Gaboury *Advances in Difference Equations* 2013, 2013:361
http://www.advancesindifferenceequations.com/content/2013/1/361

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH

Open Access

Some relations involving generalized Hurwitz-Lerch zeta function obtained by means of fractional derivatives with applications to Apostol-type polynomials

Sebastien Gaboury*

*Correspondence:
s1gabour@uqac.ca
Department of Mathematics and Computer Science, University of Quebec at Chicoutimi, Quebec, G7H 2B1, Canada

Abstract

In this paper, we present three new expansion formulas for the generalized Hurwitz-Lerch zeta function. These expansions are obtained by using Taylor-like expansions involving fractional derivatives. Finally, interesting special cases involving the Apostol-Bernoulli polynomials and the Apostol-Euler polynomials are also given.

MSC: 26A33; 11M35; 33C05; 11B68

Keywords: fractional derivatives; generalized Taylor expansion; generalized Hurwitz-Lerch zeta functions; Riemann zeta function; generalized Apostol-Bernoulli polynomials; generalized Apostol-Euler polynomials

1 Introduction

It is well known that the generalized Hurwitz-Lerch zeta function as well as its extended version have many applications in various areas of mathematics and physics. In number theory, the Riemann and Hurwitz zeta functions are closely related to Dedekind zeta functions and Artin L-functions, which play a central role in the discipline. In addition, the generalized Hurwitz-Lerch zeta functions, evaluated at negative integers, are closely related to the Apostol-Bernoulli polynomials, the Apostol-Euler polynomials and the Frobenius-Euler polynomials [1–3]. These functions are also connected to the generalized Fermi-Dirac functions and the generalized Bose-Einstein functions [1]. The generalized Fermi-Dirac and Bose-Einstein functions, which appear in quantum statistics, quantum interference and in the theory of quantum entanglement, have been introduced recently by Srivastava *et al.* [4]. Moreover, the generalized Hurwitz-Lerch zeta functions have interesting applications in geometric function theory [5]; and finally, Gupta *et al.* [6] investigated the generalized Hurwitz-Lerch zeta distribution and applied this new distribution to reliability.

The generalized Hurwitz zeta function $\zeta(s,a)$ is defined by [7, p.88 *et seq.*]

$$\zeta(s,a) := \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} \quad \bigl(\mathrm{Re}(s) > 1; a \in \mathbb{C} \setminus \mathbb{Z}_0^-; \mathbb{Z}_0^- := \{0,-1,-2,\ldots\}\bigr), \tag{1.1}$$

where

$$\zeta(s,1)=\zeta(s)=\frac{1}{2^s-1}\zeta\left(s,\frac{1}{2}\right) \tag{1.2}$$

yields the celebrated Riemann zeta function $\zeta(s)$. The Riemann zeta function is continued meromorphically to the whole complex s-plane except for a simple pole at $s=1$ with residue 1.

The Hurwitz-Lerch zeta function $\Phi(z,s,a)$ is defined, as in [7, p.121 *et seq.*], by

$$\Phi(z,s,a):=\sum_{n=0}^{\infty}\frac{z^n}{(n+a)^s}$$
$$\bigl(a\in\mathbb{C}\setminus\mathbb{Z}_0^-;s\in\mathbb{C}\text{ when }|z|<1;\operatorname{Re}(s)>1\text{ when }|z|=1\bigr). \tag{1.3}$$

Clearly, we have the following relations:

$$\Phi(1,s,a)=\zeta(s,a)\quad\text{and}\quad\Phi(1,s,1)=\zeta(s). \tag{1.4}$$

The Hurwitz-Lerch zeta function has the well-known integral representation

$$\Phi(z,s,a)=\frac{1}{\Gamma(s)}\int_0^{\infty}\frac{t^{s-1}\mathrm{e}^{-at}}{1-z\mathrm{e}^{-t}}\,dt$$
$$\bigl(\operatorname{Re}(a)>0;\operatorname{Re}(s)>0\text{ when }|z|\leq 1\ (z\neq 1);\operatorname{Re}(s)>1\text{ when }z=1\bigr). \tag{1.5}$$

Recently, Lin and Srivastava [8] investigated a more general family of Hurwitz-Lerch zeta functions. Explicitly, they introduced the function $\Phi_{\mu,\nu}^{(\rho,\sigma)}(z,s,a)$ defined by

$$\Phi_{\mu,\nu}^{(\rho,\sigma)}(z,s,a):=\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n}}{(\nu)_{\sigma n}}\frac{z^n}{(a+n)^s}$$
$$\bigl(\mu\in\mathbb{C};a,\nu\in\mathbb{C}\setminus\mathbb{Z}_0^-;\rho,\sigma\in\mathbb{R}^+;\rho<\sigma\text{ when }s,z\in\mathbb{C};$$
$$\rho=\sigma\text{ and }s\in\mathbb{C}\text{ when }|z|<1;\rho=\sigma\text{ and }\operatorname{Re}(s-\mu+\nu)>1\text{ when }|z|=1\bigr), \tag{1.6}$$

where $(\lambda)_\kappa$ denotes the Pochhammer symbol defined, in terms of the gamma function, by

$$(\lambda)_\kappa:=\frac{\Gamma(\lambda+\kappa)}{\Gamma(\lambda)}=\begin{cases}\lambda(\lambda+1)\cdots(\lambda+n-1) & (\kappa=n\in\mathbb{N};\lambda\in\mathbb{C}),\\ 1 & (\kappa=0;\lambda\in\mathbb{C}\setminus\{0\}).\end{cases} \tag{1.7}$$

It is easily seen that

$$\Phi_{\nu,\nu}^{(\sigma,\sigma)}(z,s,a)=\Phi_{\mu,\nu}^{(0,0)}(z,s,a)=\Phi(z,s,a) \tag{1.8}$$

and

$$\Phi_{\mu,1}^{(1,1)}(z,s,a)=\Phi_\mu^*(z,s,a):=\sum_{n=0}^{\infty}\frac{(\mu)_n}{n!}\frac{z^n}{(n+a)^s}. \tag{1.9}$$

The function $\Phi^*_\mu(z,s,a)$ is, in fact, a generalized Hurwith-Lerch zeta function investigated by Goyal and Laddha [9, p.100, equation (1.5)]. Another family of generalized Hurwitz-Lerch zeta functions is the one studied by Garg *et al.* [10], that is, $\Phi_{\lambda,\mu;\nu}(z,s,a)$ defined as follows:

$$\Phi_{\lambda,\mu;\nu}(z,s,a) := \sum_{n=0}^{\infty} \frac{(\lambda)_n(\mu)_n}{(\nu)_n n!} \frac{z^n}{(n+a)^s}$$

$$\bigl(\lambda,\mu \in \mathbb{C}; \nu, a \in \mathbb{C} \setminus \mathbb{Z}_0^-; s \in \mathbb{C} \text{ when } |z| < 1;$$

$$\operatorname{Re}(s+\nu-\lambda-\mu) > 1 \text{ when } |z| = 1\bigr). \tag{1.10}$$

Obviously, we see that

$$\Phi_{1,\mu;1}(z,s,a) = \Phi^*_\mu(z,s,a). \tag{1.11}$$

This family of Hurwitz-Lerch zeta functions will play an important role in the sequel.

