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Reverses of Young and Heinz inequalities for positive linear operators

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Abstract

Let A, B be invertible positive operators on a Hilbert space H. We present some improved reverses of Young type inequalities, in particular,

$$(1-\nu)^{2\nu}(A\nabla B)+(1-\nu)^{2(1-\nu)}H_{2\nu}(A,B)\geq 2(1-\nu)(A\sharp B)$$

and

$$(1-\nu)^{2\nu}H_{2\nu}(A,B)+(1-\nu)^{2(1-\nu)}(A\nabla B)\geq 2(1-\nu)(A\sharp B),$$

where $0\leq \upsilon \leq \frac{1}{2}$.

We also give some new inequalities involving the Heinz mean for the Hilbert-Schmidt norm.

1 Introduction

Let H be a Hilbert space and let $B_h(H)$ be the semi-space of all bounded linear self-adjoint operators on H. Further, let $B(H)$ and $B(H)^+$, respectively, denote the set of all bounded linear operators on a complex Hilbert space H and set of all positive operators in $B_h(H)$. The set of all positive invertible operators is denoted by $B(H)^{++}$. For $A, B \in B(H)$, A^* denotes the conjugate operator of A. An operator $A \in B(H)$ is positive, and we write $A \geq 0$, if $(Ax, x) \geq 0$ for every vector $x \in H$. If A and B are self-adjoint operators, the order relation $A \geq B$ means, as usual, that $A - B$ is a positive operator. The theory of operator means for positive (bounded linear) operators on a Hilbert space was initiated by Ando and established by him and Kubo in connection with Lowners theory for the operator monotone functions [1].

An operator mean is a binary operation σ defined on the set of strictly positive operators, if the following conditions hold:

(1) $A \leq C, B \leq D \quad \Rightarrow \quad A\sigma B \leq C\sigma D.$

(2) $A_n \downarrow A, B_n \downarrow B \quad \Rightarrow \quad A_n\sigma B_n \downarrow A\sigma B.$

(3) $T^*(A\sigma B)T \leq (T^*AT)\sigma(T^*BT) \quad \text{for } T \in B(H).$

(4) $I\sigma I = I.$

In addition, $A\sigma B = A^{\frac{1}{2}} f(A^{\frac{-1}{2}} B A^{\frac{-1}{2}}) A^{\frac{1}{2}}$ for all strictly positive operators A and B. The monotone function f is called the representing function of σ. Let $A, B \in B(H)$ be two positive operators and $\nu \in [0,1]$, then the ν-weighted arithmetic mean of A and B, denoted by $A\nabla_\nu B$, is defined as $A\nabla_\nu B = (1-\nu)A + \nu B$. If A is invertible, the ν-geometric mean of A and B, denoted by $A\sharp_\nu B$, is defined as $A\sharp_\nu B = A^{\frac{1}{2}}(A^{\frac{-1}{2}} B A^{\frac{-1}{2}})^\nu A^{\frac{1}{2}}$. In addition, if both A and B are invertible, the ν-harmonic mean of A and B, denoted by $A!_\nu B$ is defined as $A!_\nu B = ((1-\nu)A^{-1} + \nu B^{-1})^{-1}$. For more details, see Kubo and Ando [1]. When $\nu = \frac{1}{2}$, we write $A\nabla B$, $A\sharp B$, $A!B$ for brevity, respectively. The operator version of the Heinz means is denoted by

$$H_\nu(A,B) = \frac{A\sharp_\nu B + A\sharp_{1-\nu} B}{2},$$

where $A, B \in B(H)^{++}$, and $\nu \in [0,1]$. The operator version of the Heron means is denoted by

$$F_\alpha(A,B) = (1-\alpha)(A\sharp B) + \alpha(A\nabla B)$$

for $0 \le \alpha \le 1$. It is well known that if A and B are positive invertible operators, then

$$A\nabla_\nu B \ge A\sharp_\nu B \ge A!_\nu B$$

for $0 < \nu < 1$.

