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Remarks on some starlike functions

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Abstract

Let $\mathcal{A}$ be the class of functions that are analytic in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$. In this work we investigate conditions under which $|zf'(z)/f(z) - \delta| < \delta$. Next we also estimate $|\mathrm{Arg}\{f'(z)/z\}|$, $|\mathrm{Arg}\{f(z)/z^2\}|$ and $|\mathrm{Arg}\{zf'(z)/f(z)\}|$ for functions of the form $f(z) = z^2 + a_3z^3 + \cdots$ in the unit disc $|z| < 1$, which satisfy $|f''(z) - 2| < 2$. Furthermore, some geometric consequences of these results are given.

MSC: Primary 30C45; secondary 30C80

Keywords: differential subordinations; Nunokawa's lemma; starlike functions of order alpha; convex functions of order alpha; strongly starlike functions of order alpha

1 Introduction

Let $\mathcal{A}$ be the class of functions that are analytic in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$. The subclasses of $\mathcal{A}$ consisting of functions that are univalent in $\mathbb{D}$, starlike with respect to the origin and convex will be denoted by $\mathcal{S}$, $\mathcal{S}^*$ and $\mathcal{C}$, respectively. The class $\mathcal{S}^*_\alpha$ of starlike functions of order $\alpha < 1$ may be defined as

$$\mathcal{S}^*_\alpha = \left\{ f \in \mathcal{A} : \mathfrak{Re} \frac{zf'(z)}{f(z)} > \alpha, z \in \mathcal{U} \right\}.$$

The class $\mathcal{S}^*_\alpha$ and the class $\mathcal{C}_\alpha$ of convex functions of order $\alpha < 1$

$$\begin{aligned} \mathcal{K}_\alpha : &= \left\{ f \in \mathcal{A} : \mathfrak{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > \alpha, z \in \mathcal{U} \right\} \\ &= \left\{ f \in \mathcal{A} : zf' \in \mathcal{S}^*_\alpha \right\} \end{aligned}$$

were introduced by Robertson in [1]. If $\alpha \in [0;1)$, then a function in either of these sets is univalent. The convexity in one direction (it implies the univalence) of functions convex of negative order $-1/2$ was proved by Ozaki [2]. In [3] Pfaltzgraff *et al.* established that the constant $-1/2$ is, in a certain sense, the best possible. A lot of the other equivalent/sufficient conditions for univalence or for the starlikeness, or more, for the convexity in one direction, one can find in [3]. In this work we consider a similar problem, namely find α, β such that

$$\mathfrak{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) < \alpha \quad \Rightarrow \quad \left| \frac{zf'(z)}{f(z)} - \beta \right| < \beta.$$

If $\beta \in (0,1]$, it implies also the starlikeness of f.

2 Preliminaries

The following lemma is a simple generalization of Nunokawa's lemma [4], which together with the lemma from [5] has a surprising number of important applications in the theory of univalent functions.

Lemma 2.1 [6] *Let* $p(z) = 1 + \sum_{n=m\geq 2}^{\infty} c_n z^n$ *be an analytic function in* $\mathbb{D}$. *Suppose also that there exists a point* $z_0 \in \mathbb{D}$ *such that*

$$\mathfrak{Re}\{p(z)\} > 0 \quad \text{for } |z| < |z_0|$$

and

$$\mathfrak{Re}\{p(z_0)\} = 0 \quad \text{and} \quad p(z_0) \neq 0.$$

Then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik,$$

where k *is a real number and*

$$k \geq \frac{m}{2}\left(a + \frac{1}{a}\right) \geq m \geq 2 \quad \text{when } \operatorname{Arg}\{p(z_0)\} = \frac{\pi}{2}$$

and

$$k \leq -\frac{m}{2}\left(a + \frac{1}{a}\right) \leq -m \leq -2 \quad \text{when } \operatorname{Arg}\{p(z_0)\} = -\frac{\pi}{2},$$

where $|p(z_0)| = a$.

