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Global bifurcation results for general Laplacian problems

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Abstract

In this article, we consider the global bifurcation result and existence of solutions for the following general Laplacian problem,

$$\begin{cases} -(\phi(u'(t)))' = \lambda\psi(u(t)) + f(t,u,\lambda), & t \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \qquad (P)$$

where $f: [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and $\varphi, \psi: \mathbb{R} \to \mathbb{R}$ are odd increasing homeomorphisms of $\mathbb{R}$, when φ, ψ satisfy the asymptotic homogeneity conditions.

1 Introduction

In this article, we consider the following general Laplacian problem,

$$\begin{cases} -(\phi(u'(t)))' = \lambda\psi(u(t)) + f(t,u,\lambda), & t \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \qquad (P)$$

where $f: [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous with $f(t,u,0) = 0$ and $\varphi, \psi: \mathbb{R} \to \mathbb{R}$ are odd increasing homeomorphisms of $\mathbb{R}$ with $\varphi(0) = \psi(0) = 0$. We consider the following conditions;

(Φ_1) $\lim_{t\to 0} \frac{\phi(\sigma t)}{\psi(t)} = \sigma^{p-1}$, for all $\sigma \in \mathbb{R}_+$, for some $p > 1$.

(Φ_2) $\lim_{t\to 0} \frac{\phi(\sigma t)}{\psi(t)} = \sigma^{q-1}$, for all $\sigma \in \mathbb{R}_+$, for some $q > 1$.

(F_1) $f(t,u,\lambda) = o(|\psi(u)|)$ near zero, uniformly for t and λ in bounded intervals.

(F_2) $f(t,u,\lambda) = o(|\psi(u)|)$ near infinity, uniformly for t and λ in bounded intervals.

(F_3) $uf(t,u,\lambda) \ge 0$.

We note that $\varphi_r(t) = |t|^{r-2}t$, $r > 1$ are special cases of φ and ψ. We first prove following global bifurcation result.

Theorem 1.1. *Assume (Φ_1), (Φ_2), (F_1), (F_2) and (F_3). Then for any $j \in \mathbb{N}$, there exists a connected component $\mathcal{C}_j$ of the set of nontrivial solutions for (P) connecting $(0, \lambda_j(p))$ to $(\infty, \lambda_j(q))$ such that $(u, \lambda) \in \mathcal{C}_j$ implies that u has exactly j - 1 simple zeros in (0, 1), where $\lambda_j(r)$ is the j-th eigenvalue of $(\varphi_r(u'(t)))' + \lambda\varphi_r(u(t)) = 0$ and $u(0) = u(1) = 0$.*

By the aid of this theorem, we can prove the following existence result of solutions.

Theorem 1.2. *Consider problem*

$$\begin{cases} -(\phi(u'(t)))' = g(t,u), & t \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \qquad (A)$$

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where $g : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is continuous and* φ *is odd increasing homeomorphism of* $\mathbb{R}$, *which satisfy* (Φ_1) *and* (Φ_2) *with* $\varphi = \psi$. *Also* $ug(t, u) \geq 0$ *and there exist positive integers k, n with* $k \leq n$ *such that* $\mu = \lim_{s \to 0} \frac{g(t,s)}{\phi(s)} < \lambda_k(p) \leq \lambda_n(q) < \lim_{|s| \to \infty} \frac{g(t,s)}{\phi(s)} = \nu$ *uniformly in* $t \in [0,1]$. *Then for each integer j with* $k \leq j \leq n$, *problem* (*A*) *has a solution with exactly j - 1 simple zeros in* (0, 1). *Thus,* (*A*) *possesses at least n-k* + 1 *nontrivial solutions.*

In [1], the authors studied the existence of solutions and global bifurcation results for

$$\begin{cases} -(t^{N-1}\phi(u'(t)))' = t^{N-1}\lambda\psi(u(t)) + t^{N-1}f(t,\ u,\ \lambda),\ t \in (0,R), \\ u'(0) = u(R) = 0. \end{cases}$$

The main purpose of this article is to derive the same result for $N = 1$ case with Dirichlet boundary condition which was not considered in [1].

For *p*-Laplacian problems, i.e., $\varphi = \psi = \varphi_{\mathrm{p}}$, many authors have studied for the existence and multiplicity of nontrivial solutions [2-6]. In [2,5,6], the authors used fixed point theory or topological degree argument. Also global bifurcation theory was mainly employed in [3,4]. Moreover, there are some studies related to general Laplacian problems [3,7,8], but most of them are about $\varphi = \psi$ case. In [3], the authors proved some results under several kinds of boundary conditions and in [7], the authors considered a system of general Laplacian problems. In [8], the author studied global continuation result for the singular problem. In this paper, we mainly study the global bifurcation phenomenon for general Laplacian problem (*P*) and prove the existence and multiplicity result for (*A*).

This article is organized as follows: In Section 2, we set up the equivalent integral operator of (*P*) and compute the degree of this operator. In Section 3, we verify the existence of global bifurcation having bifurcation points at zero and infinity simultaneously. In Section 4, we introduce an existence result as an application of the previous result and give some examples.

2 Degree estimate

Let us consider problem (*P*) with $f \equiv 0$, i.e.,

$$\begin{cases} -(\phi(u'(t)))' = \lambda\psi(u(t)), \quad t \in (0,1), \\ u(0) = u(1) = 0. \end{cases} \qquad (\overline{P})$$

We introduce the equivalent integral operator of problem $(\overline{P})$. For this, we consider the following problem

$$\begin{cases} (\phi(u'(t)))' = h(t), \quad \text{a.e.,} \quad t \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \qquad (AP)$$

where $h \in L^1(0, 1)$. Here, a function *u* is called a solution of (*AP*) if $u \in C_0^1[0,1]$ with $\varphi(u')$ absolutely continuous which satisfies (*AP*). We note that (*AP*) is equivalently written as

$$u(t) = G(h)(t) = \int_0^t \phi^{-1}\left(a(h) + \int_0^s h(\xi)d\xi\right)ds,$$

where $a : L^1(0, 1) \to \mathbb{R}$ is a continuous function which sends bounded sets of L^1 into bounded sets of $\mathbb{R}$ and satisfying

$$\int_0^1 \phi^{-1}\left(a(h) + \int_0^s h(\xi)d\xi\right) ds = 0. \tag{1}$$

It is known that $G : L^1(0,1) \to C_0^1[0,1]$ is continuous and maps equi-integrable sets of $L^1(0, 1)$ into relatively compact sets of $C_0^1[0, 1]$. One may refer Manásevich-Mawhin [4,3] and Garcia-Huidobro-Manásevich-Ward [7] for more details. If we define the operator $T_{\phi\psi}^{\lambda} : C_0^1[0,1] \to C_0^1[0,1]$ by

$$T_{\phi\psi}^{\lambda}(u)(t) = G(-\lambda\psi(u))(t) = \int_0^t \phi^{-1}\left(a(-\lambda\psi(u)) + \int_0^s -\lambda\psi(u(\xi))d\xi\right)ds, \tag{2}$$

then $(\overline{P})$ is equivalently written as $u = T_{\phi\psi}^{\lambda}(u)$. Now let us consider p-Laplacian problem

$$\begin{cases} -(\phi_p(u'(t)))' = \lambda\phi_p(u(t)), & t \in (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{E_p}$$

