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The fourth power mean of the generalized two-term exponential sums and its upper and lower bound estimates

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Abstract
In this paper, we use the analytic method and the properties of Gauss sums to study the computational problem of one kind fourth power mean of the generalized two-term exponential sums, and give an exact computational formula for it.
MSC: Primary 11L40; 11F20

Keywords: generalized two-term exponential sums; fourth power mean; computational formula

1 Introduction

Let $q \geq 3$ be a positive integer. For any integers m and n, the generalized two-term exponential sum $C(m,n,k,\chi;q)$ is defined by

$$C(m,n,k,\chi;q) = \sum_{a=1}^{q} \chi(a) e\left(\frac{ma^k + na}{q}\right),$$

where χ denotes any Dirichlet character mod q, and $e(y) = e^{2\pi iy}$.

Regarding the upper bound estimate of $C(m,n,k,\chi;q)$, many authors have studied it and obtained a series of important results; related contents can be found in [1–5] and [6]. For example, from Weil's classical work [7] one can deduce the estimate

$$\left|C(m,0,2,\chi;p)\right| \leq 2 \cdot p^{\frac{1}{2}}$$

for $(m,p) = 1$.

Recently, Wang [8] studied the computational problem of the fourth power mean of $C(m,n,k,\chi;p)$, and proved the following conclusion:

Let p be an odd prime with $p \neq 3a+1$. Then, for any integer m with $(m,p) = 1$, we have the identity

$$\sum_{n=1}^{p} \left|C(m,n,3,\chi;p)\right|^4 = \begin{cases} p(2p^2 - 3p - 3) & \text{if } \chi \text{ is the principal character mod } p; \\ p^2(3p-7) & \text{if } \chi \text{ is the Legendre symbol mod } p; \\ p^2(2p-6) & \text{otherwise.} \end{cases}$$

Wang [9] studied the hybrid power mean of the generalized Kloosterman sums $\sum_{a=1}^{p-1}\lambda(a)\times e(\frac{ma+\overline{a}}{p})$ and $\sum_{a=1}^{p-1}\chi(a+\overline{a})$, where λ denotes a Dirichlet character mod p, and gave an interesting asymptotic formula for it. That is, she proved the following result:

Let p be an odd prime. Then, for any non-principal even character χ mod p and any character λ mod p with $\lambda \neq (\frac{*}{p})$, we have the asymptotic formula

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\lambda(a)e\left(\frac{ma+\overline{a}}{p}\right)\right|^2\cdot\left|\sum_{b=1}^{p-1}\chi(mb+\overline{b})\right|^2 = 2p^3+O\big(p^2\big).$$

In this paper, as a note of [8] and [9], we found that there exists a close relationship between the fourth power mean of $C(m,n,2,\chi;p)$ and $|\sum_{b=1}^{p-1}\chi(nb+\overline{b})|$. The main purpose of this paper is to show this point. That is, we shall prove the following theorem.

Theorem *Let p be an odd prime. Then, for any character χ_1 mod p, we have the identity*

$$\sum_{m=1}^{p}\left|\sum_{a=1}^{p-1}\chi_1(a)e\left(\frac{ma^2+a}{p}\right)\right|^4 = \begin{cases} p^3-3p^2+2(\frac{-1}{p})p^2-p-8(\frac{-1}{p})p & \text{if } \chi_1=\chi_0;\\ 2p^3-3p^2 & \text{if } \chi_1(-1)=-1;\\ 2p^3-4(\frac{-1}{p})\cdot p^2-3p^2-p\cdot|\sum_{a=1}^{p-1}\chi_1(a+\overline{a})|^2 & \text{if } \chi_1\neq\chi_0 \text{ and } \chi_1(-1)=1,\end{cases}$$

where χ_0 denotes the principal character mod p, $a\cdot\overline{a}\equiv 1$ mod p, *and $(\frac{*}{p})$ is the Legendre symbol.*

From this theorem we may immediately deduce the following corollary.

Corollary *Let p be an odd prime. Then, for any non-principal character χ_1* mod p, *we have the inequalities*

$$2p^3-11p^2\le\sum_{m=1}^{p}\left|\sum_{a=1}^{p-1}\chi_1(a)e\left(\frac{ma^2+a}{p}\right)\right|^4\le 2p^3+p^2.$$

2 Several lemmas

In this section, we shall give several lemmas, which are necessary in the proof of our theorem. Hereinafter, we shall use many properties of character sums and Gauss sums, all of these can be found in reference [10], so they will not be repeated here. First we have the following.

