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Existence of periodic solutions for p-Laplacian neutral Rayleigh equation

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Abstract

In this paper, we use topological degree theory to establish new results on the existence of periodic solutions for a p-Laplacian neutral Rayleigh equation.

Keywords: periodic solutions; neutral Rayleigh equation; p-Laplacian equation; topological degree

1 Introduction

In this paper, we consider the following second-order p-Laplacian neutral functional differential equation:

$$\bigl(\phi_p\bigl(x'(t)-cx'(t-\sigma)\bigr)\bigr)'+f\bigl(x'(t)\bigr)+\beta(t)g\bigl(x\bigl(t-\tau(t)\bigr)\bigr)=e(t), \tag{1.1}$$

where $\phi_p(x)=|x|^{p-2}x$ for $x\neq 0$ and $p>1$; σ and c are given constants with $|c|\neq 1$; $\phi_p(0)=0$, $f(0)=0$. The conjugate exponent of p is denoted by q, *i.e.* $\frac{1}{p}+\frac{1}{q}=1$. f, g, β, e, and τ are real continuous functions on $\mathbb{R}$; τ, β, and e are periodic with periodic T, $T>0$ is a constant; $\int_0^T e(t)\,dt=0$, $\int_0^T\beta(t)\neq 0$.

As we know, the p-Laplace Rayleigh equation with a deviating argument $\tau(t)$ is applied in many fields such as physics, mechanics, engineering technique fields, and so on. The existence of a periodic solution for the second-order p-Laplacian Rayleigh equations with a deviating argument as follows:

$$\bigl(\phi_p\bigl(x'(t)\bigr)\bigr)'+f\bigl(x(t)\bigr)x'(t)+g\bigl(x\bigl(t-\tau(t)\bigr)\bigr)=e(t) \tag{1.2}$$

and

$$\bigl(\phi_p\bigl(x'(t)\bigr)\bigr)'+f\bigl(x'(t)\bigr)+g\bigl(x\bigl(t-\tau(t)\bigr)\bigr)=e(t) \tag{1.3}$$

has been extensively studied in [1–4]. In recent years, Lu *et al.* [5–7] used Mawhin's continuation theory to do research to the existence of a periodic solution for p-Laplacian neutral Rayleigh equation. They obtained some existence results of periodic solutions to p-Laplacian neutral Rayleigh equations.

In the research mentioned above, the corresponding nonlinear terms of the second-order p-Laplacian Rayleigh equation did not include a variable coefficient. Only little literature discussed this kind of p-Laplacian Rayleigh equation. For more details refer to [8–11]. Here, we focus on [11] by Liang Feng *et al.* They discussed the existence of the solution to the following equation:

$$\big(\phi_p\big(x'(t) - cx'(t-r)\big)\big)' = f\big(x(t)\big)x'(t) + \beta(t)g\big(x\big(t-\tau(t)\big)\big) + e(t). \tag{1.4}$$

They established sufficient conditions for the existence of a T-periodic solution of (1.4). But their conclusions are founded on the prerequisite $\int_0^T (g(x(t-\tau(t))) + e(t))\,dt = 0$, which does not satisfy (1.1). Another significance of the paper is that the result is related to the deviating argument $\tau(t)$, while the conclusions in those existing papers mentioned above have no relation with $\tau(t)$.

2 Preliminary results

For convenience, throughout this paper, we will adopt the following assumptions:

(H_1) $\|x\|_p = (\int_0^T |x(t)|^p\,dt)^{\frac{1}{p}}$, $\|x\|_\infty = \max_{t\in[0,T]} |x(t)|$, $\|x\| = \max\{\|x\|_\infty, \|x'(t)\|_\infty\}$;
(H_2) $m_0 = \min_{t\in[0,T]} |\beta(t)|$, $m_1 = \max_{t\in[0,T]} |\beta(t)|$;
(H_3) $C_T = \{x|x \in C(\mathbb{R},\mathbb{R}), x(t+T) = x(t), \forall t \in \mathbb{R}\}$;
(H_4) $C_T^1 = \{x|x \in C^1(R,R), x(t+T) = x(t), x'(t+T) = x'(t), \forall t \in \mathbb{R}\}$.

It is obvious that C_T with norm $\|x\|_\infty$ and C_T^1 with norm $\|x\|$ are two Banach spaces.
Now we define a linear operator $A : C_T \longrightarrow C_T$, $(Ax)(t) = x(t) - cx(t-\sigma)$.
According to [12, 13], we know that the operator A has the following properties.

