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On ∗-class A contractions

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Abstract

A Hilbert space operator T belongs to $*$-class A if $|T^2| - |T^*|^2 \geq 0$. The famous Fuglede-Putnam theorem is as follows: the operator equation $AX = XB$ implies $A^*X = XB^*$ when A and B are normal operators. In this paper, firstly we prove that if T is a contraction of $*$-class A operators, then either T has a nontrivial invariant subspace or T is a proper contraction and the nonnegative operator $D = |T^2| - |T^*|^2$ is a strongly stable contraction; secondly, we show that if X is a Hilbert-Schmidt operator, A and $(B^*)^{-1}$ are $*$-class A operators such that $AX = XB$, then $A^*X = XB^*$.

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1 Introduction

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathbb{C}$ be the set of complex numbers. Let $B(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators acting on $\mathcal{H}$. For operators $T \in B(\mathcal{H})$, we shall write $\ker T$ and $\operatorname{ran} T$ for the null space and the range of T, respectively. Also, let $\sigma(T)$ denote the spectrum of T.

Recall that $T \in B(\mathcal{H})$ is called p-hyponormal for $p > 0$ if $(T^*T)^p - (TT^*)^p \geq 0$ [1]; when $p = 1$, T is called hyponormal. And T is called paranormal if $\|Tx\|^2 \leq \|T^2x\|\|x\|$ for all $x \in \mathcal{H}$ [2, 3]. And T is called normaloid if $\|T^n\| = \|T\|^n$ for all $n \in \mathbb{N}$ (equivalently, $\|T\| = r(T)$, the spectral radius of T). In order to discuss the relations between paranormal and p-hyponormal and log-hyponormal operators (T is invertible and $\log T^*T \geq \log TT^*$), Furuta *et al.* [4] introduced a very interesting class of operators: class A defined by $|T^2| - |T|^2 \geq 0$, where $|T| = (T^*T)^{\frac{1}{2}}$ which is called the absolute value of T; and they showed that the class A is a subclass of paranormal and contains p-hyponormal and log-hyponormal operators. Recently Duggal *et al.* [5] introduced $*$-class A operators (*i.e.*, $|T^2| - |T^*|^2 \geq 0$) and $*$-paranormal operators (*i.e.*, $\|T^*x\|^2 \leq \|T^2x\|\|x\|$ for all $x \in \mathcal{H}$); and they proved that a $*$-class A operator is a generalization of hyponormal operator and $*$-class A operators form a subclass of the class of $*$-paranormal operators.

A contraction is an operator T such that $\|T\| \leq 1$. A contraction T is said to be a proper contraction if $\|Tx\| < \|x\|$ for every nonzero $x \in \mathcal{H}$. A strict contraction is an operator T such that $\|T\| < 1$. A strict contraction is a proper contraction, but a proper contraction is not necessary a strict contraction, although the concepts of strict and proper contractions coincide for compact operators. A contraction T is of class $C_{0\cdot}$ if $\|T^nx\| \to 0$ when $n \to \infty$ for every $x \in \mathcal{H}$ (*i.e.*, T is a strongly stable contraction) and it is said to be of class $C_{1\cdot}$ if $\lim_{n\to\infty} \|T^nx\| > 0$ for every nonzero $x \in \mathcal{H}$. Classes $C_{\cdot 0}$ and $C_{\cdot 1}$ are defined by considering T^* instead of T, and we define the class $C_{\alpha\beta}$ for $\alpha, \beta = 0, 1$ by $C_{\alpha\beta} = C_{\alpha\cdot} \cap C_{\cdot\beta}$. An isometry is a contraction for which $\|Tx\| = \|x\|$ for every $x \in \mathcal{H}$.

In this paper, firstly we prove that if T is a contraction of $*$-class A operators, then either T has a nontrivial invariant subspace or T is a proper contraction and the nonnegative operator $D = |T^2| - |T^*|^2$ is a strongly stable contraction; secondly, we show that if X is a Hilbert-Schmidt operator, A and $(B^*)^{-1}$ are $*$-class A operators such that $AX = XB$, then $A^*X = XB^*$.

2 On $*$-class A contractions

Theorem 2.1 *If T is a contraction of $*$-class A operators, then the nonnegative operator $D = |T^2| - |T^*|^2$ is a contraction whose power sequence $\{D^n\}$ converges strongly to a projection P, and $T^*P = 0$.*

Proof Suppose that T is a contraction of $*$-class A operators, then $D = |T^2| - |T^*|^2 \geq 0$. Let $R = D^{\frac{1}{2}}$. Then for every $x \in \mathcal{H}$,

$$\begin{aligned}
\langle D^{n+1}x, x\rangle &= \|R^{n+1}x\|^2 = \langle DR^n x, R^n x\rangle \\
&= \langle |T^2| R^n x, R^n x\rangle - \langle |T^*|^2 R^n x, R^n x\rangle \\
&= \| |T^2|^{\frac{1}{2}} R^n x\|^2 - \| |T^*| R^n x\|^2 \\
&\leq \|R^n x\|^2 - \|T^* R^n x\|^2 \\
&\leq \|R^n x\|^2 \\
&= \langle D^n x, x\rangle.
\end{aligned}$$

