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Integral inequalities of Hermite-Hadamard type for functions whose third derivatives are convex

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available at the end of the article**Abstract**

In the paper, the authors establish some new inequalities of Hermite-Hadamard type for functions whose third derivatives are convex.

MSC: Primary 26D15; secondary 26A51; 41A55**Keywords:** Hermite-Hadamard's integral inequality; convex function; third derivative**1 Introduction**It is common knowledge in mathematical analysis that a function $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to be convex on an interval I if the inequality

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) \quad (1.1)$$

is valid for all $x, y \in I$ and $\lambda \in [0, 1]$. If $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is a convex function on I and $a, b \in I$ with $a < b$, then the double inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a)+f(b)}{2} \quad (1.2)$$

holds. This double inequality is known in the literature as Hermite-Hadamard's integral inequality for convex functions. The definition of convex functions and Hermite-Hadamard's integral inequality (1.2) have been generalized, refined, and extended by many mathematicians in a lot of references. Some of them may be recited as follows.

Theorem 1.1 ([1, Theorems 2.2 and 2.3]) *Let $f : I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be differentiable on I° and $a, b \in I^\circ$ with $a < b$.*(1) *If $|f'(x)|$ is a convex function on $[a, b]$, then*

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{b-a}{8} (|f'(a)| + |f'(b)|). \end{aligned} \quad (1.3)$$

(2) If $|f'(x)|^{p/(p-1)}$ for $p > 1$ is a convex function on $[a, b]$, then

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{b-a}{2(p+1)^{1/p}} \left[\frac{|f'(a)|^{p/(p-1)} + |f'(b)|^{p/(p-1)}}{2} \right]^{(p-1)/p}. \end{aligned} \tag{1.4}$$

Theorem 1.2 ([2, Theorems 2.2 and 2.3]) *Let $f : I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° and $a, b \in I^\circ$ with $a < b$. If $|f'|^{p/(p-1)}$ for $p > 1$ is convex on $[a, b]$, then*

$$\begin{aligned} \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| & \leq \frac{b-a}{16} \left(\frac{4}{p+1}\right)^{1/p} \left[(|f'(a)|^{p/(p-1)} + 3|f'(b)|^{p/(p-1)})^{(p-1)/p} \right. \\ & \left. + (3|f'(a)|^{p/(p-1)} + |f'(b)|^{p/(p-1)})^{(p-1)/p} \right] \end{aligned} \tag{1.5}$$

and

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{4} \left(\frac{4}{p+1}\right)^{1/p} [|f'(a)| + |f'(b)|]. \tag{1.6}$$

Definition 1.1 ([3]) A function $f : I \subseteq \mathbb{R} \rightarrow [0, \infty)$ is said to be quasi-convex if

$$f(\lambda x + (1-\lambda)y) \leq \sup\{f(x), f(y)\} \tag{1.7}$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$.

Theorem 1.3 ([4, Theorem 2]) *Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be differentiable on I° such that $f''' \in L([a, b])$ and $a, b \in I^\circ$ with $a < b$. If $|f'''|$ is quasi-convex on $[a, b]$, then*

$$\begin{aligned} & \left| \int_a^b f(x) dx - \frac{b-a}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] \right| \\ & \leq \frac{(b-a)^4}{1152} \left[\max \left\{ |f'''(a)|, \left| f''' \left(\frac{a+b}{2} \right) \right| \right\} + \max \left\{ \left| f''' \left(\frac{a+b}{2} \right) \right|, |f'''(b)| \right\} \right]. \end{aligned} \tag{1.8}$$

Definition 1.2 ([5]) Let $s \in (0, 1]$. A function $f : \mathbb{R}_0 \rightarrow \mathbb{R}_0$ is said to be s -convex in the second sense if

$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y) \tag{1.9}$$

for all $x, y \in I$ and $\lambda \in [0, 1]$.