Lemma 2.1 [12, 13] *If $|c| \neq 1$, then A has continuous bounded inverse on C_T and*

(1) $\|A^{-1}x\|_\infty = \frac{\|x\|_\infty}{|1-|C||}$, $\forall x \in C_T$,
(2) $(A^{-1}x)(t) = \begin{cases} \sum_{j\geq 0} c^j x(t-j\sigma), & |c| < 1, \\ -\sum_{j\geq 0} c^{-j} x(t+j\sigma), & |c| > 1, \end{cases}$
(3) $\int_0^T |(A^{-1}x)(t)|\,dt \leq \frac{1}{|1-|c||}\int_0^T |x(t)|\,dt$, $\forall x \in C_T$.

Lemma 2.2 *If $|c| \neq 1$ and $p > 1$, then*

$$\big\|A^{-1}x(t)\big\|_p \leq \frac{1}{1-|c|}\big\|x(t)\big\|_p, \quad \forall x \in C_T. \tag{2.1}$$

Proof We know that $x(t)$ is a periodic function. So $\int_0^T |x(t-j\sigma)|\,dt = \int_0^T |x(t)|\,dt$ for $j \geq 0$. When $|c| < 1$, from Lemma 2.1, we have

$$\begin{aligned}
\int_0^T \big|A^{-1}x(t)\big|^p\,dt &= \int_0^T \big|A^{-1}x(t)\big|^{p-1}\big|A^{-1}x(t)\big|\,dt \\
&= \int_0^T \big|A^{-1}x(t)\big|^{p-1}\bigg|\sum_{j\geq 0} c^j x(t-j\sigma)\bigg|\,dt \\
&\leq \sum_{j\geq 0}\big|c^j\big|\int_0^T \big|A^{-1}x(t)\big|^{p-1}\big|x(t-j\sigma)\big|\,dt \\
&\leq \sum_{j\geq 0}\big|c^j\big|\bigg(\int_0^T \big|A^{-1}x(t)\big|^{(p-1)q}\,dt\bigg)^{\frac{1}{q}}\bigg(\int_0^T \big|x(t-j\sigma)\big|^p\,dt\bigg)^{\frac{1}{p}}
\end{aligned}$$

$$= \sum_{j\geq 0} |c^j| \left(\int_0^T |A^{-1}x(t)|^p \, dt \right)^{\frac{1}{q}} \left(\int_0^T |x(t)|^p \, dt \right)^{\frac{1}{p}}$$
$$= \frac{1}{1-|c|} \left(\int_0^T |A^{-1}x(t)|^p \, dt \right)^{\frac{1}{q}} \left(\int_0^T |x(t)|^p \, dt \right)^{\frac{1}{p}}$$

which implies $(\int_0^T |A^{-1}x(t)|^p \, dt)^{\frac{1}{p}} \leq \frac{1}{1-|c|}(\int_0^T |x(t-j\sigma)|^p \, dt)^{\frac{1}{p}}$. That is to say (2.1) holds. If $|c| > 1$, we can also prove that (2.1) is true in the same way. Thus Lemma 2.2 is proved.

Now we consider the following equation in C_T^1:

$$\left(\phi_p\left(u'(t)\right)\right)' = F(u). \tag{2.2}$$

$F : C_T^1 \longrightarrow C_T$ is continuous and takes a bounded set into bounded set.

Let us define $P : C_T^1 \longrightarrow C_T$, $u| \longrightarrow u(0)$, $Q : C_T \longrightarrow C_T$, $h| \longrightarrow \frac{1}{T}\int_0^T h(s)\, ds$ and

$$H\left(h(t)\right) = \int_0^t h(s)\, ds, \quad h \in C_T.$$

It is clear that if $u \in C_T^1$ is the solution to (2.2), then u satisfies the abstract equation

$$u = Pu + QF(u) + K\left(F(u)\right),$$

where the operator $K : C_T \longrightarrow C_T^1$ is given by

$$K\left(h(t)\right) = H\left\{\phi_q\left[\alpha\left((I-Q)h\right) + H\left((I-Q)\right)\right]\right\}(t), \quad \forall t \in \mathbb{R},$$

α is a continuous function which sends bounded sets of C_T into bounded sets of $\mathbb{R}$, and it is a completely continuous mapping. For more details as regards the meaning of α, please refer to [14]. □

Lemma 2.3 [14] *Let Ω be an open bounded set in C_T^1. Suppose that the following conditions hold*:

(i) *For each $\lambda \in (0,1)$, the equation $(\phi_p(u'))' = \lambda F(u)$ has no solution on $\partial\Omega$.*

(ii) *The equation $\digamma(u) = \frac{1}{T}\int_0^T F(u(t))\, dt = 0$ has no solution on $\partial\Omega \cap \mathbb{R}$.*

(iii) *The Brouwer degree of $\digamma$, $\deg\{\digamma, \Omega \cap \mathbb{R}, 0\} \neq 0$.*

Then (2.2) *has at least one T-periodic solution in $\bar{\Omega}$.*

Lemma 2.4 [15] $\Omega \subset \mathbb{R}^n$ *is open bounded and symmetric with respect to $0 \in \Omega$. If $f \in C(\bar{\Omega}, \mathbb{R}^n)$ and $f(x) \neq \mu f(-x)$, $\forall x \in \partial\Omega$, $\mu \in [0,1]$, then* $\deg\{f, \Omega, 0\}$ *is an odd number.*

3 Main results

Theorem 3.1 *Suppose that the following conditions hold*:

(A_1) $\tau(t) \in C^1(\mathbb{R}, \mathbb{R})$, $\tau'(t) > 1$ *or* $\tau'(t) < 1$, $m_2 = \min_{t\in[0,T]} \frac{1}{1-\tau'(t)}$.