The aim of this paper is to make use of three Taylor-like expansions involving fractional derivatives to obtain some relations for the generalized Hurwitz-Lerch zeta function $\Phi^*_\mu(z,s,a)$. One of these Taylor-like expansions has been obtained by Osler [11] and the two others, more recently, by Tremblay *et al.* [12, 13]. Finally, interesting special cases of these new relations involving the Apostol-Bernoulli polynomials and the Apostol-Euler polynomials are obtained.

2 Pochhammer contour integral representation for fractional derivative and a new generalized Leibniz rule

The use of contour of integration in the complex plane provides a very powerful tool in both classical and fractional calculus. The most familiar representation for fractional derivative of order α of $z^p f(z)$ is the Riemann-Liouville integral [14–16] that is

$$D_z^\alpha z^p f(z) = \frac{1}{\Gamma(-\alpha)} \int_0^z f(\xi)\xi^p(\xi-z)^{-\alpha-1}\,\mathrm{d}\xi, \tag{2.1}$$

which is valid for $\operatorname{Re}(\alpha) < 0$, $\operatorname{Re}(p) > 1$ and where the integration is done along a straight line from 0 to z in the ξ-plane. By integrating by part m times, we obtain

$$D_z^\alpha z^p f(z) = \frac{d^m}{dz^m} D_z^{\alpha-m} z^p f(z). \tag{2.2}$$

This allows to modify the restriction $\operatorname{Re}(\alpha) < 0$ to $\operatorname{Re}(\alpha) < m$ [16]. Another used representation for the fractional derivative is the one based on the Cauchy integral formula widely used by Osler [17–20]. These two representations have been used in many interesting research papers. It appears that the less restrictive representation of fractional derivative according to parameters is Pochhammer's contour definition introduced in [21, 22].

Definition 2.1 Let $f(z)$ be analytic in a simply connected region $\mathcal{R}$. Let $g(z)$ be regular and univalent on $\mathcal{R}$, and let $g^{-1}(0)$ be an interior point of $\mathcal{R}$, then if α is not a negative integer, p is not an integer, and z is in $\mathcal{R} - \{g^{-1}(0)\}$, we define the fractional derivative of

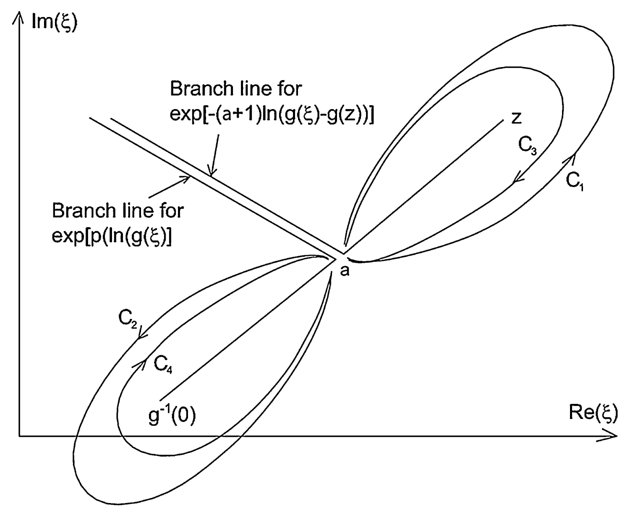

Figure 1 Pochhammer's contour.

order α of $g(z)^p f(z)$ with respect to $g(z)$ by

$$D^{\alpha}_{g(z)} g(z)^p f(z) = \frac{e^{-i\pi p}\Gamma(1+\alpha)}{4\pi \sin(\pi p)} \int_{C(z+,g^{-1}(0)+,z-,g^{-1}(0)-;F(a),F(a))} \frac{f(\xi)g(\xi)^p g'(\xi)}{(g(\xi)-g(z))^{\alpha+1}}\, d\xi. \tag{2.3}$$

For non-integers α and p, the functions $g(\xi)^p$ and $(g(\xi)-g(z))^{-\alpha-1}$ in the integrand have two branch lines which begin respectively at $\xi = z$ and $\xi = g^{-1}(0)$, and both pass through the point $\xi = a$ without crossing the Pochhammer contour $P(a) = \{C_1 \cup C_2 \cup C_3 \cup C_4\}$ at any other point as shown in Figure 1. $F(a)$ denotes the principal value of the integrand in (2.3) at the beginning and ending point of the Pochhammer contour $P(a)$ which is closed on the Riemann surface of the multiple-valued function $F(\xi)$.

Remark 2.2 In Definition 2.1, the function $f(z)$ must be analytic at $\xi = g^{-1}(0)$. However, it is interesting to note here that we could also allow $f(z)$ to have an essential singularity at $\xi = g^{-1}(0)$, and equation (2.3) would still be valid.

Remark 2.3 The Pochhammer contour never crosses the singularities at $\xi = g^{-1}(0)$ and $\xi = z$ in (2.3), then we know that the integral is analytic for all p and for all α and for z in $\mathcal{R} - \{g^{-1}(0)\}$. Indeed, the only possible singularities of $D^{\alpha}_{g(z)} g(z)^p f(z)$ are $\alpha = -1, -2, \dots$ and $p = 0, \pm 1, \pm 2, \dots$ which can directly be identified from the coefficient of the integral (2.3). However, integrating by parts N times the integral in (2.3) by two different ways, we can show that $\alpha = -1, -2, \dots$ and $p = 0, 1, 2, \dots$ are removable singularities (see [21]).

It is well known that [23, p.83, equation (2.4)]

$$D^{\alpha}_z z^p = \frac{\Gamma(1+p)}{\Gamma(1+p-\alpha)} z^{p-\alpha} \quad \bigl(\mathrm{Re}(p) > -1\bigr), \tag{2.4}$$

but adopting the Pochhammer-based representation for the fractional derivative, this last restriction becomes p not a negative integer. In view of Definition 1.6, the fractional derivative formula for the generalized Hurwitz-Lerch zeta function $\Phi_{\mu,\nu}^{(\rho,\sigma)}(z,s,a)$ with $\rho = \sigma$ [8, p.730, equation (24)] is

$$D_z^{\mu-\nu} z^{\mu-1}\Phi\left(z^\sigma,s,a\right) = \frac{\Gamma(\mu)}{\Gamma(\nu)} z^{\nu-1}\Phi_{\mu,\nu}^{(\sigma,\sigma)}\left(z^\sigma,s,a\right) \tag{2.5}$$

with $\sigma + \mu - 1$ not a negative integer.