Theorem 1.4 ([6, Theorem 3.1]) *Let $f : I \subseteq \mathbb{R}_0 \rightarrow \mathbb{R}$ be differentiable on I° , $a, b \in I^\circ$ with $a < b$, and $f''' \in L([a, b])$. If $q \geq 1$ and $|f'''|$ is s -convex in the second sense on $[a, b]$ for $s \in (0, 1]$, then*

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx - \frac{b-a}{12} [f'(b) - f'(a)] \right| \\ & \leq \frac{(b-a)^3}{192} \left[\frac{2^{2-s}(s+6+2^{s+2}s)}{(s+2)(s+3)(s+4)} \right]^{1/q} [|f'''(a)|^q + |f'''(b)|^q]^{1/q}. \end{aligned} \tag{1.10}$$

For more information on Hermite-Hadamard type inequalities, please refer to [7–19], for example, and to monographs [20, 21] and related references therein.

In this paper, we will create some new integral inequalities of Hermite-Hadamard type for functions whose third derivatives are convex.

2 Lemma

For establishing some new integral inequalities of Hermite-Hadamard type for functions whose third derivatives are convex, we need an integral identity below.

Lemma 2.1 *Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a three times differentiable mapping on I° and $a, b \in I^\circ$ with $a < b$. If $f''' \in L([a, b])$, then*

$$\begin{aligned} & f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) \, dx + \frac{b-a}{24} [f'(b) - f'(a)] \\ &= \frac{(b-a)^3}{96} \left[\int_0^1 t(1-t)(2-t) f''' \left(\frac{1-t}{2} a + \frac{1+t}{2} b \right) dt \right. \\ & \quad \left. - \int_0^1 t(1-t)(2-t) f''' \left(\frac{1+t}{2} a + \frac{1-t}{2} b \right) dt \right]. \end{aligned} \tag{2.1}$$

Proof Integrating by part and changing variable of definite integral yield

$$\begin{aligned} & \int_0^1 t(1-t)(2-t) f''' \left(\frac{1-t}{2} a + \frac{1+t}{2} b \right) dt \\ &= -\frac{2}{b-a} \int_0^1 (3t^2 - 6t + 2) f'' \left(\frac{1-t}{2} a + \frac{1+t}{2} b \right) dt \\ &= \frac{4}{(b-a)^2} \left[f'(b) + 2f' \left(\frac{a+b}{2} \right) \right] + \frac{48}{(b-a)^3} \int_0^1 (t-1) df \left(\frac{1-t}{2} a + \frac{1+t}{2} b \right) \\ &= \frac{4}{(b-a)^2} \left[f'(b) + 2f' \left(\frac{a+b}{2} \right) \right] + \frac{48}{(b-a)^3} f \left(\frac{a+b}{2} \right) \\ & \quad - \frac{48}{(b-a)^3} \int_0^1 f \left(\frac{1-t}{2} a + \frac{1+t}{2} b \right) dt \end{aligned}$$

and

$$\begin{aligned} & \int_0^1 t(1-t)(2-t) f''' \left(\frac{1+t}{2} a + \frac{1-t}{2} b \right) dt \\ &= \frac{2}{b-a} \int_0^1 (3t^2 - 6t + 2) f'' \left(\frac{1+t}{2} a + \frac{1-t}{2} b \right) dt \\ &= \frac{4}{(b-a)^2} \left[f'(a) + 2f' \left(\frac{a+b}{2} \right) \right] - \frac{48}{(b-a)^3} \int_0^1 (t-1) df \left(\frac{1+t}{2} a + \frac{1-t}{2} b \right) \\ &= \frac{4}{(b-a)^2} \left[f'(a) + 2f' \left(\frac{a+b}{2} \right) \right] - \frac{48}{(b-a)^3} f \left(\frac{a+b}{2} \right) \\ & \quad + \frac{48}{(b-a)^3} \int_0^1 f \left(\frac{1+t}{2} a + \frac{1-t}{2} b \right) dt. \end{aligned}$$

Lemma 2.1 is thus proved. □

3 Hermite-Hadamard type inequalities for convex functions

Basing on Lemma 2.1, we now start out to establish some new integral inequalities of Hermite-Hadamard type for functions whose third derivatives are convex.