To obtain inequalities for bounded self-adjoint operators on Hilbert space, we shall use the following monotonicity property for operator functions:

If $X \in B_h(H)$ with a spectrum $\operatorname{Sp}(X)$ and f, g are continuous real-valued functions on $\operatorname{Sp}(X)$, then

$$f(t) \ge g(t), \quad t \in \operatorname{Sp}(X) \quad \Rightarrow \quad f(X) \ge g(X). \tag{1.1}$$

For more details as regards this property, the reader is referred to [2].

The classical Young inequality says that if $a, b \ge 0$ and $\nu \in [0,1]$, then

$$a^\nu b^{1-\nu} \le \nu a + (1-\nu)b \tag{1.2}$$

with the equality if and only if $a = b$.

Zhao *et al.* [3] gave an inequality for the Heinz and Heron means as follows:

If A and B are two positive and invertible operators, then

$$H_\nu(A,B) \le F_{\alpha(\nu)}(A,B) \tag{1.3}$$

for $\nu \in [0,1]$, where $\alpha(\nu) = 1 - 4(\nu - \nu^2)$.

Kai in [4] gave the following Young type inequalities:

$$\nu^2 a + (1-\nu)^2 b \ge \nu^2(\sqrt{a} - \sqrt{b})^2 + \nu^{2\nu} a^\nu b^{1-\nu} \tag{1.4}$$

for $\nu \in [0, \frac{1}{2}]$. Recently, Burqan and Khandaqji [5] gave the following reverse of the scalar Young type inequality:

$$\nu^2 a + (1-\nu)^2 b \leq (1-\nu)^2(\sqrt{a}-\sqrt{b})^2 + a^\nu\big[(1-\nu)^2 b\big]^{1-\nu} \tag{1.5}$$

for $\nu \in [0, \frac{1}{2}]$. Also we have

$$\nu^2 a + (1-\nu)^2 b \leq \nu^2(\sqrt{a}-\sqrt{b})^2 + \nu^{2\nu} a^\nu b^{1-\nu} \tag{1.6}$$

for $\nu \in [\frac{1}{2}, 1]$.

2 The results and discussion

In this section, we present some converses of the Young inequality and give several refinements for matrices and operators.

2.1 Reverses of scalar Young type inequalities

First, we get reverses of the inequalities (1.4), (1.5), and (1.6).

Theorem 1 *Let $a, b > 0$ and $\nu \in [0, \frac{1}{2}]$. Then*

$$(1-\nu)^{2\nu}\big[(1-2\nu)a + 2\nu b\big] + a^{2\nu}(1-\nu)^{2(1-\nu)} b^{1-2\nu} \geq 2(1-\nu)\sqrt{ab} \tag{2.1}$$

and

$$(1-\nu)^{2(1-\nu)}\big[(2\nu)a + (1-2\nu)b\big] + a^{1-2\nu}(1-\nu)^{2\nu} b^{2\nu} \geq 2(1-\nu)\sqrt{ab}. \tag{2.2}$$

Proof If $0 \leq \nu \leq \frac{1}{2}$, then by inequality (1.2), we have

$$\begin{aligned}
&(1-\nu)^{2\nu}\big[(1-2\nu)a + 2\nu b\big] + a^{2\nu}(1-\nu)^{2(1-\nu)} b^{1-2\nu} - 2(1-\nu)\sqrt{ab} \\
&\quad \geq (1-\nu)^{2\nu}\big[a^{1-2\nu} b^{2\nu}\big] + a^{2\nu}(1-\nu)^{2(1-\nu)} b^{1-2\nu} - 2(1-\nu)\sqrt{ab} \\
&\quad = a^{1-2\nu}(1-\nu)^{2\nu} b^{2\nu} + a^{2\nu}(1-\nu)^{2(1-\nu)} b^{1-2\nu} - 2(1-\nu)\sqrt{ab} \\
&\quad = \big(a^{\frac{1-2\nu}{2}}(1-\nu)^\nu b^\nu - a^\nu(1-\nu)^{(1-\nu)} b^{\frac{1-2\nu}{2}}\big)^2 \geq 0
\end{aligned}$$