Lemma 1 *Let p be an odd prime. Then, for any integers m and n with $(mn,p)=1$, we have the identity*

$$\sum_{b=0}^{p-1}e\left(\frac{mb^2+nb}{p}\right)=\left(\frac{m}{p}\right)e\left(\frac{-\overline{4m}n^2}{p}\right)\sum_{a=0}^{p-1}e\left(\frac{a^2}{p}\right),$$

where $(\frac{x}{p})$ denotes the Legendre symbol, and $\overline{m}m\equiv 1$ mod p.

Proof See Lemma 1 in [8]. □

Lemma 2 *Let p be an odd prime,* χ_1 *be any fixed character* mod p. *Then, for any non-real character* χ mod p, *we have the identity*

$$\left|\sum_{m=1}^{p-1}\chi(m)\left|\sum_{a=1}^{p-1}\chi_1(a)e\left(\frac{ma^2+a}{p}\right)\right|^2\right|^2 = p^2\cdot\left|\sum_{a=1}^{p-1}\chi_1(a+1)\overline{\chi}(1+2\overline{a})\right|^2.$$

Proof From the properties of Gauss sums, we have

$$\begin{aligned}
&\sum_{m=1}^{p-1}\chi(m)\left|\sum_{a=1}^{p-1}\chi_1(a)e\left(\frac{ma^2+a}{p}\right)\right|^2 \\
&= \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi_1(a\overline{b})\sum_{m=1}^{p-1}\chi(m)e\left(\frac{m(a^2-b^2)+(a-b)}{p}\right) \\
&= \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi_1(a)\sum_{m=1}^{p-1}\chi(m)e\left(\frac{mb^2(a^2-1)+b(a-1)}{p}\right) \\
&= \tau(\chi)\sum_{a=1}^{p-1}\chi_1(a)\sum_{b=1}^{p-1}\overline{\chi}\left(b^2\left(a^2-1\right)\right)e\left(\frac{b(a-1)}{p}\right) \\
&= \tau(\chi)\tau\left(\overline{\chi}^2\right)\sum_{a=1}^{p-1}\chi_1(a)\overline{\chi}\left(a^2-1\right)\chi^2(a-1) \\
&= \tau(\chi)\tau\left(\overline{\chi}^2\right)\sum_{a=2}^{p-1}\chi_1(a)\overline{\chi}(a+1)\chi(a-1) \\
&= \tau(\chi)\tau\left(\overline{\chi}^2\right)\sum_{a=1}^{p-1}\chi_1(a+1)\overline{\chi}(1+2\overline{a}).
\end{aligned} \tag{1}$$

Note that $\chi_1(p-1+1)=0$, χ is a non-real character mod p, so $\overline{\chi}^2$ is also a non-principal character mod p. Therefore, $|\tau(\chi)| = |\tau(\overline{\chi}^2)| = \sqrt{p}$, so from (1) we may immediately deduce Lemma 2. □

Lemma 3 *Let p be an odd prime,* χ *be any non-principal character* mod p *with* $\chi(-1)=1$. *Then we have*

$$\left|\sum_{a=1}^{p-1}\chi(a+\overline{a})\right|^2 = \left(\frac{-1}{p}\right)\cdot\left(\sum_{a=1}^{p-1}\chi(a)\left(\frac{a^2-1}{p}\right)\right)^2$$

and

$$\left|\sum_{a=1}^{p-1}\chi(a+\overline{a})\right| \le 2\cdot\sqrt{p}.$$

Proof From the properties of quadratic residue mod p, we have

$$\begin{aligned}\sum_{a=1}^{p-1}\chi(a+\overline{a}) &= \sum_{b=1}^{p-1}\chi(b)\sum_{\substack{a=1\\ a+\overline{a}\equiv b \bmod p}}^{p-1} 1 = \sum_{b=1}^{p-1}\chi(b)\sum_{\substack{a=0\\ a^2-ba+1\equiv 0 \bmod p}}^{p-1} 1 \\ &= \sum_{b=1}^{p-1}\chi(b)\sum_{\substack{a=0\\ (2a-b)^2\equiv b^2-4 \bmod p}}^{p-1} 1 = \chi(2)\sum_{b=1}^{p-1}\chi(b)\sum_{\substack{a=0\\ a^2\equiv b^2-1 \bmod p}}^{p-1} 1 \\ &= \chi(2)\cdot\sum_{b=1}^{p-1}\chi(b)\left(1+\left(\frac{b^2-1}{p}\right)\right) = \chi(2)\cdot\sum_{b=1}^{p-1}\chi(b)\left(\frac{b^2-1}{p}\right). \end{aligned} \tag{2}$$