Theorem 3.1 *Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be three times differentiable on I° and $f''' \in L([a, b])$ for $a, b \in I^\circ$ with $a < b$. If $|f'''|^q$ for $q \geq 1$ is convex on $[a, b]$, then*

$$\begin{aligned} & \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx + \frac{b-a}{24} [f'(b) - f'(a)] \right| \\ & \leq \frac{(b-a)^3}{384} \left\{ \left[\frac{4|f'''(a)|^q + 11|f'''(b)|^q}{15} \right]^{1/q} + \left[\frac{11|f'''(a)|^q + 4|f'''(b)|^q}{15} \right]^{1/q} \right\}. \end{aligned} \quad (3.1)$$

Proof Since $|f'''|^q$ is convex on $[a, b]$, by Lemma 2.1 and Hölder's inequality, we have

$$\begin{aligned} & \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx + \frac{b-a}{24} [f'(b) - f'(a)] \right| \\ & \leq \frac{(b-a)^3}{96} \left\{ \int_0^1 t(1-t)(2-t) \left| f''' \left(\frac{1-t}{2}a + \frac{1+t}{2}b \right) \right| dt \right. \\ & \quad \left. + \int_0^1 t(1-t)(2-t) \left| f''' \left(\frac{1+t}{2}a + \frac{1-t}{2}b \right) \right| dt \right\} \\ & \leq \frac{(b-a)^3}{96} \left[\int_0^1 t(1-t)(2-t) dt \right]^{1-1/q} \\ & \quad \times \left\{ \left[\int_0^1 t(1-t)(2-t) \left| f''' \left(\frac{1-t}{2}a + \frac{1+t}{2}b \right) \right|^q dt \right]^{1/q} \right. \\ & \quad \left. + \left[\int_0^1 t(1-t)(2-t) \left| f''' \left(\frac{1+t}{2}a + \frac{1-t}{2}b \right) \right|^q dt \right]^{1/q} \right\} \\ & \leq \frac{(b-a)^3}{96} \left(\frac{1}{4} \right)^{1-1/q} \left\{ \left[\frac{1}{2} \int_0^1 t(1-t)^2(2-t) |f'''(a)|^q dt \right. \right. \\ & \quad \left. \left. + \frac{1}{2} \int_0^1 t(1-t^2)(2-t) |f'''(b)|^q dt \right]^{1/q} + \left[\frac{1}{2} \int_0^1 t(1-t^2)(2-t) |f'''(a)|^q dt \right. \right. \\ & \quad \left. \left. + \frac{1}{2} \int_0^1 t(1-t)^2(2-t) |f'''(b)|^q dt \right]^{1/q} \right\} \\ & = \frac{(b-a)^3}{384} \left\{ \left[\frac{4|f'''(a)|^q + 11|f'''(b)|^q}{15} \right]^{1/q} + \left[\frac{11|f'''(a)|^q + 4|f'''(b)|^q}{15} \right]^{1/q} \right\}. \end{aligned}$$

The proof of Theorem 3.1 is complete. □

Corollary 3.1 *Under conditions of Theorem 3.1, if $q = 1$, we have*

$$\begin{aligned} & \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx + \frac{b-a}{24} [f'(b) - f'(a)] \right| \\ & \leq \frac{(b-a)^3}{384} [|f'''(a)| + |f'''(b)|]. \end{aligned} \quad (3.2)$$

Theorem 3.2 Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be three times differentiable on I° and $f''' \in L([a, b])$ for $a, b \in I^\circ$ with $a < b$. If $|f'''|^q$ for $q > 1$ is convex on $[a, b]$ and if $q \geq r$ and $s \geq 0$, then