(A_2) *The sign of $\frac{\beta(t)}{1-\tau'(t)}$ is unchanged in the interval $[0,T]$.*

(A_3) *There exist constants $r_1 \geq 0$, $r_2 > 0$ and $k > 0$ such that*

(1) $|f(x)| \leq k + r_1|x|^{p-1}$, $\forall x \in \mathbb{R}$,

(2) $\lim_{|x|\longrightarrow\infty} \frac{|g(x)|}{|x|^{p-1}} \leq r_2$.

(A_4) *There exists a constant $d > 0$ such that $xg(x) > 0$, $\forall |x| > d$.*

Then (1.1) *has at least one solution with periodic* T *if there exists a constant* $\varepsilon > 0$ *such that the following condition holds*:

$$\left|1+|c|\right|\left(a+T^{\frac{1}{q}}\right)\left[m_1 T(r_2+\varepsilon)+r_1 T^{\frac{1}{p}}\right] < \left|1-|c|\right|^p, \tag{3.1}$$

where a *is defined in* (3.8).

Proof Consider the homotopic equation of (1.1) as follows:

$$\left(\phi_p\left(x'(t)-cx'(t-\sigma)\right)\right)' + \lambda f\left(x'(t)\right) + \lambda\beta(t)g\left(x\left(t-\tau(t)\right)\right) = \lambda e(t), \quad \lambda \in (0,1). \tag{3.2}$$

Let $x(t)$ be a possible T-periodic solution to (3.2). By integrating both sides of (3.2) over $[0, T]$, we have

$$\int_0^T \left[f\left(x'(t)\right) + \beta(t)g\left(x\left(t-\tau(t)\right)\right)\right] dt = 0. \tag{3.3}$$

Let $u(t) = t - \tau(t)$, by the condition (A_1), we know that $u(t)$ has a unique inverse denoted by $t = \gamma(u)$; noting that $\tau(0) = \tau(T)$, we get

$$\int_0^T \beta(t)g\left(x\left(t-\tau(t)\right)\right) dt = \int_{-\tau(0)}^{T-\tau(T)} \frac{\beta(\gamma(u))g(u)}{1-\tau'(\gamma(u))}\, du = \int_0^T \frac{\beta(\gamma(u))g(u)}{1-\tau'(\gamma(u))}\, du. \tag{3.4}$$

Based on the condition of (A_2) and the integral mean value theorem, there exists $\xi \in [0, T]$ such that

$$g\left(x(\xi)\right)\int_0^T \frac{\beta(\gamma(u))}{1-\tau'(\gamma(u))}\, du = -\int_0^T f\left(x'(t)\right) dt. \tag{3.5}$$

By the condition (A_3)(2) for a given $\varepsilon > 0$, $\exists \rho > d > 0$ when $|x(t)| > \rho$ such that

$$\frac{|g(x)|}{|x|^{p-1}} \geq r_2 - \varepsilon > 0, \qquad \frac{|g(x)|}{|x|^{p-1}} \leq r_2 + \varepsilon. \tag{3.6}$$

Now we can claim that there are two constants a and b such that

$$\left|x(\xi)\right| \leq a\|x\|_p + b, \tag{3.7}$$

where

$$a = \begin{cases} \left[\frac{r_1}{(r_2-\varepsilon)m_0 m_2 T^{\frac{1}{q}}}\right]^{\frac{1}{p-1}} 2^{\frac{2-p}{p-1}}, & p-1<1, \\ \left[\frac{r_1}{(r_2-\varepsilon)m_0 m_2 T^{\frac{1}{q}}}\right]^{\frac{1}{p-1}}, & p-1>1, \end{cases} \tag{3.8}$$

$$b = \begin{cases} \left[\frac{k}{(r_2-\varepsilon)m_0 m_2}\right]^{\frac{1}{p-1}} 2^{\frac{2-p}{p-1}}, & p-1<1, \\ \left[\frac{k}{(r_2-\varepsilon)m_0 m_2}\right]^{\frac{1}{p-1}}, & p-1>1. \end{cases} \tag{3.9}$$

In the following, we prove the above claim in two cases.