Note that

$$\sum_{b=1}^{p-1}\overline{\chi}(b)\left(\frac{b^2-1}{p}\right) = \sum_{b=1}^{p-1}\chi(b)\left(\frac{\overline{b}^2-1}{p}\right) = \left(\frac{-1}{p}\right)\sum_{b=1}^{p-1}\chi(b)\left(\frac{b^2-1}{p}\right),$$

so from (2) we may immediately deduce the identity

$$\left|\sum_{a=1}^{p-1}\chi(a+\overline{a})\right|^2 = \left(\frac{-1}{p}\right)\left(\sum_{a=1}^{p-1}\chi(a)\left(\frac{a^2-1}{p}\right)\right)^2.$$

The estimate

$$\left|\sum_{a=1}^{p-1}\chi(a+\overline{a})\right| \leq 2\cdot\sqrt{p}$$

follows from Lemma 1 of [9]. This proves Lemma 3. □

3 Proof of Theorem

In this section, we shall give two different proofs of our theorem. First, if χ_1 is a non-principal character mod p, then from Lemma 1 we have

$$\begin{aligned}&\left|\sum_{a=1}^{p-1}\chi_1(a)e\left(\frac{ma^2+a}{p}\right)\right|^2 \\ &\quad= \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi_1(a\overline{b})e\left(\frac{m(a^2-b^2)+a-b}{p}\right) \\ &\quad= \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi_1(a)e\left(\frac{mb^2(a^2-1)+b(a-1)}{p}\right) \\ &\quad= p-1+\chi_1(-1)\sum_{b=1}^{p-1}e\left(\frac{-2b}{p}\right)+\sum_{a=2}^{p-2}\chi_1(a)\sum_{b=1}^{p-1}e\left(\frac{mb^2(a^2-1)+b(a-1)}{p}\right) \\ &\quad= p-1-\chi_1(-1)+\sum_{a=2}^{p-2}\chi_1(a)\sum_{b=0}^{p-1}e\left(\frac{mb^2(a^2-1)+b(a-1)}{p}\right)-\sum_{a=2}^{p-2}\chi_1(a)\end{aligned}$$

$$= p + \sum_{a=2}^{p-2} \chi_1(a) \sum_{b=0}^{p-1} e\left(\frac{m(a+1)\overline{a-1} \cdot b^2 + b}{p}\right) - \sum_{a=1}^{p-1} \chi_1(a)$$
$$= p + G(p) \cdot \sum_{a=2}^{p-2} \chi_1(a) \left(\frac{m(a+1)\overline{a-1}}{p}\right) e\left(\frac{\overline{4m} \cdot \overline{a+1}(a-1)}{p}\right)$$
$$= p + G(p) \cdot \sum_{a=2}^{p-2} \chi_1(a) \left(\frac{m(a^2-1)}{p}\right) e\left(\frac{\overline{4m} \cdot \overline{a+1}(a-1)}{p}\right), \tag{3}$$

where $G(p) = \sum_{a=0}^{p-1} e(\frac{a^2}{p})$ and $G^2(p) = (\frac{-1}{p}) \cdot p$ (see Theorem 7.5.4 of [5]).