$$\begin{aligned} & \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx + \frac{b-a}{24} [f'(b) - f'(a)] \right| \\ & \leq \frac{(b-a)^3}{96} \left[2B\left(\frac{2q-r-1}{q-1}, \frac{2q-s-1}{q-1}\right) - B\left(\frac{3q-r-2}{q-1}, \frac{2q-s-1}{q-1}\right) \right]^{1-1/q} \\ & \quad \times \left(\frac{1}{2}\right)^{1/q} \left\{ [2B(r+1, s+2) - B(r+2, s+2)] |f'''(a)|^q \right. \\ & \quad + [2B(r+1, s+1) + B(r+2, s+1) - B(r+3, s+1)] |f'''(b)|^q \\ & \quad + [2B(r+1, s+1) + B(r+2, s+1) - B(r+3, s+1)] |f'''(a)|^q \\ & \quad \left. + [2B(r+1, s+2) - B(r+2, s+2)] |f'''(b)|^q \right\}, \end{aligned}$$

where $B(x, y)$ is the classical Beta function, which may be defined for $\Re(x) > 0$ and $\Re(y) > 0$ by

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt. \tag{3.3}$$

Proof By Lemma 2.1, Hölder’s inequality, and the convexity of $|f'''|^q$ on $[a, b]$, we have

$$\begin{aligned} & \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx + \frac{b-a}{24} [f'(b) - f'(a)] \right| \\ & \leq \frac{(b-a)^3}{96} \left\{ \int_0^1 t(1-t)(2-t) \left| f''' \left(\frac{1-t}{2}a + \frac{1+t}{2}b \right) \right| dt \right. \\ & \quad \left. + \int_0^1 t(1-t)(2-t) \left| f''' \left(\frac{1+t}{2}a + \frac{1-t}{2}b \right) \right| dt \right\} \\ & \leq \frac{(b-a)^3}{96} \left(\int_0^1 t^{(q-r)/(q-1)} (1-t)^{(q-s)/(q-1)} (2-t) dt \right)^{1-1/q} \\ & \quad \times \left\{ \left(\int_0^1 t^r (1-t)^s (2-t) \left| f''' \left(\frac{1-t}{2}a + \frac{1+t}{2}b \right) \right|^q dt \right)^{1/q} \right. \\ & \quad \left. + \left(\int_0^1 t^r (1-t)^s (2-t) \left| f''' \left(\frac{1+t}{2}a + \frac{1-t}{2}b \right) \right|^q dt \right)^{1/q} \right\} \\ & \leq \frac{(b-a)^3}{96} \left[2B\left(\frac{2q-r-1}{q-1}, \frac{2q-s-1}{q-1}\right) - B\left(\frac{3q-r-2}{q-1}, \frac{2q-s-1}{q-1}\right) \right]^{1-1/q} \\ & \quad \times \left\{ \left[\frac{1}{2} \int_0^1 t^r (1-t)^{s+1} (2-t) |f'''(a)|^q dt \right. \right. \\ & \quad \left. + \frac{1}{2} \int_0^1 t^r (1+t) (1-t)^s (2-t) |f'''(b)|^q dt \right]^{1/q} \\ & \quad \left. + \left[\frac{1}{2} \int_0^1 t^r (1+t) (1-t)^s (2-t) |f'''(a)|^q dt \right. \right. \\ & \quad \left. \left. + \frac{1}{2} \int_0^1 t^r (1-t)^{s+1} (2-t) |f'''(b)|^q dt \right]^{1/q} \right\} \end{aligned}$$

$$\begin{aligned}
 &= \frac{(b-a)^3}{96} \left[2B\left(\frac{2q-r-1}{q-1}, \frac{2q-s-1}{q-1}\right) - B\left(\frac{3q-r-2}{q-1}, \frac{2q-s-1}{q-1}\right) \right]^{1-1/q} \\
 &\quad \times \left(\frac{1}{2}\right)^{1/q} \{ [[2B(r+1, s+2) - B(r+2, s+2)] |f'''(a)|^q \\
 &\quad + [2B(r+1, s+1) + B(r+2, s+1) - B(r+3, s+1)] |f'''(b)|^q]^{1/q} \\
 &\quad + [[2B(r+1, s+1) + B(r+2, s+1) - B(r+3, s+1)] |f'''(a)|^q \\
 &\quad + [2B(r+1, s+2) - B(r+2, s+2)] |f'''(b)|^q]^{1/q} \}.
 \end{aligned}$$