By the similar argument, we can also get the equivalent integral operator of problem (E_p), which is known by Garcia-Huidobro-Manásevich-Schmitt [1]. Let us define $T_p^{\lambda} : C_0^1[0,1] \to C_0^1[0,1]$ by

$$T_p^{\lambda}(u)(t) = \int_0^t \phi_p^{-1}\left(a_p\left(-\lambda\phi_p(u)\right)\right) + \int_0^s -\lambda\phi_p(u(\xi))d\xi)ds, \tag{3}$$

where $a_p : L^1(0, 1) \to \mathbb{R}$ is a continuous function which sends bounded sets of L^1 into bounded sets of $\mathbb{R}$ and satisfying

$$\int_0^1 \phi_p^{-1}\left(a_p(h) + \int_0^s h(\xi)d\xi\right) ds = 0, \text{ for all } h \in L^1(0,1).$$

Note that a_p has homogineity property, i.e., $a_p(\lambda t) = \lambda a_p(t)$. Problem (E_p) can be equiva-lently written as $u = T_p^{\lambda}(u)$. Obviously, $T_{\phi\psi}^{\lambda}$ and T_p^{λ} are completely continuous.

The main purpose of this section is to compute the Leray-Schauder degree of $I - T_{\phi\psi}^{\lambda}$. Following Lemma is for the property of φ and ψ with asymptotic homogeneity condition (Φ_1) and (Φ_2), which is very useful for our analysis. The proof can be modified from Proposition 4.1 in [9].

Lemma 2.1. *Assume that φ, ψ are odd increasing homeomorphisms of $\mathbb{R}$ which satisfy* (Φ_1) *and* (Φ_2). *Then, we have*

$$(i) \lim_{t\to 0} \frac{\phi^{-1}(\sigma t)}{\psi^{-1}(t)} = \phi_p^{-1}(\sigma), \text{ for all } \sigma \in \mathbb{R}_+, \text{ for some } p > 1, \tag{4}$$

and

$$(ii) \lim_{|t|\to\infty} \frac{\phi^{-1}(\sigma t)}{\psi^{-1}(t)} = \phi_p^{-1}(\sigma), \text{ for all } \sigma \in \mathbb{R}_+, \text{ for some } q > 1. \tag{5}$$

To compute the degree, we will make use of the following well-known fact [10].

Lemma 2.2. *If* λ *is not an eigenvalue of* (E_p), $p > 1$ and $r > 0$, *then*

$$\deg\left(I - T_p^{\lambda}, B(0,r), 0\right) = \begin{cases} 1 & if \lambda < \lambda_1(p), \\ (-1)^k & if \lambda \in (\lambda_k(p), \lambda_{k+1}(p)). \end{cases} \quad (6)$$

Now, let us compute $\deg\left(I - T_{\phi\psi}^{\lambda}, B(0,r), 0\right)$ when λ is not an eigenvalue of (E_p).

Theorem 2.3. *Assume that* φ, ψ *are odd increasing homeomorphisms of* $\mathbb{R}$ *which satisfy* (Φ_1) *and* (Φ_2)*. then,*

(i) *The Leray-Schauder degree of* $I - T_{\phi\psi}^{\lambda}$ *is defined for* $B(0, \varepsilon)$*, for all sufficiently small* ε*.*

Moreover, we have

$$\deg\left(I - T_{\phi\psi}^{\lambda}, B(0,\varepsilon), 0\right) = \begin{cases} 1 & if \lambda < \lambda_1(p), \\ (-1)^m & if \lambda \in (\lambda_m(p), \lambda_{m+1}(p)). \end{cases} \quad (7)$$

(*ii*) *The Leray-Schauder degree of* $I - T_{\phi\psi}^{\lambda}$ *is defined for* $B(0, M)$*, for all sufficiently large* M*, and*

$$\deg\left(I - T_{\phi\psi}^{\lambda}, B(0,M), 0\right) = \begin{cases} 1 & if \lambda < \lambda_1(q), \\ (-1)^l & if \lambda \in (\lambda_l(q), \lambda_{l+1}(q)). \end{cases} \quad (8)$$

Proof: We give the proof for assertion (*i*). Proof for the latter case is similar. Define $T^{\lambda} : C_0^1[0,1] \times [0,1] \to C_0^1[0,1]$ by $T^{\lambda}(u,\tau) = \tau T_{\phi\psi}^{\lambda}(u) + (1-\tau) T_p^{\lambda}(u)$. We claim that the Leray-Schauder degree for $I - T^{\lambda}(\cdot, \tau)$ is defined for $B(0, \varepsilon)$ in $C_0^1[0,1]$ for all small ε. Indeed, suppose there exist sequences $\{u_n\}$, $\{\tau_n\}$ and $\{\varepsilon_n\}$ with $\varepsilon_n \to 0$ and $\|u_n\|_0 = \varepsilon_n$ such that $u_n = T^{\lambda}(u_n, \tau_n)$, i.e.,

$$\begin{aligned} u_n(t) = {} & \tau_n \int_0^t \phi^{-1}\left(a(-\lambda\psi(u_n)) + \int_0^s -\lambda\psi(u_n(\xi))d\xi\right)ds \\ & + (1-\tau_n)\int_0^t \phi_p^{-1}\left(a_p\left(-\lambda\phi_p(u_n)\right) + \int_0^s -\lambda\phi_p(u_n(\xi))d\xi\right)ds. \end{aligned}$$

Setting $v_n(t) = \frac{u_n(t)}{\varepsilon_n}$, we have $\|v_n\|_0 = 1$,

$$\begin{aligned} v_n(t) = {} & \frac{\tau_n}{\varepsilon_n}\int_0^t \phi^{-1}\left(a(-\lambda\psi(u_n)) + \int_0^s -\lambda\psi(u_n(\xi))d\xi\right)ds \\ & + (1-\tau_n)\int_0^t \phi_p^{-1}\left(a_p\left(-\lambda\phi_p(v_n)\right) + \int_0^s -\lambda\phi_p\left(v_n(\xi)\right)d\xi\right)ds, \end{aligned}$$

and

$$\begin{aligned} v'_n(t) = {} & \frac{\tau_n}{\varepsilon_n}\phi^{-1}\left(a(-\lambda\psi(u_n)) + \int_0^t -\lambda\psi(u_n(\xi))d\xi\right) \\ & + (1-\tau_n)\phi_p^{-1}\left(a_p\left(-\lambda\phi_p(v_n)\right) + \int_0^s -\lambda\phi_p\left(v_n(\xi)\right)d\xi\right). \end{aligned}$$

Now, we show that $\{v'_n\}$ is uniformly bounded. Since $\|v_n\|_0 = 1$, $\int_0^t -\lambda\phi_p(v_n(\xi))d\xi \le \lambda$. Moreover, there exists C_1 such that $a_p(-\lambda\varphi_p(v_n)) \le C_1$. These results imply the uniform boundedness of $\phi_p^{-1}\left(a_p(-\lambda\phi_p(v_n)) + \int_0^t -\lambda\phi_p(v_n(\xi))d\xi\right)$.