From (3) and the definition of Gauss sums, we may immediately deduce

$$\sum_{m=1}^{p-1} \left| \sum_{a=1}^{p-1} \chi_1(a) e\left(\frac{ma^2+a}{p}\right) \right|^4$$
$$= p^2(p-1) + 2pG(p) \sum_{m=1}^{p-1} \sum_{a=2}^{p-2} \chi_1(a) \left(\frac{m(a^2-1)}{p}\right) e\left(\frac{\overline{4m} \cdot \overline{a+1}(a-1)}{p}\right)$$
$$+ G^2(p) \sum_{a=2}^{p-2} \sum_{b=2}^{p-2} \chi_1(ab) \left(\frac{(a^2-1)(b^2-1)}{p}\right)$$
$$\times \sum_{m=1}^{p-1} e\left(\frac{\overline{4m} \cdot (\overline{a+1}(a-1) + \overline{b+1}(b-1))}{p}\right)$$
$$= p^2(p-1) + 2pG^2(p) \sum_{a=2}^{p-2} \chi_1(a) \left(\frac{(a^2-1)(a-1)\overline{a+1}}{p}\right)$$
$$+ pG^2(p) \sum_{a=2}^{p-2} \chi_1(1) \left(\frac{(a^2-1)(\overline{a}^2-1)}{p}\right) - G^2(p) \left(\sum_{a=2}^{p-2} \chi_1(a) \left(\frac{a^2-1}{p}\right)\right)^2$$
$$= p^2(p-1) + 2pG^2(p) \sum_{a=2}^{p-2} \chi_1(a) + pG^2(p) \left(\frac{-1}{p}\right)(p-3)$$
$$- G^2(p) \left(\sum_{a=2}^{p-2} \chi_1(a) \left(\frac{a^2-1}{p}\right)\right)^2. \tag{4}$$

If χ_1 is a non-principal character mod p with $\chi_1(-1) = -1$, then note that

$$\sum_{a=2}^{p-2} \chi_1(a) \left(\frac{a^2-1}{p}\right) = \sum_{a=1}^{p-1} \chi_1(a) \left(\frac{a^2-1}{p}\right) = 0,$$
$$G^2(p) = \left(\sum_{a=0}^{p-1} e\left(\frac{a^2}{p}\right)\right)^2 = \left(\frac{-1}{p}\right) \cdot p$$

and

$$\left| \sum_{a=1}^{p-1} \chi_1(a) e\left(\frac{0 \cdot a^2 + a}{p}\right) \right|^4 = \left| \sum_{a=1}^{p-1} \chi_1(a) e\left(\frac{a}{p}\right) \right|^4 = p^2.$$

From (4) we have

$$\sum_{m=0}^{p-1}\left|\sum_{a=1}^{p-1}\chi_1(a)e\left(\frac{ma^2+a}{p}\right)\right|^4 = 2p^3-3p^2. \tag{5}$$

If χ_1 is a non-principal character mod p with $\chi_1(-1)=1$, then from (4) and Lemma 3 we have

$$\begin{aligned}
&\sum_{m=0}^{p-1}\left|\sum_{a=1}^{p-1}\chi_1(a)e\left(\frac{ma^2+a}{p}\right)\right|^4 \\
&\quad = 2p^3-4\left(\frac{-1}{p}\right)\cdot p^2-3p^2-\left(\frac{-1}{p}\right)\cdot p\cdot\left(\sum_{a=1}^{p-1}\chi_1(a)\left(\frac{a^2-1}{p}\right)\right)^2 \\
&\quad = 2p^3-4\left(\frac{-1}{p}\right)\cdot p^2-3p^2-p\cdot\left|\sum_{a=1}^{p-1}\chi_1(a+\overline{a})\right|^2.
\end{aligned} \tag{6}$$

If $\chi_1=\chi_0$ is the principal character mod p, then from the method of proving (3) and (4) we have

$$\sum_{m=0}^{p-1}\left|\sum_{a=1}^{p-1}e\left(\frac{ma^2+a}{p}\right)\right|^4 = p^3-3p^2+2\left(\frac{-1}{p}\right)p^2-p-8\left(\frac{-1}{p}\right)p. \tag{7}$$

Combining (5), (6) and (7), we may immediately deduce our theorem.