The proof of Theorem 3.2 is completed. □

Corollary 3.2 *Under conditions of Theorem 3.2,*

(1) *if $r = 0$, we have*

$$\begin{aligned}
 &\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx + \frac{b-a}{24} [f'(b) - f'(a)] \right| \\
 &\leq \frac{(b-a)^3}{96} \left[2B\left(\frac{2q-1}{q-1}, \frac{2q-s-1}{q-1}\right) \right. \\
 &\quad \left. - B\left(\frac{3q-2}{q-1}, \frac{2q-s-1}{q-1}\right) \right]^{1-1/q} \left(\frac{1}{2(s+1)(s+2)(s+3)} \right)^{1/q} \\
 &\quad \times \{ [(s+1)(2s+5)|f'''(a)|^q + (2s^2+11s+13)|f'''(b)|^q]^{1/q} \\
 &\quad + [(2s^2+11s+13)|f'''(a)|^q + (s+1)(2s+5)|f'''(b)|^q]^{1/q} \};
 \end{aligned}$$

(2) *if $s = 0$, we have*

$$\begin{aligned}
 &\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx + \frac{b-a}{24} [f'(b) - f'(a)] \right| \\
 &\leq \frac{(b-a)^3}{96} \left[2B\left(\frac{2q-r-1}{q-1}, \frac{2q-1}{q-1}\right) \right. \\
 &\quad \left. - B\left(\frac{3q-r-2}{q-1}, \frac{2q-1}{q-1}\right) \right]^{1-1/q} \left[\frac{1}{2(r+1)(r+2)(r+3)} \right]^{1/q} \\
 &\quad \times \{ [(r+5)|f'''(a)|^q + (2r^2+11r+13)|f'''(b)|^q]^{1/q} \\
 &\quad + [(2r^2+11r+13)|f'''(a)|^q + (r+5)|f'''(b)|^q]^{1/q} \};
 \end{aligned}$$

(3) *if $r = s = 0$, we have*

$$\begin{aligned}
 &\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx + \frac{b-a}{24} [f'(b) - f'(a)] \right| \\
 &\leq \frac{(b-a)^3}{96} \left[2B\left(\frac{2q-1}{q-1}, \frac{2q-1}{q-1}\right) - B\left(\frac{3q-2}{q-1}, \frac{2q-1}{q-1}\right) \right]^{1-1/q} \left(\frac{3}{2}\right)^{1/q} \\
 &\quad \times \left\{ \left[\frac{5|f'''(a)|^q + 13|f'''(b)|^q}{18} \right]^{1/q} + \left[\frac{13|f'''(a)|^q + 5|f'''(b)|^q}{18} \right]^{1/q} \right\};
 \end{aligned}$$

(4) if $r = q$, we have

$$\begin{aligned} & \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx + \frac{b-a}{24} [f'(b) - f'(a)] \right| \\ & \leq \frac{(b-a)^3}{96} \left[\frac{(5q-2s-3)(q-1)}{(q-s)^2 + (5q-3s-2)(q-1)} \right]^{1-1/q} \left(\frac{1}{2}\right)^{1/q} \\ & \quad \times \{ [[2B(q+1, s+2) - B(q+2, s+2)] |f'''(a)|^q \\ & \quad + [2B(q+1, s+1) + B(q+2, s+1) - B(q+3, s+1)] |f'''(b)|^q]^{1/q} \\ & \quad + [[2B(q+1, s+1) + B(q+2, s+1) - B(q+3, s+1)] |f'''(a)|^q \\ & \quad + [2B(q+1, s+2) - B(q+2, s+2)] |f'''(b)|^q]^{1/q} \}; \end{aligned}$$