Let

$$q_n(t) = \frac{1}{\varepsilon_n}\phi^{-1}\left(a(-\lambda\psi(u_n) + \int_0^t -\lambda\psi(u_n(\xi))d\xi\right),$$

and

$$d_n(t) = \int_0^t \lambda\psi(u_n(\xi))d\xi.$$

Then $d_n \in C[0, 1]$, and

$$||d_n||_0 = \max_{t\in[0,1]}|\int_0^t \lambda\psi(u_n(\xi))d\xi| \le \int_0^1 \lambda\psi(||u_n||_0)d\xi \le \lambda\psi(\varepsilon_n).$$

Since $\int_0^1 \phi^{-1}\left(a(-\lambda\psi(u_n)) - d_n(s)\right) ds = 0$, we have

$$|a(-\lambda\psi(u_n))| \le \lambda\psi(\varepsilon_n).$$

Otherwise, $\int_0^1 \phi^{-1}\left(a(-\lambda\psi(u_n)) - d_n(s)\right) ds < 0$ (or > 0). Now, we show that $\frac{1}{\varepsilon_n}\phi^{-1}(2\lambda\psi(\varepsilon_n))$ is bounded. Indeed, suppose that it is not true, i.e., $\frac{1}{\varepsilon_n}\phi^{-1}(2\lambda\psi(\varepsilon_n)) \to \infty$ as $n \to \infty$. Then, for arbitrary $A > 0$, there exists $N_0 \in \mathbb{N}$ such that $\frac{1}{\varepsilon_n}\phi^{-1}(2\lambda\psi(\varepsilon_n)) \ge A$, for all $n > N_0$. This implies that $2\lambda \ge \frac{\phi(A\varepsilon_n)}{\psi(\varepsilon_n)}$ for all $n > N_0$. However, $\frac{\phi(A\varepsilon_n)}{\psi(\varepsilon_n)} \to \phi_p(A)$ as $n \to \infty$. This is a contradiction. Thus by the above inequality, we get

$$\frac{1}{\varepsilon_n}\phi^{-1}\left(a(-\lambda\psi(u_n)) + \int_0^t -\lambda\psi(u_n(\xi))d\xi\right) \le \frac{1}{\varepsilon_n}\phi^{-1}\left(2\lambda\psi(\varepsilon_n)\right) \le C_2,$$

for some $C_2 > 0$. Therefore, $\{v_n'\}$ is uniformly bounded. By the Arzela-Ascoli Theorem, $\{v_n\}$ has a uniformly convergent subsequence in $C[0,1]$ relabeled as the original sequence so let $\lim_{n\to\infty}$, $v_n = v$. Now, we claim that $q_n(t) \to q(t)$, where

$$q(t) = \phi_p^{-1}\left(a_p(-\lambda\phi_p(v)) + \int_0^t -\lambda\phi_p(v(\xi))d\xi\right).$$

Clearly,

$$\begin{aligned} q_n(t) &= \frac{1}{\varepsilon_n}\phi^{-1}\left(a(-\lambda\psi(u_n)) + \int_0^t -\lambda\psi(u_n(\xi))d\xi\right) \\ &= \frac{\phi^{-1}\left(\left(\frac{a(-\lambda\psi(u_n))}{\phi(\varepsilon_n)} + \int_0^t -\lambda\frac{\psi(u_n(\xi)\varepsilon_n)}{\phi(\varepsilon_n)}d\xi\right)\phi(\varepsilon_n)\right)}{\psi^{-1}(\phi(\varepsilon_n))}\frac{\psi^{-1}(\phi(\varepsilon_n))}{\phi^{-1}(\phi(\varepsilon_n))}. \end{aligned}$$

Since $|a(-\lambda\psi(u_n))| \le \lambda\psi(\varepsilon_n)$, $\frac{a(-\lambda\psi(u_n))}{\phi(\varepsilon_n)}$ has a convergent subsequence. Without loss of generality, we say that the sequence $\{\frac{a(-\lambda\psi(u_n))}{\phi(\varepsilon_n)}\}$ converges to d. Also by the facts that $\frac{\psi(v_n(\xi)\varepsilon_n)}{\phi(\varepsilon_n)} \to \phi_p(v(\xi))$ as $n \to \infty$, $\varphi(\varepsilon_n) \to 0$ and (i) of Lemma 2.1, we obtain

$$q_n(t) \to \phi_p\left(d + \int_0^t -\lambda\phi_p(v(\xi))d\xi\right).$$

Since $\int_0^1 \phi^{-1}\left(a(-\lambda\psi(u_n)) + \int_0^s -\lambda\psi(u_n(\xi))d\xi\right) ds = 0$,

$$\frac{1}{\varepsilon_n}\int_0^1 \phi^{-1}\left(a(-\lambda\psi(u_n)) + \int_0^s -\lambda\psi(u_n(\xi))d\xi\right) ds = 0.$$

Thus $\int_0^1 \phi_p^{-1}\left(d + \int_0^s -\lambda\phi_p(v(\xi))d\xi\right) ds = 0$ and by the definition of a_p, $d = a_p(-\lambda\varphi_p(v))$. Therefore, we can easily see that

$$v(t) = \int_0^t \phi_p^{-1}\left(a_p(-\lambda\phi_p(v)) + \int_0^s -\lambda\phi_p(v(\xi))d\xi\right) ds,$$

and

$$\int_0^1 \phi_p^{-1}\left(a_p(-\lambda\phi_p(v)) + \int_0^s -\lambda\phi_p(v(\xi))d\xi\right) ds = 0.$$

Consequently, v is a solution of (E_p). Since $\lambda \notin \{\lambda_n(p)\}$, $v \equiv 0$ and this fact yields a contradiction. By the properties of the Leray-Schauder degree, we get

$$\begin{aligned} &\deg\left(I - T_p^{\lambda}, B(0,\varepsilon), 0\right) = \deg\left(I - T^{\lambda}(\cdot, 0), B(0,\varepsilon), 0\right) \\ &= \deg\left(I - T^{\lambda}(\cdot, 1), B(0,\varepsilon), 0\right) = \deg\left(I - T_{\phi\psi}^{\lambda}, B(0,\varepsilon), 0\right) \end{aligned}$$

and the proof is completed by Lemma 2.2.

3 Existence of unbounded continuum

We begin with this section recalling what we mean by bifurcation at zero and at infinity. Let X be a Banach space with norm $\|\cdot\|$, and let $\mathfrak{F} : X \times I \to X$ be a completely continuous operator, where I is some real interval. Consider the equation

$$u = \mathfrak{F}(u, \lambda) \tag{9}$$

Definition 3.1. *Suppose that* $\mathfrak{F}(0,\lambda) = 0$ *for all* λ *in I, and that* $\hat{\lambda} \in I$. *We say that* $(0,\hat{\lambda})$ *is a bifurcation point of* (9) *at zero if in any neighborhood of* $(0,\hat{\lambda})$ *in* $X \times I$, *there is a nontrivial solution of* (9). *Or equivalently, if there exist sequences* $\{x_n \neq 0\}$ *and* $\{\lambda_n\}$ *with* $(\|x_n\|, \lambda_n) \to (0,\hat{\lambda})$ *and such that* (x_n, λ_n) *satisfies* (9) *for each* $n \in \mathbb{N}$.