The second proof of Theorem. First, from the orthogonality of characters mod p, we have

$$\begin{aligned}
&\sum_{\chi \bmod p}\left|\sum_{m=1}^{p-1}\chi(m)\left|\sum_{a=1}^{p-1}\chi_1(a)e\left(\frac{ma^2+a}{p}\right)\right|^2\right|^2 \\
&\quad = (p-1)\cdot\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\chi_1(a)e\left(\frac{ma^2+a}{p}\right)\right|^4.
\end{aligned} \tag{8}$$

On the other hand, from Lemma 2 we have

$$\begin{aligned}
&\sum_{\chi \bmod p}\left|\sum_{m=1}^{p-1}\chi(m)\left|\sum_{a=1}^{p-1}\chi_1(a)e\left(\frac{ma^2+a}{p}\right)\right|^2\right|^2 \\
&\quad = p^2\cdot\sum_{\chi \bmod p}\left|\sum_{a=1}^{p-1}\chi_1(a+1)\overline{\chi}(1+2\overline{a})\right|^2+\left(\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\chi_1(a)e\left(\frac{ma^2+a}{p}\right)\right|^2\right)^2 \\
&\qquad +\left|\sum_{m=1}^{p-1}\left(\frac{m}{p}\right)\left|\sum_{a=1}^{p-1}\chi_1(a)e\left(\frac{ma^2+a}{p}\right)\right|^2\right|^2-p^2\cdot\left|\sum_{a=1}^{p-2}\chi_1(a+1)\chi_0(1+2\overline{a})\right|^2 \\
&\qquad -p^2\cdot\left|\sum_{a=1}^{p-2}\chi_1(a+1)\left(\frac{1+2\overline{a}}{p}\right)\right|^2 \\
&\quad \equiv p^2A+B+C-p^2D-p^2E.
\end{aligned} \tag{9}$$

Applying the orthogonality of characters $\bmod p$, we can easily deduce that

$$A = \sum_{\chi \bmod p} \left| \sum_{a=1}^{p-1} \chi_1(a+1)\overline{\chi}(1+2\overline{a}) \right|^2 = (p-1)(p-3), \tag{10}$$

$$B = \left(\sum_{m=1}^{p-1} \left| \sum_{a=1}^{p-1} \chi_1(a) e\left(\frac{ma^2+a}{p} \right) \right|^2 \right)^2$$

$$= \begin{cases} (p^2-2p-1)^2 & \text{if } \chi_1 = \chi_0; \\ p^2(p-2-\chi_1(-1))^2 & \text{if } \chi_1 \neq \chi_0. \end{cases} \tag{11}$$

From the definition and properties of Gauss sums, we have

$$C = \left| \sum_{m=1}^{p-1} \left(\frac{m}{p} \right) \left| \sum_{a=1}^{p-1} \chi_1(a) e\left(\frac{ma^2+a}{p} \right) \right|^2 \right|^2 = p \cdot \left| \sum_{a=1}^{p-1} \chi_1(a) \left(\frac{a^2-1}{p} \right) \right|^2, \tag{12}$$

$$D = \left| \sum_{a=1}^{p-2} \chi_1(a+1)\chi_0(1+2\overline{a}) \right|^2 = p-3, \tag{13}$$

$$E = \left| \sum_{a=1}^{p-2} \chi_1(a+1) \left(\frac{1+2\overline{a}}{p} \right) \right|^2 = \left| \sum_{a=1}^{p-1} \chi_1(a) \left(\frac{a^2-1}{p} \right) \right|^2. \tag{14}$$

Note that if $\chi_1(-1) = -1$, then

$$\sum_{a=1}^{p-1} \chi_1(a) \left(\frac{a^2-1}{p} \right) = 0.$$

Combining (7)-(14) and Lemma 3, we may immediately deduce the identity

$$\sum_{m=1}^{p-1} \left| \sum_{a=1}^{p-1} \chi_1(a) e\left(\frac{ma^2+a}{p} \right) \right|^4$$

$$= \begin{cases} p^3 - 3p^2 + 2(\frac{-1}{p})p^2 - p - 8(\frac{-1}{p})p - 1 & \text{if } \chi_1 = \chi_0; \\ 2p^3 - 4p^2 & \text{if } \chi_1(-1) = -1; \\ 2p^3 - 4(\frac{-1}{p}) \cdot p^2 - 4p^2 - p \cdot |\sum_{a=1}^{p-1} \chi_1(a+\overline{a})|^2 & \text{if } \chi_1 \neq \chi_0 \text{ and } \chi_1(-1) = 1. \end{cases}$$

This completes another proof of our theorem.

The corollary follows from Theorem and Lemma 3.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
LX carried out the part of Introduction, XZ carried out the proof of some lemmas, LX with XZ carried out the theorem's proof. All authors read and approved the final manuscript.

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