and similarly

$$\begin{aligned}
&(1-\nu)^{2(1-\nu)}\big[(2\nu)a + (1-2\nu)b\big] + a^{1-2\nu}(1-\nu)^{2\nu} b^{2\nu} - 2(1-\nu)\sqrt{ab} \\
&\quad \geq (1-\nu)^{2\nu}\big[a^{1-2\nu} b^{2\nu}\big] + a^{2\nu}(1-\nu)^{2(1-\nu)} b^{1-2\nu} - 2(1-\nu)\sqrt{ab} \\
&\quad = \big(a^{\frac{1-2\nu}{2}}(1-\nu)^\nu b^\nu - a^\nu(1-\nu)^{(1-\nu)} b^{\frac{1-2\nu}{2}}\big)^2 \geq 0.
\end{aligned}$$

□

Theorem 2 *Let $a, b > 0$ and $\nu \in [0, \frac{1}{2}]$. Then*

$$(1-\nu)^{2\nu} a^{1-\nu} b^\nu + (1-\nu)^{2(1-\nu)}\big[\nu a + (1-\nu)b\big] \geq 2(1-\nu)\sqrt{ab}, \tag{2.3}$$

$$(1-\nu)^{2\nu}\big[(1-\nu)a + \nu b\big] + a^\nu b^{1-\nu}(1-\nu)^{2(1-\nu)} \geq 2(1-\nu)\sqrt{ab}. \tag{2.4}$$

Proof If $0 \leq \nu \leq \frac{1}{2}$, then by inequality (1.2), we have

$$\begin{aligned}
&(1-\nu)^{2\nu}a^{1-\nu}b^{\nu} + (1-\nu)^{2(1-\nu)}\bigl[\nu a + (1-\nu)b\bigr] - 2(1-\nu)\sqrt{ab} \\
&\quad \geq (1-\nu)^{2\nu}a^{1-\nu}b^{\nu} + (1-\nu)^{2(1-\nu)}a^{\nu}b^{1-\nu} - 2(1-\nu)\sqrt{ab} \\
&\quad = (1-\nu)^{2\nu}a^{1-\nu}b^{\nu} + a^{\nu}\bigl[(1-\nu)^2 b\bigr]^{(1-\nu)} - 2(1-\nu)\sqrt{ab} \\
&\quad = \bigl(a^{\frac{1-\nu}{2}}(1-\nu)^{\nu}b^{\frac{\nu}{2}} - a^{\frac{\nu}{2}}(1-\nu)^{(1-\nu)}b^{\frac{1-\nu}{2}}\bigr)^2 \geq 0
\end{aligned}$$

and similarly

$$\begin{aligned}
&(1-\nu)^{2\nu}\bigl[(1-\nu)a + \nu b\bigr] + a^{\nu}b^{1-\nu}(1-\nu)^{2(1-\nu)} - 2(1-\nu)\sqrt{ab} \\
&\quad \geq (1-\nu)^{2\nu}a^{1-\nu}b^{\nu} + a^{\nu}\bigl[(1-\nu)^2 b\bigr]^{(1-\nu)} - 2(1-\nu)\sqrt{ab} \\
&\quad = \bigl(a^{\frac{1-\nu}{2}}(1-\nu)^{\nu}b^{\frac{\nu}{2}} - a^{\frac{\nu}{2}}(1-\nu)^{(1-\nu)}b^{\frac{1-\nu}{2}}\bigr)^2 \geq 0.
\end{aligned}$$