3 Main results

Theorem 3.1 *Assume that* $\delta \geq 3/4$ *and* m *is a positive integer such that* $m > 4\delta - 1$. *If* $f(z) = z + \sum_{n=m}^{\infty} a_n z^n$, *and* $zf'(z)/f(z)$ *are analytic in the unit disc* $\mathbb{D}$ *with* $zf'(z) \neq 2\delta f(z)$, $f'(z) \neq 0$, $z \in \mathbb{D}$ *and*

$$\mathfrak{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < \begin{cases} 2\delta + (\delta - 1/2)(m-1) & \text{for } \delta \in [3/4, 1) \text{ and } m \geq \delta/(1-\delta), m \in \mathbb{N}, \\ \frac{m-1}{2(2\delta-1)} & \text{for } \delta \geq 1 \text{ and } m > 4\delta - 1, m \in \mathbb{N}, \\ \frac{m-1}{2(2\delta-1)} & \text{for } \delta \in [3/4, 1) \text{ and } 4\delta - 1 < m < \delta/(1-\delta), m \in \mathbb{N}, \end{cases} \tag{3.1}$$

then we have

$$\left|\frac{zf'(z)}{f(z)} - \delta\right| < \delta \quad \text{for } |z| < 1.$$

Proof The function $zf'(z)/f(z)$ is analytic in $\mathbb{D}$, thus we can define the function p by

$$\frac{zf'(z)}{f(z)} - \delta = \delta \frac{p(z) + 1 - 2\delta}{p(z) - 1 + 2\delta} \quad \text{for } |z| < 1, \tag{3.2}$$

where $p(0) = 1$, and $p(z) = 1 + p_{m-1} z^{m-1} + p_m z^m + \cdots$, $z \in \mathbb{D}$.

Then it follows that

$$1+\frac{zf''(z)}{f'(z)}=\frac{2\delta p(z)}{p(z)-1+2\delta}+\frac{2\delta-1}{p(z)-1+2\delta}\frac{zp'(z)}{p(z)}. \tag{3.3}$$

If there exists a point $z_0 \in \mathbb{D}$ such that

$$\left|\frac{zf'(z)}{f(z)}-\delta\right|<\delta \quad \text{for } |z|<|z_0|$$

and

$$\left|\frac{z_0f'(z_0)}{f(z_0)}-\delta\right|=\delta,$$

then by (3.2)

$$\mathfrak{Re}\{p(z)\}>0 \quad \text{for } |z|<|z_0|$$

and

$$\mathfrak{Re}\{p(z_0)\}=0$$

and $p(z_0)\neq 0$ by (3.3). Then applying Lemma 2.1, we have

$$\frac{z_0p'(z_0)}{p(z_0)}=ik,$$

where

$$k\geq\frac{(m-1)(a^2+1)}{2a} \quad \text{when } \operatorname{Arg}\{p(z_0)\}=\frac{\pi}{2} \tag{3.4}$$

and

$$k\leq-\frac{(m-1)(a^2+1)}{2a} \quad \text{when } \operatorname{Arg}\{p(z_0)\}=-\frac{\pi}{2},$$

and where $p(z_0)=\pm ia$ and $0<a$. For the case $\operatorname{Arg}\{p(z_0)\}=\pi/2$, $p(z_0)=ia$ and $0<a$ it follows from (3.3) that

$$\begin{aligned}
&\mathfrak{Re}\left\{1+\frac{z_0f''(z_0)}{f'(z_0)}\right\}\\
&\quad=\mathfrak{Re}\frac{2\delta ia}{ia-1+2\delta}+\mathfrak{Re}\frac{(2\delta-1)ik}{ia-1+2\delta}\\
&\quad=\frac{2a^2\delta}{a^2+(2\delta-1)^2}+\frac{(2\delta-1)ak}{a^2+(2\delta-1)^2}\\
&\quad=\frac{2a^2\delta+(2\delta-1)ak}{a^2+(2\delta-1)^2}.
\end{aligned}$$

Therefore, we have from (3.4)

$$\begin{aligned}\mathfrak{Re}\left\{1+\frac{z_0f''(z_0)}{f'(z_0)}\right\} &\geq \frac{2a^2\delta+(2\delta-1)a\frac{m-1}{2}\frac{a^2+1}{a}}{a^2+(2\delta-1)^2}\\ &= \frac{4a^2\delta+(2\delta-1)(m-1)(a^2+1)}{2(a^2+(2\delta-1)^2)}\\ &= 2\delta+(\delta-1/2)(m-1)+2\delta(2\delta-1)\frac{(m-1)(1-\delta)-(2\delta-1)}{a^2+(2\delta-1)^2} \quad \text{for } a>0.\end{aligned}$$