A very interesting special case is obtained when setting $\nu = \sigma = 1$, equation (2.5) reduces to the following form:

$$\Phi_\mu^*(z,s,a) = \frac{1}{\Gamma(\mu)} D_z^{\mu-1} z^{\mu-1}\Phi(z,s,a) \tag{2.6}$$

with μ not a negative integer.

As remarked by Lin and Srivastava [8], the function $\Phi_\mu^*(z,s,a)$ is essentially a fractional derivative of the classical Hurwitz-Lerch function $\Phi(z,s,a)$. Many other interesting explicit representations for $\Phi_\mu^*(z,s,a)$ have been proven by Lin and Srivastava [8].

3 Fractional calculus theorems

In this section, we recall three important theorems related to fractional calculus that will play central roles in this work. These theorems are Taylor-like expansions in terms of different types of functions. First of all, we state the theorem obtained in 1971 by Osler [11].

Theorem 3.1 *Let $f(z)$ be an analytic function in a simply connected region $\mathcal{R}$. Let α, γ be arbitrary complex numbers and $\theta(z) = (z - z_0)q(z)$ with $q(z)$ be a regular and univalent function without zero in $\mathcal{R}$. Let a be a positive real number and $K = \{0,1,\ldots,[c],[c]$ being the largest integer not greater than $c\}$. Let b, z_0 be two points in $\mathcal{R}$ such that $b \neq z_0$, and let $\omega = \exp(2\pi i/a)$, then the following relationship*

$$\begin{aligned}&\sum_{k\in K} c^{-1}\omega^{-\gamma k} f\left(\theta^{-1}\left(\theta(z)\omega^k\right)\right)\\ &\quad= \sum_{n=-\infty}^{\infty} \frac{D_{z-b}^{cn+\gamma}[f(z)\theta'(z)[(z-z_0)/\theta(z)]^{cn+\gamma+1}]|_{z=z_0}\theta(z)^{cn+\gamma}}{\Gamma(cn+\gamma+1)}\end{aligned} \tag{3.1}$$

holds true for $|z - z_0| = |z_0|$.

In particular, if $0 < c \leq 1$ and $\theta(z) = (z - z_0)$, then $k = 0$ and the formula (3.1) reduces to

$$f(z) = c\sum_{n=-\infty}^{\infty} \frac{D_{z-b}^{cn+\gamma} f(z)|_{z=z_0}(z-z_0)^{cn+\gamma}}{\Gamma(cn+\gamma+1)}. \tag{3.2}$$

This last formula is usually called the Taylor-Riemann formula, and it has been studied in several papers [19, 24–27].

Recently, Tremblay *et al.* [13] obtained the power series of an analytic function $f(z)$ in terms of the rational expression $(\frac{z-z_1}{z-z_2})$, where z_1 and z_2 are two arbitrary points inside the region of analyticity $\mathcal{R}$ of $f(z)$. In particular, they proved the next theorem.

Theorem 3.2 (i) *Let c be real and positive, and let $\omega = e^{2\pi i/c}$.* (ii) *Let $f(z)$ be analytic in the simply connected region $\mathcal{R}$ with z_1 and z_2 being interior points of $\mathcal{R}$.* (iii) *Let the set of curves $\{C(t) \mid 0 < t \leq r\}$, $C(t) \subset \mathcal{R}$, defined by*

$$C(t) = C_1(t) \cup C_2(t) = \{z \mid |\lambda_t(z_1, z_2; z)| = |\lambda_t(z_1, z_2; (z_1 + z_2)/2)|\}, \tag{3.3}$$

where

$$\lambda_t(z_1, z_2; z) = [z - (z_1 + z_2)/2 + t(z_1 - z_2)/2][z - (z_1 + z_2)/2 - t(z_1 - z_2)/2], \tag{3.4}$$

which are lemniscates of Bernoulli type with center located at $(z_1 + z_2)/2$ and with double-loops (as seen in Figure 2); one loop $C_1(t)$ leads around the focus point $(z_1 + z_2)/2 + t(z_1 - z_2)/2$ and the other loop $C_2(t)$ encircles the focus point $(z_1 + z_2)/2 - t(z_1 - z_2)/2$, for each t such that $0 < t \leq r$. (iv) *Let $((z - z_1)(z - z_2))^\lambda = \exp\{\lambda \ln(\theta((z - z_1)(z - z_2)))\}$ denote the principal branch of that function which is continuous and inside $C(r)$, cut by the respective two branch lines $L_\pm$ defined by*

$$L_\pm = \begin{cases} \{z \mid z = (z_1 + z_2)/2 \pm t(z_1 - z_2)/2\} & \text{for } 0 \leq t \leq 1, \\ \{z \mid z = (z_1 + z_2)/2 \pm it(z_1 - z_2)/2\} & \text{for } t < 0 \end{cases} \tag{3.5}$$

such that $\ln((z-z_1)(z-z_2))$ is real where $((z-z_1)(z-z_2)) > 0$. (v) *Let $f(z)$ satisfy the conditions of Definition 2.1 for the existence of the fractional derivative of $(z - z_2)^p f(z)$ of order α for $z \in \mathcal{R} - \{L_+ \cup L_-\}$, noticed by $D^\alpha_{z-z_2}(z - z_2)^p f(z)$, where α and p are real or complex numbers.* (vi) *Let $K = \{k \mid k \in \mathbb{N}$ and $\arg(\lambda_t(z_1, z_2, (z_1 + z_2)/2)) < \arg(\lambda_t(z_1, z_2, (z_1 + z_2)/2)) + 2\pi k/a < \arg(\lambda_t(z_1, z_2, (z_1 + z_2)/2)) + 2\pi\}$. Then, for arbitrary complex numbers μ, ν, γ and for z on $C_1(1)$ defined by $\xi = \frac{z_1+z_2}{2} + \frac{z_1-z_2}{2}\sqrt{1 + e^{i\theta}}$, $-\pi < \theta < \pi$, we have*

$$\begin{aligned} &\sum_{k \in K} \frac{c^{-1}\omega^{-\gamma k} f(\phi^{-1}(\phi(z)\omega^k))(\phi^{-1}(\phi(z)\omega^k) - z_1)^\nu (\phi^{-1}(\phi(z)\omega^k) - z_2)^\mu}{(z_1 - z_2)} \\ &\quad = \sum_{n=-\infty}^{\infty} \frac{e^{i\pi c(n+1)} \sin((\mu + cn + \gamma)\pi) D^{-\nu+cn+\gamma}_{z-z_2}(z - z_2)^{\mu+cn+\gamma-1} f(z)|_{z=z_1}}{\sin((\mu - c + \gamma)\pi)\Gamma(1 - \nu + cn + \gamma)} \phi(z)^{cn+\gamma}, \end{aligned} \tag{3.6}$$

where $\phi(z) = (\frac{z-z_1}{z-z_2})$.