Thus R (and so D) is a contraction and $\{D^n\}$ is a decreasing sequence of nonnegative contractions. Hence $\{D^n\}$ converges strongly to a projection P. Moreover,

$$\sum_{n=0}^{m} \|T^* R^n x\|^2 \leq \sum_{n=0}^{m} \left(\|R^n x\|^2 - \|R^{n+1}x\|^2\right) = \|x\|^2 - \|R^{m+1}x\|^2 \leq \|x\|^2$$

for all nonnegative integers m and every $x \in \mathcal{H}$. Therefore $\|T^* R^n x\| \to 0$ as $n \to \infty$, hence we have

$$T^* P x = T^* \lim_{n\to\infty} D^n x = \lim_{n\to\infty} T^* R^{2n} x = 0$$

for every $x \in \mathcal{H}$. So that $T^*P = 0$. □

Theorem 2.2 *Let T be a contraction of $*$-class A operators. If T has no nontrivial invariant subspace, then*

(i) *T is a proper contraction*;
(ii) *the nonnegative operator $D = |T^2| - |T^*|^2$ is a strongly stable contraction* (*so that* $D \in C_{00}$).

Proof (i) Suppose that T is a $*$-class A operator, then $|T^*|^2 \leq |T^2|$. We have

$$\|T^*x\|^2 = \langle |T^*|^2 x, x\rangle \leq \langle |T^2| x, x\rangle \leq \| |T^2| x\| \|x\| = \|T^2 x\| \|x\|$$

for every $x \in \mathcal{H}$. By [6] Theorem 3.6, we have that

$$T^*Tx = \|T\|^2 x \quad \text{if and only if} \quad \|Tx\| = \|T\|\|x\|.$$

Put $\mathcal{U} = \{x \in \mathcal{H} : \|Tx\| = \|T\|\|x\|\} = \ker(|T|^2 - \|T\|^2)$, which is a subspace of $\mathcal{H}$. In the following, we shall show that $\mathcal{U}$ is an invariant subspace of T. For every $x \in \mathcal{U}$, we have

$$\begin{aligned}\|T(Tx)\|^2 &\le \|T\|^2\|Tx\|^2 = \|T\|^4\|x\|^2 = \|\|T\|^2 x\|^2 = \|T^*Tx\|^2 \\ &\le \|T^2Tx\|\|Tx\| = \|T^2Tx\|\|T\|\|x\|, \end{aligned} \tag{2.1}$$

where the second inequality holds since T is a $*$-class A operator. So, we have that

$$\|T\|^4\|x\|^2 \le \|T^2Tx\|\|T\|\|x\|,$$

that is, $\|T\|^3\|x\| \le \|T^2Tx\|$. Hence we have

$$\|T\|^3\|x\| = \|T^2Tx\|. \tag{2.2}$$

By (2.2), we have

$$\|T\|^3\|x\| = \|T^2Tx\| \le \|T\|\|T(Tx)\|, \tag{2.3}$$

that is, $\|T\|^2\|x\| \le \|T(Tx)\|$. Hence

$$\|T\|^2\|x\| = \|T(Tx)\|. \tag{2.4}$$

Hence by (2.1) and (2.4), we have

$$\|T^2Tx\|\|T\|\|x\| = \|T(Tx)\|^2.$$

So, we have that $\|T(Tx)\| = \|T\|\|Tx\|$. That is, $\mathcal{U}$ is an invariant subspace of T. Now suppose T is a contraction of $*$-class A operators. If T is a strict contraction, then it is trivially a proper contraction. If T is not a strict contraction (*i.e.*, $\|T\| = 1$) and T has no nontrivial invariant subspace, then $\mathcal{U} = \{x \in \mathcal{H} : \|Tx\| = \|x\|\} = \{0\}$ (actually, if $\mathcal{U} = \mathcal{H}$, then T is an isometry, and isometries have nontrivial invariant subspaces). Thus, for every nonzero $x \in \mathcal{H}$, $\|Tx\| < \|x\|$, so T is a proper contraction.

(ii) Let T be a contraction of $*$-class A operators. By Theorem 2.1 we have D is a contraction, $\{D^n\}$ converges strongly to a projection P, and $T^*P = 0$. So, $PT = 0$. Suppose T has no nontrivial invariant subspace. Since $\ker P$ is a nonzero invariant subspace for T whenever $PT = 0$ and $T \neq 0$, it follows that $\ker P = \mathcal{H}$. Hence $P = 0$ and so D^n converges strongly to 0, that is, $D = |T^2| - |T^*|^2$ is a strongly stable contraction. D is self-adjoint, so that $D \in C_{00}$. □

Since a self-adjoint operator T is a proper contraction if and only if T is a C_{00}-contraction, we have the following corollary by Theorem 2.2.

