U}$. By the definitions, we have

$$\mathcal{S}^* = \left\{\psi \in \mathcal{A} : \mathrm{Re}\left\{\frac{z\psi'(z)}{\psi(z)}\right\} > 0, z \in \mathcal{U}\right\},$$

$$\mathcal{C} = \left\{\psi \in \mathcal{A} : \mathrm{Re}\left\{1 + \frac{z\psi''(z)}{\psi'(z)}\right\} > 0, z \in \mathcal{U}\right\},$$

$$\mathcal{QS}^* = \left\{\psi \in \mathcal{A} : \exists g \in \mathcal{S}^* \text{ such that } \mathrm{Re}\left\{\frac{z\psi'(z)}{g(z)}\right\} > 0, z \in \mathcal{U}\right\},$$

$$\mathcal{QC} = \left\{\psi \in \mathcal{A} : \exists g \in \mathcal{C} \text{ such that } \mathrm{Re}\left\{\frac{(z\psi'(z))'}{g'(z)}\right\} > 0, z \in \mathcal{U}\right\}.$$

We observe that

$$\psi(z) \in \mathcal{C} \quad \Leftrightarrow \quad z\psi'(z) \in \mathcal{S}^* \tag{4.1}$$

and

$$\psi \in \mathcal{QC} \quad \Leftrightarrow \quad z\psi'(z) \in \mathcal{QS}^*. \tag{4.2}$$

From the above definitions it is easily to observe the following lemma.

Lemma 4.3 [25]
(i) *If $\psi \in \mathcal{C}$, and $g \in \mathcal{S}^*$ then $\psi * g \in \mathcal{S}^*$.*
(ii) *If $\psi \in \mathcal{C}$, $g \in \mathcal{S}^*$ and $p \in \mathcal{P}$ (the class of Caratheodory functions) with $p(0) = 1$, then $\psi * gp = (\psi * g)p_1$, where $p_1(\mathcal{U}) \subset$ closed convex hull of $p(\mathcal{U})$.*

Definition 4.1 The nth Cesaro means of order β, $\beta \geq 0$ of the series of the form (1.2) can be defined as

$$\tau_k^\beta(z,\psi) = \tau_k^\beta(z) * \psi(z) = \sum_{n=0}^{k} \frac{\binom{k-n+\beta}{k-n}}{\binom{k+\beta}{k}} \frac{\prod_{i=1}^{r} \Gamma(a_i' + b_i'n) z^{n+1}}{\prod_{j=1}^{s} \Gamma(a_j + b_j n) n!}, \tag{4.3}$$

where k is a positive number and $\binom{a}{b} = \frac{a!}{b!(a-b)!}$.

Clearly that $\tau_k^\beta(z)$ is the nth Cesaro mean of the geometric series $\frac{z}{1-z}$ of order β.

Lemma 4.4 [26] *Let $f \in \mathcal{A}$ such that for some $\beta \geq 0$ we have*

$$\left(\tau_k^\beta(z,f)\right)' \neq 0, \quad z \in \mathcal{U}, k \in N.$$

Then

$$\tau_1^{\beta+m}(z,f) \prec \tau_2^{\beta+m}(z,f) \prec \cdots \prec \tau_k^{\beta+m}(z,f) \prec \cdots \prec f(z), \quad m \in N.$$

Theorem 4.5 *Let $\psi \in \mathcal{A}$ be given in the form (1.2) and convex in $\mathcal{U}$ and $\tau_k^\beta(z,\psi)$ is the nth Cesaro mean of ψ, $\beta \geq 0$, then*

$$\tau_1^{\beta+m}(z,\psi) \prec \tau_2^{\beta+m}(z,\psi) \prec \cdots \prec \tau_k^{\beta+m}(z,\psi) \prec \cdots \prec \psi(z), \quad m \in N.$$