The case $0 < c \leq 1$ reduces to

$$\begin{aligned} &\frac{c^{-1} f(z)(z - z_1)^\nu (z - z_2)^\mu}{(z_1 - z_2)} \\ &\quad = \sum_{n=-\infty}^{\infty} \frac{e^{i\pi c(n+1)} \sin((\mu + cn + \gamma)\pi)}{\sin((\mu - c + \gamma)\pi)\Gamma(1 - \nu + cn + \gamma)} \\ &\qquad \times D^{-\nu+cn+\gamma}_{z-z_2}(z - z_2)^{\mu+cn+\gamma-1} f(z)\bigg|_{z=z_1} \left(\frac{z - z_1}{z - z_2}\right)^{cn+\gamma}. \end{aligned} \tag{3.7}$$

Finally, in 2007, Tremblay and Fugère [12] obtained the power series of an analytic function $f(z)$ in terms of an arbitrary function $(z - z_1)(z - z_2)$, where z_1 and z_2 are two arbitrary points inside the analyticity region $\mathcal{R}$ of $f(z)$. Explicitly, they found the following relationship.

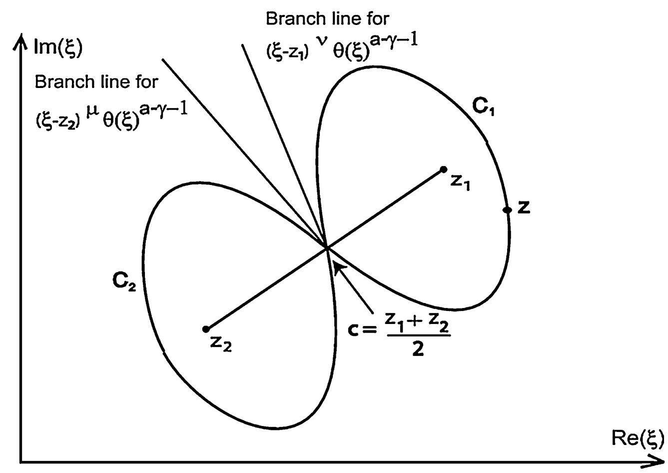

Figure 2 Multi-loops contour.

Theorem 3.3 *Assuming the assumptions of Theorem* 3.2, *the following expansion*

$$\begin{aligned}
&\sum_{k\in K} c^{-1}\omega^{-\gamma k}\biggl[f\biggl(\frac{z_1+z_2+\sqrt{\Delta_k}}{2}\biggr)\biggl(\frac{z_2-z_1+\sqrt{\Delta_k}}{2}\biggr)^{\alpha}\biggl(\frac{z_1-z_2+\sqrt{\Delta_k}}{2}\biggr)^{\beta} \\
&\qquad - e^{i\pi(\alpha-\beta)}\frac{\sin((\alpha+c-\gamma)\pi)}{\sin((\beta+c-\gamma)\pi)}f\biggl(\frac{z_1+z_2-\sqrt{\Delta_k}}{2}\biggr) \\
&\qquad \times\biggl(\frac{z_2-z_1-\sqrt{\Delta_k}}{2}\biggr)^{\alpha}\biggl(\frac{z_1-z_2-\sqrt{\Delta_k}}{2}\biggr)^{\beta}\biggr] \\
&\quad = \sum_{n=-\infty}^{\infty}\frac{\sin((\beta-cn-\gamma)\pi)}{\sin((\beta-c-\gamma)\pi)}e^{-i\pi c(n+1)}\theta(z)^{cn+\gamma} \\
&\qquad \times\frac{D_{z-z_2}^{-\alpha+cn+\gamma}[(z-z_2)^{\beta-cn-\gamma-1}(\frac{\theta(z)}{(z-z_2)(z-z_1)})^{-cn-\gamma-1}\theta'(z)f(z)]|_{z=z_1}}{\Gamma(1-\alpha+cn+\gamma)},
\end{aligned} \tag{3.8}$$

where

$$\Delta_k = (z_1-z_2)^2 + 4V\bigl(\theta(z)\omega^k\bigr), \tag{3.9}$$

$$V(z) = \sum_{r=1}^{\infty} D_z^{r-1}\bigl(q(z)^{-r}\bigr)|_{z=0}z^r/r!, \tag{3.10}$$

$$\theta(z) = (z-z_1)(z-z_2)q\bigl((z-z_1)(z-z_2)\bigr), \tag{3.11}$$

holds true.

As a special case, if we set $0 < c \leq 1$, $q(z) = 1$ ($\theta(z) = (z-z_1)(z-z_2)$) and $z_2 = 0$ in (3.8), we obtain

$$\begin{aligned}
f(z) &= cz^{-\beta}(z-z_1)^{-\alpha}\sum_{n=-\infty}^{\infty}\frac{\sin((\beta-cn-\gamma)\pi)}{\sin((\beta+c-\gamma)\pi)}e^{i\pi c(n+1)}\bigl[z(z-z_1)\bigr]^{cn+\gamma} \\
&\quad \times\frac{D_z^{-\alpha+cn+\gamma}}{\Gamma(1-\alpha+cn+\gamma)}z^{\beta-cn-\gamma-1}(z+w-z_1)f(z)\bigg|_{\substack{z=z_1\\ w=z}}.
\end{aligned} \tag{3.12}$$

4 Main expansions involving the generalized Hurwitz-Lerch zeta function $\Phi^*_\mu(z,s,a)$

In this section, we present and prove three different expansion formulas involving the generalized Hurwitz-Lerch zeta functions obtained from Theorem 3.1 to Theorem 3.3. Also, the conditions of existence are explicitly given for each expansion.

Theorem 4.1 *Assuming the assumptions of Theorem* 3.1, *the following expansion holds true for the generalized Hurwitz-Lerch zeta function* $\Phi^*_\mu(z,s,a)$:

$$\Phi^*_\mu(z,s,a) = c\sum_{n=-\infty}^{\infty}\frac{(z_0)^{-cn}(z-z_0)^{cn}}{\Gamma(cn+1)\Gamma(1-cn)}\Phi^{(1,1)}_{\mu,1-cn}(z_0,s,a)$$
$$\bigl(\mu\in\mathbb{C}; a\in\mathbb{C}\setminus\mathbb{Z}_0^-; s\in\mathbb{C}\bigr) \tag{4.1}$$

for z such that $|z-z_0| = |z_0|$, $|z|<1$ *and* $|z_0|<1$.