In the last expression, the numerator $(m-1)(1-\delta)-(2\delta-1)$ is nonnegative if and only if $\delta\in[3/4,1)$ and $m\geq\delta/(1-\delta)$ but this expression tends to 0^+ when $a\to\infty$. Therefore, in this case we have

$$\mathfrak{Re}\left\{1+\frac{z_0f''(z_0)}{f'(z_0)}\right\}\geq 2\delta+(\delta-1/2)(m-1) \quad \text{for } \delta\in[3/4,1) \text{ and } m>\delta/(1-\delta). \tag{3.5}$$

Furthermore, the numerator $(m-1)(1-\delta)-(2\delta-1)$ is negative if and only if $\delta\geq 1$ and $m\in\mathbb{N}$ or $\delta\in[3/4,1)$ and $m<\delta/(1-\delta)$. In this case the quotient decreases when $a\to 0^+$. Therefore, in this case we have

$$\begin{aligned}\mathfrak{Re}\left\{1+\frac{z_0f''(z_0)}{f'(z_0)}\right\} &\geq 2\delta+(\delta-1/2)(m-1)+\frac{2\delta\{(m-1)(1-\delta)-(2\delta-1)\}}{2\delta-1}\\ &= \frac{m-1}{2(2\delta-1)}.\end{aligned} \tag{3.6}$$

We have assumed that $m>4\delta-1$ to have the right-hand side in (3.1) greater to 1. So in this case we have

$$4\delta-1<m<\frac{\delta}{1-\delta} \quad \text{for } \delta\in[3/4,1). \tag{3.7}$$

Therefore, we can write (3.6) in the form

$$\begin{aligned}\mathfrak{Re}\left\{1+\frac{z_0f''(z_0)}{f'(z_0)}\right\} &\geq \frac{m-1}{2(2\delta-1)}\begin{cases}\text{either for} & \delta\geq 1 \text{ and } m>4\delta-1, m\in\mathbb{N},\\ \text{or for} & \delta\in[3/4,1) \text{ and } 4\delta-1<m<\delta/(1-\delta).\end{cases}\end{aligned} \tag{3.8}$$

Inequalities (3.5) and (3.8) contradict the hypothesis of Theorem 3.1, and therefore we have

$$\mathfrak{Re}\{p(z)\}>0 \quad \text{for } |z|<1. \tag{3.9}$$

Furthermore, from (3.2) and (3.9) we obtain

$$\left|\frac{zf'(z)}{f(z)}-\delta\right|=\left|\delta\frac{p(z)+1-2\delta}{p(z)-1+2\delta}\right|<\delta \quad \text{for } |z|<1. \tag{3.10}$$

For the case $\operatorname{Arg}\{p(z_0)\} = -\pi/2$, $p(z_0) = -ia$ and $0 < a$, applying the same method as above, we also have (3.9). Therefore, we get (3.10), which completes the proof of Theorem 3.1. □

Substituting $\delta = 1$ in Theorem 3.1 leads to the following corollary.

Corollary 3.2 *If $f(z) = z + \sum_{n=m}^{\infty} a_n z^n$ is analytic in the unit disc $\mathbb{D}$ and*

$$\mathfrak{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < \frac{m-1}{2},$$

then we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 \quad \text{for } |z| < 1.$$

Substituting $\delta = 3/4$, $m = 3$ in Theorem 3.1 gives the following corollary.

Corollary 3.3 *If $f(z) = z + \sum_{n=3}^{\infty} a_n z^n$ is analytic in the unit disc $\mathbb{D}$ and*

$$\mathfrak{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < \frac{13}{8},$$

then we have

$$\left|\frac{zf'(z)}{f(z)} - \frac{3}{4}\right| < \frac{3}{4} \quad \text{for } |z| < 1.$$

Substituting $\delta = 4/5$, $m = 3$ in Theorem 3.1 gives the following corollary.