Definition 3.2. *We say that* $(\infty,\hat{\lambda})$ *is a bifurcation point of* (9) *at infinity if in any neigh-borhood of* $(\infty,\hat{\lambda})$ *in* $X \times I$, *there is a nontrivial solution of* (9). *Equivalently, if there exist sequences* $\{x_n \neq 0\}$ *and* $\{\lambda_n\}$ *with* $(\|x_n\|, \lambda_n) \to (\infty,\hat{\lambda})$ *and such that* (x_n, λ_n) *satisfies* (9) *for each* $n \in \mathbb{N}$.

Let u be a solution of problem (P). Define $\mathfrak{F}(u, \lambda)$ by

$$\mathfrak{F}(u,\lambda)(t) = \int_0^t \phi^{-1}\left(a(-\lambda\psi(u) - f(\cdot, u, \lambda)) + \int_0^s -\lambda\psi(u(\xi)) - f(\xi, u, \lambda)d\xi\right) ds. \quad (10)$$

We note that (P) is written as $u = \mathfrak{F}(u, \lambda)$. It is clear that $\mathfrak{F} : C_0^1[0,1] \times \mathbb{R} \to C_0^1[0,1]$ is a completely continuous operator.

Lemma 3.3. (i) *Assume* (Φ_1) *and* (F_1). *if* $(0, \hat{\lambda})$ *is a bifurcation point of* (P), *then* $\hat{\lambda} = \lambda_n(p)$ *for some* $p \in \mathbb{N}$.

(ii) *Assume* (Φ_2) *and* (F_2). *if* $(\infty, \hat{\lambda})$ *is a bifurcation point of* (P), *then* $\hat{\lambda} = \lambda_n(q)$ *for some* $q \in \mathbb{N}$.

Proof: We prove assertion (i). Suppose that $(\infty, \hat{\lambda})$ is a bifurcation point of (P). Then there exists a sequence $\{(u_n, \lambda_n)\}$ in $C_0^1[0,1] \times \mathbb{R}$ with $(u_n, \lambda_n) \to (0, \hat{\lambda})$ and such that (u_n, λ_n) satisfies $u_n = \mathfrak{F}(u_n, \lambda_n)$ for each $n \in \mathbb{N}$. Equivalently, (u_n, λ_n) satisfies

$$u_n(t) = \int_0^t \phi^{-1}\left(a(-\lambda_n\psi(u_n) - f(\cdot, u_n, \lambda_n)) + \int_0^s -\lambda_n\psi(u_n(\xi)) - f(\xi, u_n, \lambda_n)d\xi\right) ds$$

with $\int_0^1 \phi^{-1}\left(a(-\lambda_n\psi(u_n) - f(\cdot, u_n, \lambda_n)) + \int_0^s -\lambda_n\psi(u_n(\xi)) - f(\xi, u_n, \lambda)d\xi\right) ds = 0$.

Let $\varepsilon_n = \|u_n\|_0$ and $v_n(t) = \frac{u_n(t)}{\varepsilon_n}$. Then

$$v_n(t) = \frac{1}{\varepsilon_n}\int_0^1 \phi^{-1}\left(a(-\lambda_n\psi(u_n) - f(\cdot, u_n, \lambda_n)) + \int_0^s -\lambda_n\psi(u_n(\xi)) - f(\xi, u_n, \lambda)d\xi\right) ds,$$

and

$$v_n'(t) = \frac{1}{\varepsilon_n}\phi^{-1}\left(a(-\lambda_n\psi(u_n) - f(\cdot, u_n, \lambda_n)) + \int_0^s -\lambda_n\psi(u_n(\xi)) - f(\xi, u_n, \lambda)d\xi\right).$$

Now, define $d_n(t) = \int_0^t -\lambda_n\psi(u_n(\xi)) - f(\xi, u_n, \lambda_n)d\xi$. Since $f(t, u, \lambda) = o(|\psi(u)|)$ near zero, uniformly for t and λ, for some constants K_1 and K_2.

$$\begin{aligned}||d_n||_0 &= \max_{t\in[0,1]}\left|\int_0^t \lambda_n\psi(u_n(\xi)) + f(\xi, u_n, \lambda_n)d\xi\right| \\ &\leq \max_{t\in[0,1]}\int_0^t |\lambda_n\psi(u_n(\xi))| + |f(\xi, u_n, \lambda_n)|d\xi \\ &\leq \int_0^1 \lambda_n\psi(||u_n||_0) + K_1\psi(||u_n||_0)d\xi \\ &\leq K_2\psi(\varepsilon_n).\end{aligned}$$

Since $\int_0^1 \phi^{-1}\left(a(-\lambda_n\psi(u_n) - f(\cdot, u_n, \lambda_n)) - d_n(s)\right) ds = 0$, we have

$$|a(-\lambda_n\psi(u_n) - f(\cdot, u_n, \lambda_n))| \leq K_2\psi(\varepsilon_n).$$

Otherwise, $\int_0^1 \phi^{-1}\left(a(-\lambda_n\psi(u_n) - f(\cdot, u_n, \lambda_n)) - d_n(s)\right) ds > 0$ or < 0. Now, let us verify that $\frac{1}{\varepsilon_n}\phi^{-1}(2K_2\psi(\varepsilon_n))$ is bounded. If $\frac{1}{\varepsilon_n}\phi^{-1}(2K_2\psi(\varepsilon_n)) \to \infty$ as $n \to \infty$ then for arbitrary $A > 0$, there exists $N_0 \in \mathbb{N}$ such that

$$\frac{1}{\varepsilon_n}\phi^{-1}(2K_2\psi(\varepsilon_n)) \geq A, \text{ for all } n \geq N_0.$$

This implies that $2K_2 \geq \frac{\phi(A\varepsilon_n)}{\psi(\varepsilon_n)}$, for all $n \geq N_0$. This is impossible. Thus

$$\frac{1}{\varepsilon_n}\phi^{-1}\left(a(-\lambda_n\psi(u_n) - f(\cdot, u_n, \lambda_n)) + \int_0^t -\lambda_n\psi(u_n(\xi)) - f(\xi, u_n, \lambda_n)d\xi\right) \leq K_3.$$

Consequently, $\{v_n'\}$ is uniformly bounded and by the Arzela-Ascoli Theorem, $\{v_n\}$ has a uniformly convergent subsequence in $C[0,1]$. Let $v_n \to v$ in $C[0,1]$. Now claim that

$$v(t) = \int_0^t \phi_p^{-1}\left(a_p(-\hat{\lambda}\phi_p(v)) + \int_0^s -\hat{\lambda}\phi_p(v(\xi))d\xi\right)ds.$$

Clearly,

$$\begin{aligned} v'_n(t) &= \frac{1}{\varepsilon_n}\phi^{-1}\left(a(-\lambda_n\psi(u_n) - f(\cdot, u_n, \lambda_n)) + \int_0^t -\lambda_n\psi(u_n(\xi)) - f(\xi, u_n, \lambda_n)d\xi\right) \\ &= \frac{\phi^{-1}(h_n(t)\psi(\varepsilon_n))}{\psi^{-1}(\psi(\varepsilon_n))}, \end{aligned}$$

where $h_n(t) = \dfrac{a(-\lambda_n\psi(u_n) - f(\cdot, u_n, \lambda_n))}{\psi(\varepsilon_n)} + \displaystyle\int_0^t -\lambda_n \frac{\psi(u_n(\xi))\phi(\varepsilon_n)}{\phi(\varepsilon_n)\psi(\varepsilon_n)} - \frac{f(\xi, u_n, \lambda_n)}{\psi(\varepsilon_n)}d\xi$.