Proof Setting $f(z) = \Phi^*_\mu(z,s,a)$ in Theorem 3.1 with $b=\gamma=0$, $0<c\le 1$ and $\theta(z) = z-z_0$, we have

$$\Phi^*_\mu(z,s,a) = c\sum_{n=-\infty}^{\infty}\frac{D_z^{cn}\Phi^*_\mu(z,s,a)|_{z=z_0}(z-z_0)^{cn}}{\Gamma(1+cn)} \tag{4.2}$$

for $z_0\neq 0$ and z such that $|z-z_0| = |z_0|$.

Using relation (2.4), we find

$$\begin{aligned}
D_z^{cn}\Phi^*_\mu(z,s,a)|_{z_0} &= D_z^{cn}\sum_{k=0}^{\infty}\frac{(\mu)_k}{k!}\frac{z^k}{(k+a)^s}\bigg|_{z=z_0}\\
&= \sum_{k=0}^{\infty}\frac{(\mu)_k}{k!}\frac{\Gamma(1+k)z_0^{k-cn}}{\Gamma(1+k-cn)(k+a)^s}\\
&= \frac{z_0^{-cn}\Phi^{(1,1)}_{\mu,1-cn}(z_0,s,a)}{\Gamma(1-cn)}.
\end{aligned}\tag{4.3}$$

Combining (4.2) and (4.3) yields (4.1). With the help of (1.6), we see that equation (4.1) holds true if it satisfies the conditions of Theorem 3.1 as well as the following conditions: $|z|<1$, $|z_0|<1$, $\mu\in\mathbb{C}$, $s\in\mathbb{C}$, $a\in\mathbb{C}\setminus\mathbb{Z}_0^-$. Moreover, the condition $1-cn\in\mathbb{C}\setminus\mathbb{Z}_0^-$ is now unnecessary because of the presence of the term $\frac{1}{\Gamma(1-cn)}$. □

Theorem 4.2 *Assuming the hypotheses of Theorem* 3.2, *the following expansion holds true for the generalized Hurwitz-Lerch zeta function* $\Phi^*_\mu(z,s,a)$:

$$\begin{aligned}
\Phi^*_\mu(z,s,a) &= cz^{-\alpha}(z-z_1)^{-\nu}z_1^{\alpha+\nu}\sum_{n=-\infty}^{\infty}\frac{e^{i\pi c(n+1)}\sin((\alpha+cn+\gamma)\pi)\Gamma(\alpha+cn+\gamma)}{\sin((\alpha-c+\gamma)\pi)\Gamma(1-\nu+cn+\gamma)\Gamma(\alpha+\nu)}\\
&\quad\times\Phi_{\mu,\alpha+cn+\gamma;\alpha+\nu}(z_1,s,a)\left(\frac{z-z_1}{z}\right)^{cn+\gamma}\\
&\quad\bigl(\mu,\nu,\gamma,\alpha\in\mathbb{C}; a,\alpha+cn+\gamma\in\mathbb{C}\setminus\mathbb{Z}_0^-; s\in\mathbb{C}\bigr)
\end{aligned}\tag{4.4}$$

for z on $C_1(1)$ (*defined by* $z=\frac{z_1}{2}+\frac{z_1}{2}\sqrt{1+e^{i\theta}}$, $-\pi<\theta<\pi$), $z\neq 0$, $|z|<1$ *and* $|z_1|<1$.

Proof Taking $f(z) = \Phi^*_\mu(z,s,a)$ in Theorem 3.2 with $z_2 = 0$, $\mu = \alpha$ and $0 < c \le 1$ gives

$$\begin{aligned}\Phi^*_\mu(z,s,a) &= c(z-z_1)^{-\nu} z^{-\alpha} z_1 \sum_{n=-\infty}^{\infty} \frac{e^{i\pi c(n+1)} \sin((\alpha+cn+\gamma)\pi)}{\sin((\alpha-c+\gamma)\pi)\Gamma(1-\nu+cn+\gamma)} \\ &\quad \times D_z^{-\nu+cn+\gamma} z^{\alpha+cn+\gamma-1} \Phi^*_\mu(z,s,a)\Big|_{z=z_1} \left(\frac{z-z_1}{z}\right)^{cn+\gamma}. \end{aligned} \tag{4.5}$$

With the help of relations (2.4) and (1.10), we have

$$\begin{aligned} &D_z^{-\nu+cn+\gamma} z^{\alpha+cn+\gamma-1} \Phi^*_\mu(z,s,a)|_{z=z_1} \\ &\quad = \sum_{k=0}^{\infty} \frac{(\mu)_k}{k!} \frac{D_z^{-\nu+cn+\gamma} z^{k+\alpha+cn+\gamma-1}}{(a+k)^s}\bigg|_{z=z_1} \\ &\quad = z_1^{\alpha+\nu} \sum_{k=0}^{\infty} \frac{(\mu)_k}{k!} \frac{\Gamma(\alpha+cn+\gamma+k)}{\Gamma(\alpha+\nu+k)} \frac{z_1^k}{(a+k)^s} \\ &\quad = z_1^{\alpha+\nu} \frac{\Gamma(\alpha+cn+\gamma)}{\Gamma(\alpha+\nu)} \Phi_{\mu,\alpha+cn+\gamma;\alpha+\nu}(z_1,s,a). \end{aligned} \tag{4.6}$$

Combining (4.5) and (4.6) gives (4.4). Using (1.6) and (1.10), we find that equation (4.4) holds true if it satisfies the conditions of Theorem 3.2 as well as the following conditions: $|z| < 1$, $|z_1| < 1$, $\mu, \nu, \gamma, \alpha \in \mathbb{C}$, $s \in \mathbb{C}$, $a \in \mathbb{C} \setminus \mathbb{Z}_0^-$. Moreover, the condition $\alpha + \nu \in \mathbb{C} \setminus \mathbb{Z}_0^-$ is now unnecessary because of the presence of the term $\frac{1}{\Gamma(\alpha+\nu)}$, but we must add the condition $\alpha + cn + \gamma \in \mathbb{C} \setminus \mathbb{Z}_0^-$ because of the presence of the term $\Gamma(\alpha + cn + \gamma)$. □