Proof First, we note that $\tau_k^\beta(z,\psi') = (\tau_k^\beta(z,\psi))'$, for $k \in N$, $z \in \mathcal{U}$ and $\beta \geq 0$ Let $\varphi(z) = \sum_{n=0}^{\infty}(n+1)z^{n+1}$ be defined such that

$$\begin{aligned} z\psi'(z) &= \varphi(z) * \psi(z) \\ &= \sum_{n=0}^{\infty} \frac{n+1}{n!} \frac{\prod_{i=r}^{r} \Gamma(a_i' + nb_i')}{\prod_{j=1}^{s} \Gamma(a_j + nb_j)} z^{n+1}. \end{aligned}$$

Then

$$\begin{aligned} \tau_k^\beta(z,\psi') &= \psi'(z) * \tau_k^\beta(z) \\ &= \frac{z\psi'(z) * z\tau_k^\beta}{z} \\ &= \frac{\psi'(z) * \varphi(z) * z\tau_k^\beta}{z} \\ &= \frac{\psi(z) * z(z\tau_k^\beta)'}{z}. \end{aligned}$$

In view of Lemma 4.3, the relation (4.2) and the fact that $z\tau_k^\beta$ is convex yield that there exists a function $g \in \mathcal{S}*$ and $p \in \mathcal{P}$ with $p(0) = 1$ such that

$$\frac{\psi(z) * z(z\tau_k^\beta)'}{z} = \frac{\psi(z) * gp(z)}{z} = \frac{(\psi(z) * g(z))p_1(z)}{z} \neq 0.$$

It is known that $\operatorname{Re}\{p_1(z)\} > 0$ and that $\psi(z) * g(z) = 0$ if and only if $z = 0$. So we have $\tau_k^\beta(z,\psi') \neq 0$. By using Lemma 4.4, we obtain

$$\tau_1^{\beta+m}(z,\psi) \prec \tau_2^{\beta+m}(z,\psi) \prec \cdots \prec \tau_k^{\beta+m}(z,\psi) \prec \cdots \prec \psi(z), \quad m \in N. \qquad \square$$

Now, we consider the nth Cesaro means for functions in the class $\mathcal{M}\mathcal{T}_s^r[a_i', a_j, b_i', b_j, \Phi, \Psi, \alpha]$.

Let $f \in \mathcal{M}\mathcal{T}_s^r[a_i', a_j, b_i', b_j, \Phi, \Psi, \alpha]$, then the nth Cesaro means of f of order β defined by the form

$$\tau_k^\beta(z,f) = \sum_{n=1}^{k} \frac{\binom{k-n+\beta}{k-n}}{\binom{k+\beta}{k}} a_n z^n.$$

Theorem 4.6 *If $f \in \mathcal{M}\mathcal{T}_s^r[a_i', a_j, b_i', b_j, \Phi, \Psi, \alpha]$, then the series $\tau_k^\beta(z,f) \in \mathcal{M}\mathcal{T}_s^r[a_i', a_j, b_i', b_j, \Phi, \Psi, \alpha]$.*

Proof Since,

$$\frac{\binom{k-n+\beta}{k-n}}{\binom{k+\beta}{k}} = \frac{k!(k-n+\beta)}{(k-n)!(k+\beta)!} \leq 1$$

and by considering the sufficient condition for f to be in the class $\mathcal{M}\mathcal{T}_s^r[a_i', a_j, b_i', b_j, \Phi, \Psi, \alpha]$ we get

$$\sum_{n=2}^{\infty} \frac{\lambda_n - (1-\alpha)\mu_n}{\alpha} \frac{\binom{k-n+\beta}{k-n}}{\binom{k+\beta}{k}} |\Upsilon_n||a_n| \leq 1. \qquad \square$$

Conclusion

We have studied a class of analytic functions defined by means of the familiar quasi hypergemetric functions. The necessary and sufficient conditions for a function to be in the class are obtained. Several properties for functions belonging to this class are derived. Cesaro results are also being considered. Few other results related to Cesaro means can be seen in [27–31].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The first author is currently a Ph.D. student under supervision of the second author and jointly worked on deriving the results. All authors read and approved the final manuscript.

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