Corollary 3.4 *If $f(z) = z + \sum_{n=3}^{\infty} a_n z^n$ is analytic in the unit disc $\mathbb{D}$ and*

$$\mathfrak{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < \frac{5}{2},$$

then we have

$$\left|\frac{zf'(z)}{f(z)} - \frac{4}{5}\right| < \frac{4}{5} \quad \text{for } |z| < 1.$$

As a supplement to the above results recall here the known result [7, p.61] that if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is analytic in the unit disc $\mathbb{D}$ and

$$\mathfrak{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \prec R_{1,1}(z) = \frac{1+z}{1-z} + \frac{2z}{1-z^2},$$

then

$$\mathfrak{Re}\frac{zf'(z)}{f(z)} > 0 \quad \text{for } |z| < 1.$$

Note that the open door function $R_{1,1}(z)$ maps $\mathbb{D}$ onto the complex plane with slits along the half-lines $\mathfrak{Re}\{w\} = 0$, and $|\mathfrak{Im}\{w\}| \geq \sqrt{3}$. Next we give the bounds for $|\operatorname{Arg}\{zf'(z)/f(z)\}|$.

Theorem 3.5 *Let* $f(z) = z^2 + \sum_{n=3}^{\infty} a_n z^n$ *be analytic in the unit disc* $\mathbb{D}$. *If*

$$\left|f''(z) - 2\right| < 2 \quad \textit{for } |z| < 1, \tag{3.11}$$

then

$$\left|\frac{f'(z)}{z} - 2\right| < 1 \quad \textit{for } |z| < 1. \tag{3.12}$$

Proof By the Schwarz lemma we have

$$\left|f''\left(te^{i\varphi}\right) - 2\right| \le 2t, \quad t \in [0,1).$$

Let $z = re^{i\varphi}$, $r \in [0,1)$, and let φ be fixed. Using this we obtain

$$\begin{aligned}
\left|\frac{f'(z)}{z} - 2\right| &= \frac{|f'(z) - 2z|}{|z|} \\
&= \frac{|\int_0^z (f''(u) - 2)\,\mathrm{d}u|}{|z|} \\
&= \frac{|\int_0^r (f''(te^{i\varphi}) - 2)\,\mathrm{d}(te^{i\varphi})|}{|re^{i\varphi}|} \\
&= \frac{|\int_0^r e^{i\varphi}(f''(te^{i\varphi}) - 2)\,\mathrm{d}t|}{|re^{i\varphi}|} \\
&\le \frac{\int_0^r |e^{i\varphi}(f''(te^{i\varphi}) - 2)|\,\mathrm{d}t}{|re^{i\varphi}|} \\
&\le \frac{\int_0^r 2t\,\mathrm{d}t}{r} \\
&= \frac{r^2}{r} = r < 1.
\end{aligned}$$

Therefore, we obtain (3.12). □

For the function $f(z) = z^3/3 + z^2$, condition (3.11) is satisfied while (3.12) becomes $|z| < 1$ in the unit disc, which shows that the constant 1 in (3.12) cannot be replaced by a smaller one. A simple geometric observation yields the following corollary.

Corollary 3.6 *Let* $f(z) = z^2 + \sum_{n=3}^{\infty} a_n z^n$ *be analytic in the unit disc* $\mathbb{D}$. *If*

$$\left|f''(z) - 2\right| < 2 \quad \textit{for } |z| < 1, \tag{3.13}$$

then

$$\left|\operatorname{Arg}\left\{\frac{f'(z)}{z}\right\}\right| < \frac{\pi}{6} \quad \textit{for } |z| < 1. \tag{3.14}$$

Using the same method as in the proof of Theorem 3.5, we can obtain the following result.

Theorem 3.7 *Let* $f(z) = z^2 + \sum_{n=3}^{\infty} a_n z^n$ *be analytic in the unit disc* $\mathbb{D}$. *If*

$$\left|f''(z) - 2\right| < 2 \quad \text{for } |z| < 1, \tag{3.15}$$

then

$$\left|\frac{f(z)}{z^2} - 1\right| < \frac{1}{3} \quad \text{for } |z| < 1. \tag{3.16}$$

For the function $f(z) = z^3/3 + z^2$, condition (3.15) is satisfied while (3.16) becomes $|z/3| < 1/3$ in the unit disc, which shows that the constant 1/3 in (3.16) cannot be replaced by a smaller one. A simple geometric observation yields the following corollary.