Theorem 4.3 *Assuming the hypotheses of Theorem* 3.3, *the following expansion holds true for the generalized Hurwitz-Lerch zeta function* $\Phi^*_\mu(z,s,a)$:

$$\begin{aligned}\Phi^*_\mu(z,s,a) &= cz^{-\beta+\gamma}(z-z_1)^{-\alpha+\gamma} z_1^{\beta+\alpha-2cn-2\gamma-1} \sum_{n=-\infty}^{\infty} \frac{\sin((\beta-cn-\gamma)\pi) e^{i\pi c(n+1)}}{\sin((\beta+c-\gamma)\pi)\Gamma(1-\alpha+cn+\gamma)} \\ &\quad \times \frac{\Gamma(\beta-cn-\gamma)[z(z-z_1)]^{cn}}{\Gamma(\beta+\alpha-2cn-2\gamma)} \bigg[(z-z_1)\Phi_{\mu,\beta-cn-\gamma;\alpha+\beta-2cn-2\gamma}(z_1,s,a) \\ &\quad + \frac{(\beta-cn-\gamma)}{(\alpha+\beta-2cn-2\gamma)} z_1 \Phi_{\mu,1+\beta-cn-\gamma;1+\alpha+\beta-2cn-2\gamma}(z_1,s,a)\bigg] \\ &\quad \big(\mu,\beta,\gamma,\alpha \in \mathbb{C}; a, \beta-cn-\gamma \in \mathbb{C}\setminus\mathbb{Z}_0^-; s \in \mathbb{C}\big) \end{aligned} \tag{4.7}$$

for z on $C_1(1)$ (*defined by* $z = \frac{z_1}{2} + \frac{z_1}{2}\sqrt{1+e^{i\theta}}$, $-\pi < \theta < \pi$), $z \neq 0$, $|z| < 1$ *and* $|z_1| < 1$.

Proof Putting $f(z) = \Phi^*_\mu(z,s,a)$ in Theorem 3.3 with $z_2 = 0$, $0 < c \le 1$, $q(z) = 1$ and $\theta(z) = (z-z_1)(z-z_2)$ yields

$$\begin{aligned}\Phi^*_\mu(z,s,a) &= cz^{-\beta}(z-z_1)^{-\alpha} \sum_{n=-\infty}^{\infty} \frac{\sin((\beta-cn-\gamma)\pi)}{\sin((\beta+c-\gamma)\pi)} e^{I\pi c(n+1)} \big[z(z-z_1)\big]^{cn+\gamma} \\ &\quad \times \frac{D_z^{-\alpha+cn+\gamma}}{\Gamma(1-\alpha+cn+\gamma)} z^{\beta-cn-\gamma-1}(z+w-z_1)\Phi^*_\mu(z,s,a)\Big|_{\substack{z=z_1\\ w=z}}. \end{aligned} \tag{4.8}$$

With the help of relations (2.4) and (1.10), we have

$$\begin{aligned}
&D_z^{-\alpha+cn+\gamma} z^{\beta-cn-\gamma-1}(z+w-z_1)\Phi_\mu^*(z,s,a)\big|_{\substack{z=z_1\\ w=z}} \\
&\quad= D_z^{-\alpha+cn+\gamma} z^{\beta-cn-\gamma}\Phi_\mu^*(z,s,a)\big|_{z=z_1} + (z-z_1)D_z^{-\alpha+cn+\gamma} z^{\beta-cn-\gamma-1}\Phi_\mu^*(z,s,a)\big|_{z=z_1} \\
&\quad= z_1^{\beta+\alpha-2cn-2\gamma}\frac{\Gamma(1+\beta-cn-\gamma)}{\Gamma(1+\beta+\alpha-2cn-2\gamma)}\Phi_{\mu,1+\beta-cn-\gamma;1+\beta+\alpha-2cn-2\gamma}(z_1,s,a) \\
&\qquad+ (z-z_1)z_1^{\beta+\alpha-2cn-2\gamma-1}\frac{\Gamma(\beta-cn-\gamma)}{\Gamma(\beta+\alpha-2cn-2\gamma)}\Phi_{\mu,\beta-cn-\gamma;\beta+\alpha-2cn-2\gamma}(z_1,s,a).
\end{aligned} \tag{4.9}$$

Combining (4.8) and (4.9) gives (4.7). Using (1.6) and (1.10), we find that equation (4.7) holds true if it satisfies the conditions of Theorem 3.3 as well as the following conditions: $|z| < 1$, $|z_1| < 1$, $\mu,\beta,\gamma,\alpha \in \mathbb{C}$, $s \in \mathbb{C}$, $a \in \mathbb{C}\setminus\mathbb{Z}_0^-$. Moreover, the conditions $\alpha+\beta-2cn-2\gamma \in \mathbb{C}\setminus\mathbb{Z}_0^-$ and $1+\alpha+\beta-2cn-2\gamma \in \mathbb{C}\setminus\mathbb{Z}_0^-$ are now unnecessary because of the presence of the terms $\frac{1}{\Gamma(\alpha+\beta-2cn-2\gamma)}$ and $\frac{1}{\Gamma(1+\alpha+\beta-2cn-2\gamma)}$, but we must add the condition $\beta-cn-\gamma \in \mathbb{C}\setminus\mathbb{Z}_0^-$ because of the presence of the term $\Gamma(\beta-cn-\gamma)$. □

5 Special cases

This section is devoted to special cases of Theorem 4.1 to Theorem 4.3. We first recall definitions of Apostol-Bernoulli and Apostol-Euler polynomials and their connections to the generalized Hurwitz-Lerch zeta function $\Phi_\mu^*(z,s,a)$. Next, we give some expansion formulas involving these polynomials.

Definition 5.1 The generalized Apostol-Bernoulli polynomials $\mathcal{B}_n^{(\alpha)}(z;\lambda)$ are defined, for $\lambda, z \in \mathbb{C}$, by the following generating function [28, 29]:

$$\left(\frac{t}{\lambda e^t-1}\right)^\alpha e^{zt} = \sum_{n=0}^\infty \mathcal{B}_n^{(\alpha)}(z;\lambda)\frac{t^n}{n!}, \qquad \left|t+\log(\lambda)\right| < 2\pi$$
$$(\alpha \in \mathbb{C},\ \text{if } \lambda = 1; \alpha \in \mathbb{N}, n \geq \alpha, \text{if } \lambda \neq 1). \tag{5.1}$$

Definition 5.2 The generalized Apostol-Euler polynomials $\mathcal{E}_n^{(\alpha)}(z;\lambda)$ are defined, for λ, α and $z \in \mathbb{C}$, by the following generating function [30]:

$$\left(\frac{2}{\lambda e^t+1}\right)^\alpha e^{zt} = \sum_{n=0}^\infty \mathcal{E}_n^{(\alpha)}(z;\lambda)\frac{t^n}{n!}, \qquad \left|t+\log(\lambda)\right| < \pi. \tag{5.2}$$

Recently, Bayad and Chikhi [1] established the following relationship between the Apostol-Euler polynomials $\mathcal{E}_n^{(\alpha)}(z;\lambda)$ and the generalized Hurwitz-Lerch zeta function $\Phi_\mu^*(z,s,a)$.