□

Corollary 1 *Let $a, b > 0$ and $\nu \in [0, \frac{1}{2}]$. Then*

$$\begin{aligned}
&(1-\nu)^{2\nu}\left(\frac{a+b}{2}\right) + (1-\nu)^{2(1-\nu)}\left(\frac{a^{2\nu}b^{1-2\nu} + a^{1-2\nu}b^{2\nu}}{2}\right) \\
&\quad \geq 2(1-\nu)(\sqrt{ab}), \qquad (2.5) \\
&(1-\nu)^{2(1-\nu)}\left(\frac{a+b}{2}\right) + (1-\nu)^{2\nu}\left(\frac{a^{2\nu}b^{1-2\nu} + a^{1-2\nu}b^{2\nu}}{2}\right) \\
&\quad \geq 2(1-\nu)(\sqrt{ab}), \qquad (2.6) \\
&(1-\nu)^{2\nu}\left(\frac{a+b}{2}\right) + (1-\nu)^{2(1-\nu)}\left(\frac{a^{\nu}b^{1-\nu} + a^{1-\nu}b^{\nu}}{2}\right) \\
&\quad \geq 2(1-\nu)(\sqrt{ab}). \qquad (2.7)
\end{aligned}$$

2.2 Reverses of operator Young type inequalities

We begin this section with the reverses of Young type inequalities for operators.

Theorem 3 *Let $A, B \in B(H)^{++}$ and $\nu \in [0, \frac{1}{2}]$. Then*

$$(1-\nu)^{2\nu}(A\nabla B) + (1-\nu)^{2(1-\nu)}H_{2\nu}(A,B) \geq 2(1-\nu)(A\sharp B)$$

and

$$(1-\nu)^{2\nu}H_{2\nu}(A,B) + (1-\nu)^{2(1-\nu)}(A\nabla B) \geq 2(1-\nu)(A\sharp B).$$

Proof If $\nu \in [0, \frac{1}{2}]$, the inequality (2.1), for $a = 1$, $b > 0$, becomes

$$(1-\nu)^{2\nu}\bigl[(1-2\nu) + 2\nu b\bigr] + (1-\nu)^{2(1-\nu)}b^{1-2\nu} \geq 2(1-\nu)\sqrt{b}. \qquad (2.8)$$

The operator $X = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ has a positive spectrum. According to rule (1.1), we can insert $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ in the above inequality, *i.e.*, we have

$$(1-\nu)^{2\nu}\big[(1-2\nu)+2\nu A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\big]+(1-\nu)^{2(1-\nu)}\big(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\big)^{1-2\nu}$$
$$\geq 2(1-\nu)\big(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\big)^{\frac{1}{2}}. \tag{2.9}$$

Finally, if we multiply inequality (2.9) by $A^{\frac{1}{2}}$ on the left and right, we get

$$(1-\nu)^{2\nu}\big[(1-2\nu)A+2\nu B\big]+(1-\nu)^{2(1-\nu)}(A\sharp_{1-2\nu}B)\geq 2(1-\nu)(A\sharp B). \tag{2.10}$$

By replacing A by B and B by A, we have

$$(1-\nu)^{2\nu}\big[(1-2\nu)B+2\nu A\big]+(1-\nu)^{2(1-\nu)}(A\sharp_{2\nu}B)\geq 2(1-\nu)(A\sharp B), \tag{2.11}$$

and by the sum of (2.10) and (2.11), we have

$$(1-\nu)^{2\nu}(A\nabla B)+(1-\nu)^{2(1-\nu)}H_{2\nu}(A,B)\geq 2(1-\nu)(A\sharp B).$$

Similarly, the inequality (2.2) implies that

$$(1-\nu)^{2\nu}H_{2\nu}(A,B)+(1-\nu)^{2(1-\nu)}(A\nabla B)\geq 2(1-\nu)(A\sharp B). \qquad \square$$

Using the same strategy as in the proof of Theorem 3 and inequalities (2.3) and (2.4), we get the following theorems.