(5) if $s = q$, we have

$$\begin{aligned} & \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx + \frac{b-a}{24} [f'(b) - f'(a)] \right| \\ & \leq \frac{(b-a)^3}{96} \left(\frac{(q-1)(4q-r-3)}{(2q-r-1)(3q-r-2)} \right)^{1-1/q} \left(\frac{1}{2}\right)^{1/q} \\ & \quad \times \{ [[2B(r+1, q+2) - B(r+2, q+2)] |f'''(a)|^q \\ & \quad + [2B(r+1, q+1) + B(r+2, q+1) - B(r+3, q+1)] |f'''(b)|^q]^{1/q} \\ & \quad + [[2B(r+1, q+1) + B(r+2, q+1) - B(r+3, q+1)] |f'''(a)|^q \\ & \quad + [2B(r+1, q+2) - B(r+2, q+2)] |f'''(b)|^q]^{1/q} \}; \end{aligned}$$

(6) if $r = s = q$, we have

$$\begin{aligned} & \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx + \frac{b-a}{24} [f'(b) - f'(a)] \right| \\ & \leq \frac{3^{-1/q}(b-a)^3}{64} \{ [[2B(q+1, q+2) - B(q+2, q+2)] |f'''(a)|^q \\ & \quad + [2B(q+1, q+1) + B(q+2, q+1) - B(q+3, q+1)] |f'''(b)|^q]^{1/q} \\ & \quad + [[2B(q+1, q+1) + B(q+2, q+1) - B(q+3, q+1)] |f'''(a)|^q \\ & \quad + [2B(q+1, q+2) - B(q+2, q+2)] |f'''(b)|^q]^{1/q} \}. \end{aligned}$$

Theorem 3.3 Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be three times differentiable on I° and $f''' \in L([a, b])$ for $a, b \in I^\circ$ with $a < b$. If $|f'''|^q$ is convex on $[a, b]$ for $q > 1$ and $q \geq \ell \geq 0$, then

$$\begin{aligned} & \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx + \frac{b-a}{24} [f'(b) - f'(a)] \right| \\ & \leq \frac{(b-a)^3}{96} \left[\frac{(2^{\xi+2} + 1)\xi - 2^{\xi+2} + 5}{\xi^3 + 6\xi^2 + 11\xi + 6} \right]^{1-1/q} \left[\frac{1}{(\ell+1)(\ell+2)(\ell+3)(\ell+4)} \right]^{1/q} \\ & \quad \times \{ [(2^{\ell+1}\ell^2 - (2^{\ell+1} + 1)\ell + 2^{\ell+3} - 7) |f'''(a)|^q \\ & \quad + ((2^{\ell+1} + 1)\ell^2 + 2(7 \times 2^\ell + 5)\ell - 3 \times 2^{\ell+3} + 27) |f'''(b)|^q]^{1/q} \} \end{aligned}$$

$$\begin{aligned}
 &+ \left[(2^{\ell+1} + 1)\ell^2 + 2(7 \times 2^\ell + 5)\ell - 3 \times 2^{\ell+3} + 27 \right] |f'''(a)|^q \\
 &+ (2^{\ell+1}\ell^2 - (2^{\ell+1} + 1)\ell + 2^{\ell+3} - 7) |f'''(b)|^q \Big]^{1/q},
 \end{aligned}$$

where $\xi = \frac{q-\ell}{q-1}$.

Proof By Lemma 2.1, Hölder’s inequality, and the convexity of $|f'''|^q$ on $[a, b]$, we have