Since $\dfrac{a(-\lambda_n\psi(u_n) - f(\cdot, u_n, \lambda_n))}{\psi(\varepsilon_n)}$ is bounded, considering a subsequence if necessary, we may assume that sequence $\left\{\dfrac{a(-\lambda_n\psi(u_n) - f(\cdot, u_n, \lambda_n))}{\psi(\varepsilon_n)}\right\}$ converges to d as $n \to \infty$. This implies that

$$v_n'(t) \to \phi_p^{-1}\left(d + \int_0^t -\hat{\lambda}\phi_p(v(\xi))d\xi\right) \text{ as } n \to \infty,$$

and thus $v(t) = \int_0^t \phi_p^{-1}\left(d + \int_0^s -\hat{\lambda}\phi_p(v(\xi))d\xi\right)ds$. Since $v_n(1) = 0$ for all $n, d = a_p\left(-\hat{\lambda}\phi_p(v)\right)$ and v is a solution of (E_p). Consequently, $\hat{\lambda}$ must be an eigenvalue of the p-Laplacian operator.

The converse of first part of Theorem 3.3 is true in our problem.

Lemma 3.4. *Assume* (Φ_1) *and* (F_1). *If* μ *is an eigenvalue of* (E_p), *then* $(0, \mu)$ *is a bifurcation point.*

Proof: Suppose that $(0, \mu)$ is not a bifurcation point of (P). Then there is a neighborhood of $(0, \mu)$ containing no nontrivial solutions of (P). In particular, we may choose an ε-ball B_ε such that there are no solutions of (P) on $\partial B_\varepsilon \times [\mu - \varepsilon, \mu + \varepsilon]$ and μ is the only eigenvalue of (E_p) on $[\mu - \varepsilon, \mu + \varepsilon]$. Let $\Phi(u, \lambda) = u - \mathfrak{F}(u, \lambda)$. Then $\deg(\Phi(\cdot, \lambda), B(0,\varepsilon), 0)$ is well-defined for λ with $|\lambda - \mu| \leq \varepsilon$. Moreover, from the homotopy invariance theorem,

$$\deg(\Phi(\cdot, \lambda), B(0, \varepsilon), 0) \equiv \text{constant, for all } \lambda \text{ with} |\lambda - \mu| \leq \varepsilon.$$

Now, we claim that

$$\deg(\Phi(\cdot,\mu-\varepsilon),B(0,\varepsilon),0)=\deg(\Phi_p(\cdot,\mu-\varepsilon),B(0,\varepsilon),0),$$

where $\Phi_p(u,\mu-\varepsilon)=u-T_p^{\mu-\varepsilon}(u)$. Define $H^{\mu-\varepsilon}:C_0^1[0,1]\times[0,1]\to C_0^1[0,1]$ by

$$H^{\mu-\varepsilon}(u,\tau)(t)=\tau\mathfrak{F}(u,\mu-\varepsilon)(t)+(1-\tau)T_p^{\mu-\varepsilon}(u)(t).$$

We know that $\mathfrak{F}(\cdot,\mu-\varepsilon)$ and $T_p^{\mu-\varepsilon}$ are completely continuous. To apply the homotopy invariance theorem, we need to show that $0\notin u$ - $H^{\mu-\varepsilon}(u,\tau)(\partial B_\varepsilon)$ to guarantee well-definedness of deg(I-$H^{\mu-\varepsilon}(\cdot,\tau)$, $B(0,\varepsilon)$, 0). Suppose that this is not the case, then there exist sequences $\{u_n\}$, $\{\tau_n\}$ and $\{\varepsilon_n\}$ with $\varepsilon_n\to 0$ and $||u_n||_0=\varepsilon_n$ such that $u_n=H^{\mu-\varepsilon}(u_n,\tau_n)$, i.e.,

$$\begin{aligned}u_n(t)=\tau_n\int_0^t\phi^{-1}\Big(&a(-(\mu-\varepsilon)\psi(u_n)-f(\cdot,u_n,\mu-\varepsilon))\\&+\int_0^s-(\mu-\varepsilon)\psi(u_n(\xi))-f(\xi,u_n,\mu-\varepsilon)d\xi\Big)ds\\+(1-\tau_n)\int_0^t\phi_p^{-1}&\left(a_p(-(\mu-\varepsilon)\phi_p(u_n))+\int_0^s-(\mu-\varepsilon)\phi_p(u_n(\xi))d\xi\right)ds.\end{aligned}$$

Setting $v_n(t)=\frac{u_n(t)}{\varepsilon_n}$, we have that $\|v_n\|_0=1$ and

$$\begin{aligned}v_n(t)=\frac{\tau_n}{\varepsilon_n}\int_0^t\phi^{-1}\Big(&a(-(\mu-\varepsilon)\psi(u_n)-f(\cdot,u_n,\mu-\varepsilon))\\&+\int_0^s-(\mu-\varepsilon)\psi(u_n(\xi))-f(\xi,u_n,\mu-\varepsilon)d\xi\Big)ds\\+(1-\tau_n)\int_0^t\phi_p^{-1}&\left(a_p(-(\mu-\varepsilon)\phi_p(v_n))+\int_0^s-(\mu-\varepsilon)\phi_p(v_n(\xi))d\xi\right)ds.\end{aligned}$$

Hence, we Obtain that

$$\begin{aligned}v'_n(t)=\frac{\tau_n}{\varepsilon_n}\int_0^t\phi^{-1}\Big(&a(-(\mu-\varepsilon)\psi(u_n)-f(\cdot,u_n,\mu-\varepsilon))\\&+\int_0^s-(\mu-\varepsilon)\psi(u_n(\xi))-f(\xi,u_n,\mu-\varepsilon)d\xi\Big)\\+(1-\tau_n)\phi_p^{-1}&\left(a_p(-(\mu-\varepsilon)\phi_p(v_n))+\int_0^s-(\mu-\varepsilon)\phi_p(v_n(\xi))d\xi\right),\end{aligned}$$

and we see that $\{v'_n\}$ is uniformly bounded. Therefore, by the Arzela-Ascoli Theorem, $\{v_n\}$ has a uniformly convergent subsequence in C[0,1]. Without loss of generality, let $v_n\to v$. Moreover, using the fact that

$$\begin{aligned}&\frac{1}{\varepsilon_n}\phi^{-1}\left(a(-(\mu-\varepsilon)\psi(u_n)-f(\cdot,u_n,\mu-\varepsilon))+\int_0^t-(\mu-\varepsilon)\psi(u_n(\xi))-f(\xi,u_n,\mu-\varepsilon)d\xi\right)\\&\quad\to\phi_p^{-1}\left(a_p(-(\mu-\varepsilon)\phi_p(v))+\int_0^t-(\mu-\varepsilon)\phi_p(v(\xi))d\xi\right),\end{aligned}$$

we can obtain that

$$v(t) = \int_0^t \phi_p^{-1}\left(a_p(-(\mu-\varepsilon)\phi_p(v)) + \int_0^s -(\mu-\varepsilon)\phi_p(v(\xi))d\xi\right)ds.$$