Theorem 4 *Let* $A,B\in B(H)^{++}$ *and* $\nu\in[0,\frac{1}{2}]$. *Then*

$$(1-\nu)^{2\nu}H_{\nu}(A,B)+(1-\nu)^{2(1-\nu)}(A\nabla B)\geq 2(1-\nu)(A\sharp B).$$

Theorem 5 *Let* $A,B\in B(H)^{++}$ *and* $\nu\in[0,\frac{1}{2}]$ *Then*

$$(1-\nu)^{2(1-\nu)}H_{\nu}(A,B)+(1-\nu)^{2\nu}(A\nabla B)\geq 2(1-\nu)(A\sharp B).$$

2.3 Reverses of Young type inequalities for matrices

In the following, let $M_n(C)$ be the space of all $n\times n$ complex matrices. For Hermitian matrices $A,B\in M_n(C)$, we write $A\geq 0$ if A is positive semidefinite, $A>0$ if A is positive definite, and $A\geq B$ if $A-B\geq 0$.

The Hilbert-Schmidt (or Frobenius) norm of $A=[a_{ij}]\in M_n(C)$ is denoted by $\|A\|_2 = (\sum_{j=1}^{n}s_j^2(A))^{\frac{1}{2}}$, where $s_1(A)\geq s_2(A)\geq\cdots\geq s_n(A)$ are the singular values of A, which are the eigenvalues of the positive semidefinite matrix $|A|=(AA^*)^{\frac{1}{2}}$, arranged in decreasing order and repeated according to multiplicity. It is well known that the Hilbert-Schmidt norm is unitarily invariant.

For more information on matrix versions of the Young inequality (1.2) the reader is referred to [6]. In this section, we will discuss the reverse Heinz mean inequality for unitarily invariant norms.

A matrix version of the inequality (1.3) is

$$\left\| \frac{A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}}{2} \right\| \le \left\| \bigl(1-\alpha(\nu)\bigr)A^{\frac{1}{2}}XB^{\frac{1}{2}} + \alpha(\nu)\left(\frac{AX+XB}{2}\right) \right\|,$$

which was introduced by Bhatia [7].

The matrix version of the inequality

$$a\sharp b \le H_{\nu}(a,b) \le a\nabla b,$$

was proved by Bhatia and Davis [8], saying that if $0 \le \nu \le 1$, then

$$\bigl\| A^{\frac{1}{2}}XB^{\frac{1}{2}} \bigr\| \le \left\| \frac{A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}}{2} \right\| \le \left\| \frac{AX+XB}{2} \right\|. \tag{2.12}$$

The second part of the inequality (2.12) is known as the Heinz inequality. Let $0 \le \nu \le 1$, $r_0 = \min\{\nu, 1-\nu\}$, Kittaneh [9] gave a refinement of the Heinz inequality as follows:

$$\left\| \frac{A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}}{2} \right\| \le 2r_0 \bigl\| A^{\frac{1}{2}}XB^{\frac{1}{2}} \bigr\| + (1-2r_0)\left\| \frac{AX+XB}{2} \right\|. \tag{2.13}$$

Meanwhile, Kittaneh and Manasarah [10] also obtained two refinements of the Heinz inequality for the Hilbert-Schmidt norm as follows:

$$\left\| \frac{A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}}{2} \right\|_2^2 \le \left\| \frac{AX+XB}{2} \right\|_2^2 - 2r_0 \left\| \frac{AX-XB}{2} \right\|_2^2, \tag{2.14}$$

$$\left\| \frac{A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}}{2} \right\|_2 \le \left\| \frac{AX+XB}{2} \right\|_2 - r_0\bigl(\sqrt{\|AX\|_2} - \sqrt{\|XB\|_2}\bigr)^2. \tag{2.15}$$

He *et al.* [11] proved that

$$\left\| \frac{A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}}{2} \right\|^2 \le 2r_0 \bigl\| A^{\frac{1}{2}}XB^{\frac{1}{2}} \bigr\|^2 + (1-2r_0)\left\| \frac{AX+XB}{2} \right\|^2.$$

It is weaker than the inequality (2.13) and it is equivalent to the inequality (2.14) for the Hilbert-Schmidt norm [11]. Zhan [12] proved that if $\frac{1}{4} \le \nu \le \frac{3}{4}$ and $-2 < t \le 2$, then

$$\left\| \frac{A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}}{2} \right\| \le \frac{1}{t+2} \bigl\| tA^{\frac{1}{2}}XB^{\frac{1}{2}} + AX + XB \bigr\|. \tag{2.16}$$