Theorem 5.3 *Let λ be a complex number such that $|\lambda| \leq 1$ and $\lambda \neq -1$. Let μ and z be two complex numbers such that* $\operatorname{Re}(\mu) > 0$ *and* $\operatorname{Re}(z) > 0$, *then for all non-negative integers n, we have*

$$\mathcal{E}_n^{(\mu)}(z;\lambda) = 2^\mu \Phi_\mu^*(-\lambda,-n,z). \tag{5.3}$$

From the generating function of Apostol-Euler polynomials (5.2), we can find, after simple calculations, the following relationship between the Apostol-Euler polynomials and the Apostol-Bernoulli polynomials:

$$\mathcal{B}_n^{(l)}(z;\lambda) = l!\binom{n}{l}\left(\frac{-1}{2}\right)^l \mathcal{E}_{n-l}^{(l)}(z;-\lambda) \quad (l \in \mathbb{N}_0, n \geq l). \tag{5.4}$$

Thus, combining (5.4) and Theorem 5.3, we obtain the next connection between the Apostol-Bernoulli polynomials $\mathcal{B}_n^{(\alpha)}(z;\lambda)$ and the generalized Hurwitz-Lerch zeta function $\Phi_\mu^*(z,s,a)$

$$\mathcal{B}_n^{(l)}(z;\lambda) = l!\binom{n}{l}(-1)^l \Phi_l^*(\lambda, l-n, z) \quad \bigl(|\lambda| \leq 1; \lambda \neq 1; \operatorname{Re}(z) > 0; l \in \mathbb{N}_0; n \geq l\bigr). \tag{5.5}$$

Now let us shift our focus to some special cases of Theorems 4.1, 4.2 and Theorem 4.3 given in the forms of corollaries.

Corollary 5.4 *Assuming the assumptions of Theorem* 4.1, *the following expansion holds true for the Apostol-Bernoulli polynomials* $\mathcal{B}_n^{(l)}(z;\lambda)$:

$$\begin{aligned}\mathcal{B}_n^{(l)}(z;\lambda) &= cl!\binom{n}{l}(-1)^l \sum_{k=-\infty}^{\infty} \frac{(z_0)^{-ck}(\lambda - z_0)^{ck}}{\Gamma(ck+1)\Gamma(1-ck)} \Phi_{l,1-ck}^{(1,1)}(z_0, l-n, z) \\ &\bigl(0 < c \leq 1; l \in \mathbb{N}_0; \operatorname{Re}(z) > 0; n \in \mathbb{N}; n \geq l\bigr)\end{aligned} \tag{5.6}$$

for z such that $|\lambda - z_0| = |z_0|$, $|\lambda| < 1$ *and* $|z_0| < 1$.

Proof Making the substitutions $\mu = l$ ($l \in \mathbb{N}_0$), $s = l - n$ ($n \in \mathbb{N}$), $a = z$ ($\operatorname{Re}(z) > 0$), $z = \lambda$ ($\lambda \in \mathbb{C}$) in Theorem 4.1 and with the use of (5.5), the result follows. □

Corollary 5.5 *Assuming the assumptions of Theorem* 4.2, *the following expansion holds true for the Apostol-Bernoulli polynomials* $\mathcal{B}_n^{(l)}(z;\lambda)$:

$$\begin{aligned}\mathcal{B}_n^{(l)}(z;\lambda) &= cl!\binom{n}{l}(-1)^l \lambda^{-\alpha}(\lambda - z_1)^{-\nu} z_1^{\alpha+\nu} \sum_{k=-\infty}^{\infty} \frac{e^{i\pi c(k+1)} \sin((\alpha + ck + \gamma)\pi)}{\sin((\alpha - c + \gamma)\pi)\Gamma(\alpha+\nu)} \\ &\quad \times \frac{\Gamma(\alpha + ck + \gamma)}{\Gamma(1 - \nu + ck + \gamma)} \Phi_{l,\alpha+ck+\gamma;\alpha+\nu}(z_1, l-n, z)\left(\frac{\lambda - z_1}{\lambda}\right)^{ck+\gamma} \\ &\quad \bigl(0 < c \leq 1; l \in \mathbb{N}_0; \nu, \gamma, \alpha \in \mathbb{C}; \operatorname{Re}(z) > 0; \\ &\quad \alpha + cn + \gamma \in \mathbb{C} \setminus \mathbb{Z}_0^-; n \in \mathbb{N}; n \geq l\bigr)\end{aligned} \tag{5.7}$$

for λ *on* $C_1(1)$ (*defined by* $\lambda = \frac{z_1}{2} + \frac{z_1}{2}\sqrt{1 + e^{i\theta}}$, $-\pi < \theta < \pi$), $\lambda \neq 0$, $|\lambda| < 1$ *and* $|z_1| < 1$.

Proof Setting $\mu = l$ ($l \in \mathbb{N}_0$), $s = l - n$ ($n \in \mathbb{N}$), $a = z$ ($\operatorname{Re}(z) > 0$), $z = \lambda$ ($\lambda \in \mathbb{C}$) in Theorem 4.2 and using (5.5), the result follows. □

Corollary 5.6 *Assuming the assumptions of Theorem* 4.3, *the following expansion holds true for the Apostol-Euler polynomials* $\mathcal{E}_n^{(\mu)}(z;\lambda)$:

$$\begin{aligned}\mathcal{E}_n^{(\mu)}(z;\lambda) &= c2^{\mu}(-\lambda)^{-\beta+\gamma}(-\lambda-z_1)^{-\alpha+\gamma}z_1^{\beta+\alpha-2\gamma-1} \\ &\quad\times\sum_{k=-\infty}^{\infty}\frac{\sin((\beta-ck-\gamma)\pi)}{\sin((\beta+c-\gamma)\pi)}\frac{\Gamma(\beta-ck-\gamma)\mathrm{e}^{i\pi c(k+1)}[\lambda(\lambda+z_1)]^{ck}}{\Gamma(\beta+\alpha-2ck-2\gamma)\Gamma(1-\alpha+ck+\gamma)} \\ &\quad\times z_1^{-2ck}\Biggl[-(\lambda+z_1)\Phi_{\mu,\beta-ck-\gamma;\alpha+\beta-2ck-2\gamma}(z_1,-n,z) \\ &\quad+\frac{(\beta-ck-\gamma)}{(\alpha+\beta-2ck-2\gamma)}z_1\Phi_{\mu,1+\beta-ck-\gamma;1+\alpha+\beta-2ck-2\gamma}(z_1,-n,z)\Biggr] \\ &\quad\bigl(\operatorname{Re}(\mu)>0;\beta,\gamma,\alpha\in\mathbb{C};\operatorname{Re}(a)>0,\beta-cn-\gamma\in\mathbb{C}\setminus\mathbb{Z}_0^-;n\in\mathbb{N}\bigr)\end{aligned}\tag{5.8}$$

for $-\lambda$ *on* $C_1(1)$ (*defined by* $\lambda=\frac{-z_1}{2}-\frac{z_1}{2}\sqrt{1+\mathrm{e}^{i\theta}}$, $-\pi<\theta<\pi$), $\lambda\neq 0$, $|\lambda|<1$ *and* $|z_1|<1$.