This implies $v \equiv 0$ and this is a contradiction. Consequently, deg(I - $H^{\mu-\varepsilon}(\cdot, \tau)$, B(0, ε), 0) is well defined. Therefore, by the homotopy invariance theorem,

$$\deg(\Phi(\cdot, \mu-\varepsilon), B(0,\varepsilon), 0) = \deg(\Phi_p(\cdot, \mu-\varepsilon), B(0,\varepsilon), 0).$$

Similarly,

$$\deg(\Phi(\cdot, \mu+\varepsilon), B(0,\varepsilon), 0) = \deg(\Phi_p(\cdot, \mu-\varepsilon), B(0,\varepsilon), 0).$$

Let μ is k-th eigenvalue of (E_p). Then by Lemma 2.2, we get

$$\deg(\Phi(\cdot, \mu-\varepsilon), B(0,\varepsilon), 0) = (-1)^{k-1} \text{ and } \deg(\Phi_p(\cdot, \mu+\varepsilon), B(0,\varepsilon), 0) = (-1)^k.$$

This is a contradiction to the fact deg($\Phi(\cdot$, μ - ε), B(0,ε),0) = deg($\Phi(\cdot$, μ + ε), B(0, ε), 0).

Thus (0, μ) is a bifurcation point of (P).

Now, we shall adopt Rabinowitz's standard arguement [11]. Let $\mathfrak{S}$ denote the closure of the set of nontrivial solutions of (P) and $\mathcal{S}_k^+$ denote the set $u \in C_0^1[0,1]$ such that u has exactly k - 1 simple zeros in (0,1), $u > 0$ near 0, and all zeros of u in [0,1] are simple. Let $\mathcal{S}_k^- = -\mathcal{S}_k^+$ and $\mathcal{S}_k = \mathcal{S}_k^+ \cup \mathcal{S}_k^-$. We note that the sets $\mathcal{S}_k^+, \mathcal{S}_k^-$ and $\mathcal{S}_k$ are open in $C_0^1[0,1]$. Moreover, let $\mathcal{C}_k$ denote the component of $\mathfrak{S}$ which meets (0, μ_k), where $\mu_k = \lambda_k(p)$. By the similar argument of Theorem 1.10 in [11], we can show the existence of two types of components $\mathcal{C}$ emanating from (0, μ) contained in $\mathfrak{S}$, when μ is an eigenvalue of (E_p); either it is unbounded or it contains $(0, \hat{\mu})$, where $\hat{\mu}(\neq \mu)$ is an eigenvalue of (E_p). The existence of a neighborhood $\mathcal{O}_k$ of (0, μ_k) such that $(u, \lambda) \in \mathfrak{S} \cap \mathcal{O}_k$ and $u \not\equiv 0$ imply $u \in \mathcal{S}_k$ is also proved in [11]. Actually, only the first alternative is possible as shall be shown next.

Lemma 3.5. *Assume* (Φ_1), (Φ_2), *and* (F_1). *Then,* $\mathcal{C}_k$ *is unbounded in* $\mathcal{S}_k \times \mathbb{R}$.

Proof: Suppose $\mathcal{C}_k \subset (\mathcal{S}_k \times \mathbb{R}) \cup \{(0, \mu_k)\}$. Then since $\mathcal{S}_k \cap \mathcal{S}_j = \emptyset$ for $j \neq k$, it follows from the above facts, $\mathcal{C}_k$ must be unbounded in $\mathcal{S}_k \times \mathbb{R}$. Hence, Lemma 3.5 will be established once we show $\mathcal{C}_k \not\subset (\mathcal{S}_k \times \mathbb{R}) \cup \{(0, \mu_k)\}$ is impossible. It is clear that $\mathcal{C}_k \cap \mathcal{O}_k \subset (\mathcal{S}_k \times \mathbb{R}) \cup \{(0, \mu_k)\}$. Hence if $\mathcal{C}_k \not\subset (\mathcal{S}_k \times \mathbb{R}) \cup \{(0, \mu_k)\}$, then there exists $(\mu, \lambda) \in \mathcal{C}_k \cap (\partial\mathcal{S}_k \times \mathbb{R})$ with $(u, \lambda) \neq (0, \mu_k)$ and $(u, \lambda) = \lim_{n\to\infty}(u_n, \lambda_n)$, $u_n \in \mathcal{S}_k$. *If* $u \in \partial\mathcal{S}_k$, $u \equiv 0$ because u dose not have double zero. Henceforth $\lambda = \mu_j$, $j \neq k$. But then, $(u_n, \lambda_n) \in (\mathcal{S}_k \times \mathbb{R}) \cap \mathcal{O}_j$ for large n which is impossible by the fact that $(u_n, \lambda_n) \in \mathfrak{S} \cap \mathcal{O}_j$ implies $u_n \in \mathcal{S}_j$. The proof is complete.

Lemma 3.6. *Assume* (Φ_1), (Φ_2), ($F1$), *and* (F_3). *Then for each* $k \in \mathbb{N}$, *there exists a constant* $M_k \in (0, \infty)$ *such that* $\lambda \le M_k$ *for every* λ *with* $(u, \lambda) \in \mathcal{C}_k$.

Proof: Suppose it is not true, then there exists a sequence $\{(u_n, \lambda_n)\} \subset \mathcal{C}_k$ such that $\lambda_n \to \infty$. Let ρ_{j_n} be the jth zero of u_n. Then there exists $j \in \{1,..., k - 1\}$ such that $|\rho_{(j+1)_n} - \rho_{j_n}| \ge \frac{1}{k}$. Thus for each n, there exists $\sigma_{j_n} \in (\rho_{j_n}, \rho_{(j+1)_n})$ such that $u_n'(\sigma_{j_n}) = 0$. Let $u_n(t) > 0$ for all $t \in (\rho_{j_n}, \rho_{(j+1)_n})$. Suppose $\sigma_{j_n} \in \left(\rho_{j_n}, \frac{\rho_{j_n} + 3\rho_{(j+1)_n}}{4}\right)$.