$$\begin{aligned}
 &\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx + \frac{b-a}{24} [f'(b) - f'(a)] \right| \\
 &\leq \frac{(b-a)^3}{96} \left\{ \int_0^1 t(1-t)(2-t) \left| f''' \left(\frac{1-t}{2}a + \frac{1+t}{2}b \right) \right| dt \right. \\
 &\quad \left. + \int_0^1 t(1-t)(2-t) \left| f''' \left(\frac{1+t}{2}a + \frac{1-t}{2}b \right) \right| dt \right\} \\
 &\leq \frac{(b-a)^3}{96} \left(\int_0^1 t(1-t)(2-t)^{(q-\ell)/(q-1)} dt \right)^{1-1/q} \\
 &\quad \times \left\{ \left(\int_0^1 t(1-t)(2-t)^\ell \left| f''' \left(\frac{1-t}{2}a + \frac{1+t}{2}b \right) \right|^q dt \right)^{1/q} \right. \\
 &\quad \left. + \left(\int_0^1 t(1-t)(2-t)^\ell \left| f''' \left(\frac{1+t}{2}a + \frac{1-t}{2}b \right) \right|^q dt \right)^{1/q} \right\} \\
 &\leq \frac{(b-a)^3}{96} \left(\int_0^1 t(1-t)(2-t)^{(q-\ell)/(q-1)} dt \right)^{1-1/q} \\
 &\quad \times \left\{ \left[\frac{1}{2} \int_0^1 t(1-t)(2-t)^\ell [(1-t)|f'''(a)|^q + (1+t)|f'''(b)|^q] dt \right]^{1/q} \right. \\
 &\quad \left. + \left[\frac{1}{2} \int_0^1 t(1-t)(2-t)^\ell [(1+t)|f'''(a)|^q + (1-t)|f'''(b)|^q] dt \right]^{1/q} \right\} \\
 &= \frac{(b-a)^3}{96} \left[\frac{(2^{\xi+2} + 1)\xi - 2^{\xi+2} + 5}{\xi^3 + 6\xi^2 + 11\xi + 6} \right]^{1-1/q} \left[\frac{1}{(\ell+1)(\ell+2)(\ell+3)(\ell+4)} \right]^{1/q} \\
 &\quad \times \left\{ [(2^{\ell+1}\ell^2 - (2^{\ell+1} + 1)\ell + 2^{\ell+3} - 7) |f'''(a)|^q \right. \\
 &\quad + ((2^{\ell+1} + 1)\ell^2 + 2(7 \times 2^\ell + 5)\ell - 3 \times 2^{\ell+3} + 27) |f'''(b)|^q]^{1/q} \\
 &\quad + [(2^{\ell+1} + 1)\ell^2 + 2(7 \times 2^\ell + 5)\ell - 3 \times 2^{\ell+3} + 27) |f'''(a)|^q \\
 &\quad \left. + (2^{\ell+1}\ell^2 - (2^{\ell+1} + 1)\ell + 2^{\ell+3} - 7) |f'''(b)|^q]^{1/q} \right\}.
 \end{aligned}$$

The proof of Theorem 3.3 is complete. □

Corollary 3.3 Under conditions of Theorem 3.3.

(1) if $\ell = 0$, we have

$$\begin{aligned}
 &\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx + \frac{b-a}{24} [f'(b) - f'(a)] \right| \\
 &\leq \frac{(b-a)^3}{96} \left(\frac{1}{6}\right)^{1/q} \left[\frac{(q-1)^2(6q + 2^{(3q-2)/(q-1)} - 5)}{(2q-1)(3q-2)(4q-3)} \right]^{1-1/q} \\
 &\quad \times \left\{ \left[\frac{|f'''(a)|^q + 3|f'''(b)|^q}{4} \right]^{1/q} + \left[\frac{3|f'''(a)|^q + |f'''(b)|^q}{4} \right]^{1/q} \right\}, \tag{3.4}
 \end{aligned}$$

(2) if $\ell = q$, we have

$$\begin{aligned} & \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx + \frac{b-a}{24} [f'(b) - f'(a)] \right| \\ & \leq \frac{(b-a)^3}{96} \left(\frac{1}{6}\right)^{1-1/q} \left[\frac{1}{(q+1)(q+2)(q+3)(q+4)} \right]^{1/q} \\ & \quad \times \{ [(2^{q+1}q^2 - (2^{q+1} + 1)q + 2^{q+3} - 7)|f'''(a)|^q \\ & \quad + ((2^{q+1} + 1)q^2 + 2(7 \times 2^q + 5)q - 3 \times 2^{q+3} + 27)|f'''(b)|^q]^{1/q} \\ & \quad + [((2^{q+1} + 1)q^2 + 2(7 \times 2^q + 5)q - 3 \times 2^{q+3} + 27)|f'''(a)|^q \\ & \quad + (2^{q+1}q^2 - (2^{q+1} + 1)q + 2^{q+3} - 7)|f'''(b)|^q]^{1/q} \}. \end{aligned}$$

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the manuscript and read and approved the final manuscript.

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