Proof Replacing z by $-\lambda$ ($\lambda\in\mathbb{C}$), s by $-n$ ($n\in\mathbb{N}$), $a=z$ ($\operatorname{Re}(z)>0$) in Theorem 4.3 and appealing to (5.3) gives the result. □

Competing interests
The author declares that he has no competing interests.

Acknowledgements
The author wishes to thank referees for valuable suggestions and comments.

Received: 22 August 2013 Accepted: 19 November 2013 Published: 12 Dec 2013

References

1. Bayad, A, Chikhi, J: Reduction and duality of the generalized Hurwitz-Lerch zetas. Fixed Point Theory Appl. **2013**, 82 (2013)
2. Garg, M, Jain, K, Srivastava, HM: Some relationships between the generalized Apostol-Bernoulli polynomials and Hurwitz-Lerch zeta functions. Integral Transforms Spec. Funct. **17**(11), 803-815 (2006)
3. Srivastava, HM: Some formulas for the Bernoulli and Euler polynomials at rational arguments. Math. Proc. Camb. Philos. Soc. **129**, 77-84 (2000)
4. Srivastava, HM, Chaudhry, MA, Qadir, A, Tassaddiq, A: Some extensions of the Fermi-Dirac and Bose-Einstein functions with applications to the family of the zeta and related functions. Russ. J. Math. Phys. **18**(1), 107-121 (2011)
5. Raducanu, D, Srivastava, HM: A new class of analytic functions defined by means of a convolution operator involving the Hurwitz-Lerch zeta function. Integral Transforms Spec. Funct. **18**, 933-943 (2007)
6. Gupta, PL, Gupta, RC, Ong, S-H, Srivastava, HM: A class of Hurwitz-Lerch zeta distributions and their applications in reliability. Appl. Math. Comput. **196**, 521-531 (2008)
7. Srivastava, HM, Choi, J: Series Associated with Zeta and Related Functions. Kluwer Academic, Dordrecht (2001)
8. Lin, S-D, Srivastava, HM: Some families of the Hurwitz-Lerch zeta functions and associated fractional derivative and other integral representations. Appl. Math. Comput. **154**, 725-733 (2004)
9. Goyal, SP, Laddha, RK: On the generalized Riemann zeta function and the generalized Lambert transform. Ganita Sandesh **11**, 99-108 (1997)
10. Garg, M, Jain, K, Kalla, SL: A further study of general Hurwitz-Lerch zeta function. Algebras Groups Geom. **25**, 311-319 (2008)
11. Osler, TJ: Taylor's series generalized for fractional derivatives and applications. SIAM J. Math. Anal. **2**, 37-48 (1971)
12. Tremblay, R, Fugère, BJ: The use of fractional derivatives to expand analytical functions in terms of quadratic functions with applications to special functions. Appl. Math. Comput. **187**, 507-529 (2007)
13. Tremblay, R, Gaboury, S, Fugère, B-J: Taylor-like expansion in terms of a rational function obtained by means of fractional derivatives. Integral Transforms Spec. Funct. **24**(1), 50-64 (2013)
14. Erdelyi, A: An integral equation involving Legendre polynomials. SIAM J. Appl. Math. **12**, 15-30 (1964)
15. Liouville, J: Mémoire sur le calcul des différentielles à indices quelconques. J. Éc. Polytech. **13**, 71-162 (1832)
16. Riesz, M: L'intégrale de Riemann-Liouville et le problème de Cauchy. Acta Math. **81**, 1-233 (1949)
17. Osler, TJ: Fractional derivatives of a composite function. SIAM J. Math. Anal. **1**, 288-293 (1970)
18. Osler, TJ: Leibniz rule for the fractional derivatives and an application to infinite series. SIAM J. Appl. Math. **18**, 658-674 (1970)
19. Osler, TJ: Leibniz rule, the chain rule and Taylor's theorem for fractional derivatives. PhD thesis, New York University (1970)
20. Osler, TJ: Fractional derivatives Leibniz rule. Am. Math. Mon. **78**, 645-649 (1971)

21. Lavoie, J-L, Osler, TJ, Tremblay, R: Fundamental properties of fractional derivatives via Pochhammer integrals. In: Fractional Calculus and Its Applications. Lecture Notes in Mathematics. Springer, Berlin (1976)
22. Tremblay, R: Une contribution à la théorie de la dérivée fractionnaire. PhD thesis, Laval University, Canada (1974)
23. Miller, KS, Ross, B: An Introduction of the Fractional Calculus and Fractional Differential Equations. Wiley, New York (1993)
24. Hardy, GH: Riemann's forms of Taylor's series. J. Lond. Math. Soc. **20**, 48-57 (1945)
25. Heaviside, O: Electromagnetic Theory, vol. 2. Dover, New York (1950)
26. Riemann, B: Versuch einer allgemeinen auffassung der integration und differentiation. The Collection Works of Bernhard Riemann. Dover, New York (1953)
27. Watanabe, Y: Zum riemannschen binomischen lehrsatz. Proc. Phys. Math. Soc. Jpn. **14**, 22-35 (1932)
28. Luo, Q-M, Srivastava, HM: Some generalizations of the Apostol-Bernoulli and Apostol-Euler polynomials. J. Math. Anal. Appl. **308**(1), 290-302 (2005)
29. Srivastava, HM, Kurt, B, Simsek, Y: Some families of Genocchi type polynomials and their interpolation functions. Integral Transforms Spec. Funct. **23**(12), 939-940 (2012)
30. Luo, Q-M: Apostol-Euler polynomials of higher order and Gaussian hypergeometric functions. Taiwan. J. Math. **10**(4), 917-925 (2006)

10.1186/1687-1847-2013-361
Cite this article as: Gaboury: Some relations involving generalized Hurwitz-Lerch zeta function obtained by means of fractional derivatives with applications to Apostol-type polynomials. *Advances in Difference Equations* 2013, 2013:361