It is also a refinement of the Heinz inequality for matrices. Zou [13] proved that if $0 \le \nu \le 1$, then

$$\left\| \frac{A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}}{2} \right\|_2^2 \le \left\| \frac{AX+XB}{2} \right\|_2^2 - a\nu(1-\nu)\left\| \frac{AX-XB}{2} \right\|_2^2,$$

which is an improvement of (2.14). Zou also, in [14], has proved another Heinz inequality for the Hilbert-Schmidt norm as follows.

Theorem 6 *Let $\alpha(\nu) = 1 - 4(\nu - \nu^2)$ and $0 \le \nu \le 1$. Then*

$$\left\| \frac{A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu}{2} \right\|_2 \le \left\| \bigl(1-\alpha(\nu)\bigr)A^{\frac{1}{2}}XB^{\frac{1}{2}} + \alpha(\nu)\frac{AX+XB}{2} \right\|_2.$$

Kittaneh and Manasarah in [6] have showed that if A and B are positive definite matrices, $X \in M_n$ and $\nu \in [0,1]$, then

$$\begin{aligned} &\|(1-\nu)AX + \nu XB\|_2^2 \le \|A^{1-\nu}XB^\nu\|_2^2 + R^2\|AX - XB\|_2^2, \\ &\|AX + XB\|_2^2 \le \|A^{1-\nu}XB^\nu + A^\nu XB^{1-\nu}\|_2^2 + 2R\|AX - XB\|_2^2, \end{aligned}$$

where $R = \max\{\upsilon, 1-\upsilon\}$.

Bakherad and Moslehian [15] improved the Young inequality and obtained the following inequalities:

$$\|AX + XB\|_2^2 + 2(\nu - 1)\|AX - XB\|_2^2 \le \|A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu\|_2^2,$$

where A and B are positive definite matrices, $X \in M_n(C)$ and $\nu > 1$.

Theorem 7 *Let $A, B, X \in M_n$ such that A and B are positive definite; if $\nu \in [0, \frac{1}{2}]$, then*

$$\begin{aligned} &\left\| (1-\nu)^{2(1-\nu)} \frac{A^{2\nu}XB^{1-2\nu} + A^{1-2\nu}XB^{2\nu}}{2} + (1-\nu)^{2\nu}\frac{AX+XB}{2} \right\|_2 \\ &\quad \ge 2(1-\nu)\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\|_2. \end{aligned}$$

Proof Since A and B are positive semidefinite, it follows by the spectral theorem that there exist unitary matrices $U, V \in M_n$ such that

$$A = U\Gamma_1 U^* \quad \text{and} \quad B = V\Gamma_2 V^*,$$

where

$$\Gamma_1 = \operatorname{diag}(\lambda_1, \ldots, \lambda_n), \qquad \Gamma_2 = \operatorname{diag}(\mu_1, \ldots, \mu_n), \quad \lambda_i, \mu_i \ge 0, i = 1, \ldots, n.$$

Let

$$Y = U^*XV = [y_{ij}],$$

then

$$\begin{aligned} &\frac{A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu}{2} \\ &\quad = \frac{(U\Gamma_1U^*)^\nu X(V\Gamma_2V^*)^{1-\nu} + (U\Gamma_1U^*)^{1-\nu}X(V\Gamma_2V^*)^\nu}{2} \\ &\quad = \frac{(U\Gamma_1^\nu U^*)X(V\Gamma_2^{1-\nu}V^*) + (U\Gamma_1^{1-\nu}U^*)X(V\Gamma_2^\nu V^*)}{2} \end{aligned}$$