Corollary 3.8 *Let* $f(z) = z^2 + \sum_{n=3}^{\infty} a_n z^n$ *be analytic in the unit disc* $\mathbb{D}$. *If*

$$\left|f''(z) - 2\right| < 2 \quad \text{for } |z| < 1, \tag{3.17}$$

then

$$\left|\operatorname{Arg}\left\{\frac{f(z)}{z^2}\right\}\right| < \sin^{-1}\frac{1}{3} \quad \text{for } |z| < 1. \tag{3.18}$$

Using Corollaries 3.6 and 3.8 together, we obtain the next one.

Corollary 3.9 *Let* $f(z) = z^2 + \sum_{n=3}^{\infty} a_n z^n$ *be analytic in the unit disc* $\mathbb{D}$. *If*

$$\left|f''(z) - 2\right| < 2 \quad \text{for } |z| < 1, \tag{3.19}$$

then

$$\left|\operatorname{Arg}\left\{\frac{zf'(z)}{f(z)}\right\}\right| < \frac{\pi}{6} + \sin^{-1}\frac{1}{3} \approx 0.8634 \quad \text{for } |z| < 1. \tag{3.20}$$

Proof From (3.12) and from (3.16), we have

$$\begin{aligned}
\left|\operatorname{Arg}\left\{\frac{zf'(z)}{f(z)}\right\}\right| &= \left|\operatorname{Arg}\left\{\frac{f'(z)}{z}\frac{z^2}{f(z)}\right\}\right| \\
&\le \left|\operatorname{Arg}\left\{\frac{f'(z)}{z}\right\}\right| + \left|\operatorname{Arg}\left\{\frac{f(z)}{z^2}\right\}\right| \\
&< \frac{\pi}{6} + \sin^{-1}\frac{1}{3} \\
&\approx 0.8634.
\end{aligned}$$

□

Recall the class $\mathcal{SS}^*(\beta)$ of strongly starlike functions of order β, $0 < \beta \le 1$,

$$\mathcal{SS}^*(\beta) := \left\{f \in \mathcal{A} : \left|\operatorname{Arg}\frac{zf'(z)}{f(z)}\right| < \frac{\beta\pi}{2}, z \in \mathbb{U}\right\},$$

which was introduced in [8] and [9]. Therefore, Corollary 3.9 says that if f satisfies the assumptions, then it is 2-valently strongly starlike of order at least 0.8634.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors jointly worked on the results, and they read and approved the final manuscript.

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References

1. Robertson, MS: On the theory of univalent functions. Ann. Math. **37**, 374-408 (1936)
2. Ozaki, S: On the theory of multivalent functions. Sci. Rep. Tokyo Bunrika Daigaku **4**, 45-86 (1941)
3. Pfaltzgraff, JA, Reade, MO, Umezawa, T: Sufficient conditions for univalence. Ann. Fac. Sci. Kinshasa Zaïre Sect. Math.-Phys. **2**, 94-100 (1976)
4. Nunokawa, M: On properties of non-Carathéodory functions. Proc. Jpn. Acad. Ser. A **68**(6), 152-153 (1992)
5. Nunokawa, M: On the order of strongly starlikeness of strongly convex functions. Proc. Jpn. Acad. Ser. A **69**(7), 234-237 (1993)
6. Nunokawa, M, Sokół, J: New conditions for starlikeness and strongly starlikeness of order alpha (submitted)
7. Miller, SS, Mocanu, PT: Differential Subordinations: Theory and Applications. Series of Monographs and Textbooks in Pure and Applied Mathematics, vol. 225. Dekker, New York (2000)
8. Stankiewicz, J: Quelques problèmes extrêmaux dans les classes des fonctions α-angulairement étoilées. Ann. Univ. Mariae Curie-Skłodowska, Sect. A **20**, 59-75 (1966)
9. Brannan, DA, Kirwan, WE: On some classes of bounded univalent functions. J. Lond. Math. Soc. **1**(2), 431-443 (1969)

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