Corollary 2.3 *Let T be a contraction of $*$-class A operators. If T has no nontrivial invariant subspace, then both T and the nonnegative operator $D = |T^2| - |T^*|^2$ are proper contractions.*

3 The Fuglede-Putnam theorem for $*$-class A operators

The famous Fuglede-Putnam theorem is as follows [3, 7, 8].

Theorem 3.1 *Let A and B be normal operators and X be an operator such that $AX = XB$, then $A^*X = XB^*$.*

The Fuglede-Putnam theorem was first proved in the case $A = B$ by Fuglede [7] and then a proof in the general case was given by Putnam [8]. Berberian [9] proved that the Fuglede theorem was actually equivalent to that of Putnam by a nice operator matrix derivation trick. Rosenblum [10] gave an elegant and simple proof of the Fuglede-Putnam theorem by using Liouville's theorem. There were various generalizations of the Fuglede-Putnam theorem to nonnormal operators; we only cite [11–14]. For example, Radjabalipour [13] showed that the Fuglede-Putnam theorem holds for hyponormal operators; Uchiyama and Tanahashi [14] showed that the Fuglede-Putnam theorem holds for p-hyponormal and log-hyponormal operators. If let $X \in B(\mathcal{H})$ be Hilbert-Schmidt class, Mecheri and Uchiyama [15] showed that normality in the Fuglede-Putnam theorem can be replaced by A and B^* class A operators. In this paper, we show that if X is a Hilbert-Schmidt operator, A and $(B^*)^{-1}$ are $*$-class A operators such that $AX = XB$, then $A^*X = XB^*$.

Let $\mathcal{C}_2(\mathcal{H})$ denote the Hilbert-Schmidt class. For each pair of operators $A, B \in B(\mathcal{H})$, there is an operator $\Gamma_{A,B}$ defined on $\mathcal{C}_2(\mathcal{H})$ via the formula $\Gamma_{A,B}(X) = AXB$ in [11]. Obviously, $\|\Gamma_{A,B}\| \leq \|A\|\|B\|$. The adjoint of $\Gamma_{A,B}$ is given by the formula $\Gamma^*_{A,B}(X) = A^*XB^*$; see details [11].

Let $A \otimes B$ denote the tensor product on the product space $\mathcal{H} \otimes \mathcal{H}$ for non-zero $A, B \in B(\mathcal{H})$. In [5], Duggal *et al.* give a necessary and sufficient condition for $A \otimes B$ to be a $*$-class A operator.

Lemma 3.2 (see [5]) *Let $A, B \in B(\mathcal{H})$ be non-zero operators. Then $A \otimes B$ belongs to $*$-class A operators if and only if A and B belong to $*$-class A operators.*

Theorem 3.3 *Let A and $B \in B(\mathcal{H})$. Then $\Gamma_{A,B}$ is a $*$-class A operator on $\mathcal{C}_2(\mathcal{H})$ if and only if A and B^* belong to $*$-class A operators.*

Proof The unitary operator $U : \mathcal{C}_2(\mathcal{H}) \to \mathcal{H} \otimes \mathcal{H}$ by a map $(x \otimes y)^* \to x \otimes y$ induces the $*$-isomorphism $\Psi : B(\mathcal{C}_2(\mathcal{H})) \to B(\mathcal{H} \otimes \mathcal{H})$ by a map $X \to UXU^*$. Then we can obtain $\Psi(\Gamma_{A,B}) = A \otimes B^*$; see details [16]. This completes the proof by Lemma 3.2. □

Lemma 3.4 (see [17]) *Let $T \in B(\mathcal{H})$ be a $*$-class A operator. If $\lambda \neq 0$ and $(T - \lambda)x = 0$ for some $x \in \mathcal{H}$, then $(T - \lambda)^*x = 0$.*

Now we are ready to extend the Fuglede-Putnam theorem to $*$-class A operators.

Theorem 3.5 *Let A and $(B^*)^{-1}$ be $*$-class A operators. If $AX = XB$ for $X \in \mathcal{C}_2(\mathcal{H})$, then $A^*X = XB^*$.*

Proof Let Γ be defined on $\mathcal{C}_2(\mathcal{H})$ by $\Gamma Y = AYB^{-1}$. Since A and $(B^{-1})^* = (B^*)^{-1}$ are $*$-class A operators, we have that Γ is a $*$-class A operator on $\mathcal{C}_2(\mathcal{H})$ by Theorem 3.3. Moreover, we have $\Gamma X = AXB^{-1} = X$ because of $AX = XB$. Hence X is an eigenvector of Γ. By Lemma 3.4 we have $\Gamma^* X = A^* X(B^{-1})^* = X$, that is, $A^*X = XB^*$. The proof is complete. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of the present article. And they also read and approved the final manuscript.

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