$$\begin{aligned}
&= \frac{U\Gamma_1^{\nu}(U^*XV)\Gamma_2^{1-\nu}V^* + U\Gamma_1^{1-\nu}(U^*XV)\Gamma_2^{\nu}V^*}{2} \\
&= U\left(\frac{\Gamma_1^{\nu}Y\Gamma_2^{1-\nu} + \Gamma_1^{1-\nu}Y\Gamma_2^{\nu}}{2}\right)V^*.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left\|\frac{A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}}{2}\right\|_2^2 &= \left\|\frac{\Gamma_1^{\nu}Y\Gamma_2^{1-\nu} + \Gamma_1^{1-\nu}Y\Gamma_2^{\nu}}{2}\right\|_2^2 \\
&= \sum_{i,j=1}^{n}\left(\frac{\lambda_i^{\nu}\mu_j^{1-\nu} + \lambda_i^{1-\nu}\mu_j^{\nu}}{2}\right)^2 |y_{ij}|^2.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\sum_{i,j=1}^{n}(\sqrt{\lambda_i\mu_j})^2|y_{ij}|^2 = \left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2, \\
&\sum_{i,j=1}^{n}\left(\frac{\lambda_i + \mu_j}{2}\right)^2|y_{ij}|^2 = \left\|\frac{AX + XB}{2}\right\|_2^2.
\end{aligned}$$

It follows from the inequality (2.5) that

$$\begin{aligned}
&\left\|(1-\nu)^{2(1-\nu)}\frac{A^{2\nu}XB^{1-2\nu} + A^{1-2\nu}XB^{2\nu}}{2} + (1-\nu)^{2\nu}\frac{AX + XB}{2}\right\|_2^2 \\
&\quad= \sum_{i,j=1}^{n}\left((1-\nu)^{2(1-\nu)}\frac{\lambda_i^{2\nu}\mu_j^{1-2\nu} + \lambda_i^{1-2\nu}\mu_j^{2\nu}}{2} + (1-\nu)^{2\nu}\frac{\lambda_i + \mu_j}{2}\right)^2|y_{ij}|^2 \\
&\quad\geq \sum_{i,j=1}^{n}\bigl(2(1-\nu)\sqrt{\lambda_i\mu_j}\bigr)^2|y_{ij}|^2 = \bigl(2(1-\nu)\bigr)^2\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2.
\end{aligned}$$

□

Using the same strategy as in the proof of Theorem 7 and inequalities (2.6) and (2.7), we get the following theorems.

Theorem 8 *Let $A, B, X \in M_n$ such that A and B are positive definite if $\nu \in [0, \frac{1}{2}]$, then*

$$\begin{aligned}
&\left\|(1-\nu)^{2\nu}\frac{A^{2\nu}XB^{1-2\nu} + A^{1-2\nu}XB^{2\nu}}{2} + (1-\nu)^{2(1-\nu)}\frac{AX + XB}{2}\right\|_2 \\
&\quad\geq \left\|2(1-\nu)A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2.
\end{aligned}$$

Theorem 9 *Let $A, B, X \in M_n$ such that A and B are positive definite if $\nu \in [0, \frac{1}{2}]$, then*

$$\begin{aligned}
&\left\|(1-\nu)^{2(1-\nu)}\frac{A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}}{2} + (1-2\nu)^{2\nu}\frac{AX + XB}{2}\right\|_2 \\
&\quad\geq \left\|2(1-\nu)A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2.
\end{aligned}$$

3 Conclusions

In the present paper we got reverses of the scalar Young type inequality and using them we obtained the reverses of Young type inequalities for operators. Then we considered the reverse Heinz mean inequality for unitarily invariant norms and established the following inequality and several related results:

$$\left\|(1-\nu)^{2(1-\nu)}\frac{A^{2\nu}XB^{1-2\nu}+A^{1-2\nu}XB^{2\nu}}{2}+(1-\nu)^{2\nu}\frac{AX+XB}{2}\right\|_2 \geq 2(1-\nu)\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2,$$

where $A, B, X \in M_n$ such that A and B are positive definite and $\nu \in [0, \frac{1}{2}]$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors conceived of the study, and participated in the design and coordination. All authors read and approved the final manuscript.

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