Then by integrating the equation in (*P*) from σ_{j_n} to $t \in [\sigma_{j_n}, \rho_{(j+1)_n}]$, we see that u_n satisfies

$$u_n(t) = \int_t^{\rho_{(j+1)_n}} \phi^{-1}\left(\int_{\sigma_{j_n}}^{s} \lambda_n \psi(u_n(\xi)) + f(\xi, u_n, \lambda_n) d\xi\right) ds.$$

For $t \in \left[\frac{\rho_{j_n} + 4\rho_{(j+1)_n}}{5}, \frac{\rho_{j_n} + 5\rho_{(j+1)_n}}{6}\right]$,

$$\begin{aligned} u_n(t) &\geq \int_{\frac{\rho_{j_n}+5\rho_{(j+1)_n}}{6}}^{\rho_{(j+1)_n}} \phi^{-1}\left(\int_{\sigma_{j_n}}^{t} \lambda_n \psi(u_n(t)) d\xi\right) ds \\ &\geq \int_{\frac{\rho_{j_n}+5\rho_{(j+1)_n}}{6}}^{\rho_{(j+1)_n}} \phi^{-1}\left(\int_{\frac{\rho_{j_n}+3\rho_{(j+1)_n}}{4}}^{\frac{\rho_{j_n}+4\rho_{(j+1)_n}}{5}} \lambda_n \psi(u_n(t)) d\xi\right) ds \\ &= \frac{\rho_{(j+1)_n} - \rho_{j_n}}{6} \phi^{-1}\left(\frac{\rho_{(j+1)_n} - \rho_{j_n}}{20} \lambda_n \psi(u_n(t))\right) \\ &\geq \frac{1}{6k} \phi^{-1}\left(\frac{1}{20k} \lambda_n \psi(u_n(t))\right). \end{aligned}$$

Thus

$$\frac{\phi(6ku_n(t))}{\psi(u_n(t))} \geq \frac{\lambda_n}{20k}, \tag{11}$$

The left side of (11) is bounded and independent on n, but the right side goes to ∞ as $n \to \infty$. This is impossible. Now, if $\sigma_{j_n} \in \left(\frac{\rho_{j_n} + 3\rho_{(j+1)_n}}{4}, \rho_{(j+1)_n}\right)$, then by integrating the equation in (*P*) from $t \in [\rho_{j_n}, \sigma_{j_n}]$ to σ_{j_n}, we see that u_n satisfies

$$u_n(t) = \int_{\rho_{j_n}}^{t} \phi^{-1}\left(\int_{s}^{\sigma_{j_n}} \lambda_n \psi(u_n(t)) + f(\xi, u_n, \lambda_n) d\xi\right) ds.$$

For $t \in \left[\frac{\rho_{j_n} + \rho_{(j+1)_n}}{2}, \frac{\rho_{j_n} + 2\rho_{(j+1)_n}}{3}\right]$

$$\begin{aligned} u_n(t) &\geq \int_{\rho_{j_n}}^{t} \phi^{-1}\left(\int_{t}^{\sigma_{j_n}} \lambda_n \psi(u_n(t)) d\xi\right) ds \\ &\geq \int_{\sigma_{j_n}}^{\frac{\rho_{j_n}+\rho_{(j+1)_n}}{2}} \phi^{-1}\left(\int_{\frac{\rho_{j_n}+2\rho_{(j+1)_n}}{3}}^{\frac{\rho_{j_n}+3\rho_{(j+1)_n}}{4}} \lambda_n \psi(u_n(t)) d\xi\right) ds \\ &= \frac{\rho_{(j+1)_n} - \rho_{j_n}}{2} \phi^{-1}\left(\frac{\rho_{(j+1)_n} - \rho_{j_n}}{12} \lambda_n \psi(u_n(t))\right) \\ &\geq \frac{1}{2k} \phi^{-1}\left(\frac{1}{12k} \lambda_n \psi(u_n(t))\right). \end{aligned}$$

From the above argument, we get

$$\frac{\phi(2ku_n(t))}{\psi(u_n(t))} \geq \frac{\lambda_n}{12k}. \tag{12}$$

This is impossible because the left side is bounded and independent on n, but the right side goes to infinity as n goes to infinity. We can get similar results when $u_n(t) <$

0. Indeed, if $\sigma_{j_n} \in \left(\rho_{j_n}, \dfrac{\rho_{j_n} + 3\rho_{(j+1)_n}}{4}\right)$, then we have

$$\frac{\phi(6k|u_n(t)|)}{\psi(|u_n(t)|)} \geq \frac{\lambda_n}{20k}. \tag{13}$$

Also if $\sigma_{j_n} \in \left(\frac{\rho_{jn}+3\rho_{(j+1)n}}{4}, \rho_{(j+1)_n}\right)$, then we have

$$\frac{\phi(2k|u_n(t)|)}{\psi(|u_n(t)|)} \geq \frac{\lambda_n}{12k}. \tag{14}$$

Since both (13) and (14) are impossible, there is no sequence $\{(u_n, \lambda_n)\} \subset \mathcal{C}_k$ satisfying $\lambda_n \to \infty$. Consequently, there exists an $M_k \in (0, \infty)$ such that $\lambda \leq M_k$.

Proof of Theorem 1.1

By Lemmas 3.3, 3.4, and 3.5, for any $j \in \mathbb{N}$, there exists an unbounded connected component $\mathcal{C}_j$ of the set of nontrivial solutions emanating from $(0, \lambda_j(p))$ such that $(u, \lambda) \in \mathcal{C}_j$ implies u has exactly j - 1 simple zeros in (0,1). From Lemma 3.6, there is an M_j such that $(u, \lambda) \in \mathcal{C}_j$ implies that $\lambda \leq M_j$, and there are no nontrivial solutions of (P) for $\lambda = 0$, it follows that for any $M > 0$, there is $(u, \lambda) \in \mathcal{C}_j$ such that $\|u\|_1 > M$. Hence, we can choose subsequence $\{(u_n, \lambda_n)\} \subset \mathcal{C}_j$ such that $\lambda_n \to \hat{\lambda}$ and $\|u_n\|_1 \to \infty$. Thus, $(\infty, \hat{\lambda})$ is a bifurcation point and $\hat{\lambda} = \lambda_j(q)$.

4 Application and some examples

Proof of Theorem 1.2

Let us consider the bifurcation problem

$$\begin{cases} -(\phi(u'(t)))' = \lambda\phi(u(t)) + g(t, u), & t \in (0, 1), \\ u(0) = u(1) = 0 \end{cases}$$

Put $f(t,u,\lambda) = -\mu\varphi(u) + g(t, u)$. We can easily see that $f(t,u,\lambda) = o(|\varphi(u)|)$ near zero uniformly for t and λ in bounded intervals. The equation in (A_g) can be equivalently changed into the following equation

$$\begin{cases} -(\phi(u'(t)))' = (\lambda + \mu)\phi(u(t)) + f(t, u, \lambda), & t \in (0, 1), \\ u(0) = u(1) = 0. \end{cases} \tag{A_f}$$

By the similar argument in the proof of Theorem 1.1, for each $k \leq j \leq n$, there is a connected branch $\mathcal{C}_j$ of solutions to (A_f) emanating from $(0, \lambda_j(p) - \mu)$ which is unbounded in $C_0^1[0, 1] \times \mathbb{R}$ and such that $(u, \lambda) \in \mathcal{C}_j$ implies that u has exactly j - 1 simple zeros in (0,1). From the fact $ug(t, u) \geq 0$, it can be proved that there is an $M_j > 0$ such that $(u, \lambda) \in \mathcal{C}_j$ implies that $\lambda \leq M_j$, by the same argument as in the proof of Lemma 3.6. Since there is a constant $K_g > 0$ such that $g(t, s) \leq K_g\varphi(s)$ for all $(t, s) \in [0,1] \times \mathbb{R}$, if $(u, \lambda) \in \mathcal{C}_j$, then $\lambda > -K_g$. Hence $\mathcal{C}_j$ will bifurcate from infinity also, which can only happen for $\lambda = \lambda_j(q) - \nu$. Since $\lambda_j(q) - \nu < 0 < \lambda_j(p) - \mu$ and $\mathcal{C}_j$ is connected, there exists $u \neq 0$ such that $(u, 0) \in \mathcal{C}_j$. This u is a solution of (A). Since this is true for every such j, (A) has at least $n - k + 1$ nontrivial solutions.

Finally, we illustrate several examples of Theorems 1.1 and 1.2.

Example 4.1. Define φ, ψ, f by

$$\phi(u) = \begin{cases} u + u^2 + u^3, \ if\ u \geq 0, \\ u - u^2 + u^3, \ if\ u < 0, \end{cases} \qquad \psi(u) = \begin{cases} u + 2u^2 + u^3, \ if\ u \geq 0, \\ u - 2u^2 + u^3, \ if\ u < 0, \end{cases}$$

$$f(t, u, \lambda) = \begin{cases} \lambda u^2, \ \ if\ u \geq 0, \\ -\lambda u^2, \ if\ u < 0. \end{cases}$$

Then φ and ψ are odd increasing homeomorphisms of $\mathbb{R}$ and

$$\lim_{u\to 0} \frac{\phi(\sigma u)}{\psi(u)} = \sigma = \phi_2(\sigma), \quad \lim_{|u|\to\infty} \frac{\phi(\sigma u)}{\psi(u)} = \sigma^3 = \phi_4(\sigma).$$

Moreover, f satisfies $f(t, u, \lambda) = o(|\psi(u)|)$ near zero and infinity, uniformly in t and λ, and $uf(t,u, \lambda) \geq 0$. Therefore, all hypotheses of Theorem 1.1 are satisfied.

Example 4.2. Define φ, g by

$$\phi(u) = \begin{cases} u + u^2, \ if\ u \geq 0, \\ u - u^2, \ if\ u < 0, \end{cases} \qquad g(t, u) = \begin{cases} \pi u + \pi^3 u^2, \ if\ u \geq 0, \\ \pi u - \pi^3 u^2, \ if\ u < 0. \end{cases}$$

Then φ is odd increasing homeomorphism of $\mathbb{R}$ and

$$\lim_{u\to 0} \frac{\phi(\sigma u)}{\phi(u)} = \sigma = \phi_2(\sigma), \quad \lim_{|u|\to\infty} \frac{\phi(\sigma u)}{\phi(u)} = \sigma^2 = \phi_3(\sigma).$$

Moreover, $ug(t,u) \geq 0$ and

$$\lim_{u\to\infty} \frac{g(t, u)}{\phi(u)} = \pi, \quad \lim_{|u|\to\infty} \frac{g(t, u)}{\phi(u)} = \pi^3.$$

Thus we can check on the fact that

$$\lim_{u\to\infty} \frac{g(t, u)}{\phi(u)} < \lambda_1(2) = \pi^2 < \lambda_1(3) \quad = \left(\frac{4\sqrt{3}\pi}{9}\right)^3 < \lim_{|u|\to\infty} \frac{g(t, u)}{\phi(u)}$$

All hypotheses of Theorem 1.2 for $k = n = 1$ are satisfied so that (A) possesses at least one nontrivial solution.

Example 4.3. Define φ, g by

$$\phi(u) = \begin{cases} u^2 \ln(u + 1), \quad if\ u \geq 0, \\ -u^2 \ln(-u + 1), \ if\ u < 0, \end{cases} \qquad g(t, u) = \begin{cases} 2^{10} u^2 \ln(u + 1)\tan^{-1} u, \ \ if\ u \geq 0, \\ 2^{10} u^2 \ln(-u + 1)\tan^{-1} u, \ if\ u < 0. \end{cases}$$

Then φ is odd increasing homeomorphism of $\mathbb{R}$ and

$$\lim_{u\to 0} \frac{\phi(\sigma u)}{\phi(u)} = \sigma^3 = \phi_4(\sigma), \quad \lim_{|u|\to\infty} \frac{\phi(\sigma u)}{\phi(u)} = \sigma^2 = \phi_3(\sigma).$$

Moreover, $ug(t,u) \geq 0$ and

$$\lim_{u\to\infty} \frac{g(t, u)}{\phi(u)} = 0, \quad \lim_{|u|\to\infty} \frac{g(t, u)}{\phi(u)} = 2^9 \pi.$$

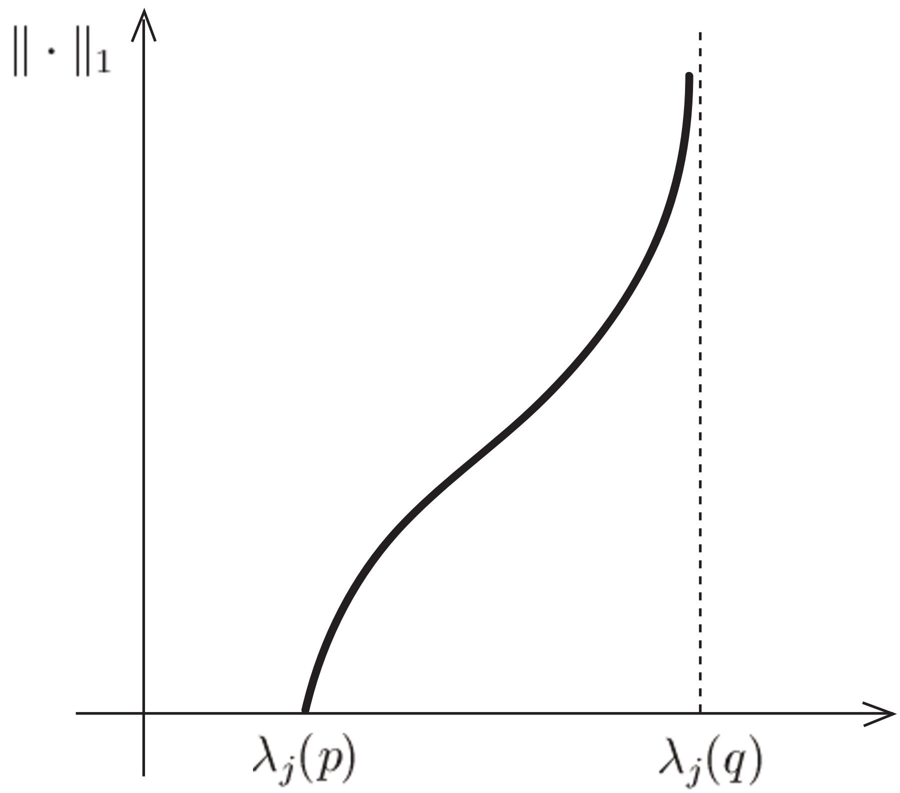

Figure 1 A connected component $\mathcal{C}_j$ of (*P*).

Thus we can check on the fact that

$$\lim_{u\to\infty}\frac{g(t,u)}{\phi(u)} < \lambda_1(4) = \left(\frac{\sqrt{2\pi}}{2}\right)^4 < \lambda_3(3) \quad = \left(\frac{4\sqrt{3\pi}}{3}\right)^3 < \lim_{|u|\to\infty}\frac{g(t,u)}{\phi(u)}$$

All hypotheses of Theorem 1.2 for $k = 1$ and $n = 3$ are satisfied so that (A) possesses at least three nontrivial solutions.

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