Case 1. If $|x(\xi)| \leq \rho$, $\xi \in [0, T]$, then (3.7) holds.

Case 2. If $|x(\xi)| > \rho$, $\xi \in [0, T]$, by (3.5), (3.6), and the condition $(A_3)(1)$, we have

$$\begin{aligned}|r_2 - \varepsilon|\left|x(\xi)\right|^{p-1} m_0 m_2 T &\le \left|g\left(x(\xi)\right)\right| \int_0^T \frac{\beta(\gamma(u))}{1 - \tau'(\gamma(u))}\,du \\ &\le \int_0^T \left|f\left(x'(t)\right)\right| dt \le kT + r_1 \int_0^T \left|x'(t)\right|^{p-1} dt \\ &\le kT + r_1 T^{\frac{1}{p}} \left(\int_0^T \left|x'(t)\right|^p dt\right)^{\frac{p-1}{p}} = kT + r_1 T^{\frac{1}{p}} \left\|x'\right\|_p^{p-1}, \\ \left|x(\xi)\right|^{p-1} &\le \frac{k}{(r_2 - \varepsilon) m_0 m_2} + \frac{r_1}{(r_2 - \varepsilon) m_0 m_2 T^{\frac{1}{q}}} \left\|x'\right\|_p^{p-1}.\end{aligned}$$

When $0 < p - 1 \le 1$, according to the Minkowski inequality, we have

$$\left|x(\xi)\right| \le \left[\frac{k}{(r_2 - \varepsilon) m_0 m_2}\right]^{\frac{1}{p-1}} 2^{\frac{2-p}{p-1}} + \left[\frac{r_1}{(r_2 - \varepsilon) m_0 m_2 T^{\frac{1}{q}}}\right]^{\frac{1}{p-1}} 2^{\frac{2-p}{p-1}} \left\|x'\right\|_p.$$

When $p - 1 > 1$, from $(a + b)^{\frac{1}{m}} \le (a)^{\frac{1}{m}} + (b)^{\frac{1}{m}}$, $a, b \in [0, +\infty)$, $m > 1$, we have

$$\left|x(\xi)\right| \le \left[\frac{k}{(r_2 - \varepsilon) m_0 m_2}\right]^{\frac{1}{p-1}} + \left[\frac{r_1}{(r_2 - \varepsilon) m_0 m_2 T^{\frac{1}{q}}}\right]^{\frac{1}{p-1}} \left\|x'\right\|_p.$$

Therefore, (3.7) is also satisfied for case 2.

For any $t \in \mathbb{R}$, there exists $t_0 \in [0, T]$, such that $t = kT + t_0$, where k is an integer. Then

$$\left|x(t)\right| = \left|x(t_0)\right| \le \left|x(\xi)\right| + \int_0^T \left|x'(s)\right| ds, \quad \xi \in [0, T].$$

By (3.7), we have

$$\|x\|_\infty \le a\left\|x'\right\|_p + b + T^{\frac{1}{q}} \left\|x'\right\|_p = \left(a + T^{\frac{1}{q}}\right)\left\|x'\right\|_p + b. \tag{3.10}$$

At first, we prove that there is a constant R_1 such that

$$\|x\|_\infty \le R_1. \tag{3.11}$$

By (3.10), we only need to prove that $\|x'\|_p$ is bounded in order to prove (3.11).

If $\|x'\|_p = 0$, then $\|x'\|_p$ is obviously bounded.

If $\frac{b}{a + T^{\frac{1}{q}}\|x'\|_p} \ge h$, then $\|x'\|_p \le \frac{b - ah}{hT^{\frac{1}{q}}}$, that is, $\|x'\|_p$ is bounded as well.

If $\frac{b}{a + T^{\frac{1}{q}}\|x'\|_p} < h$, we prove that $\|x'\|_p$ is bounded in the following.

By multiplying both sides of (3.2) by $A(x(t)) = x(t) - cx(t-\sigma)$ and integrating them over $[0,T]$, we have

$$\begin{aligned}
&\left|\int_0^T \left[\varphi_p\left(Ax'\right)\right]' A\left(x(t)\right)dt\right| \\
&\quad = \left|\varphi_p\left(Ax'\right)A\left(x(t)\right)\big|_0^T - \int_0^T \varphi_p\left(Ax'\right)Ax'\,dt\right| \\
&\quad = \int_0^T \left|Ax'\right|^p dt = \left\|Ax'\right\|_p^p \\
&\quad = \left|\lambda\int_0^T A\left(x(t)\right)\left[f\left(x'(t)\right) + \beta(t)g\left(x\left(t-\tau(t)\right)\right) - e(t)\right]dt\right| \\
&\quad \le \left(1+|c|\right)\|x\|_\infty \int_0^T \left[\left|f\left(x'(t)\right| + \left|\beta(t)g\left(x\left(t-\tau(t)\right)\right)\right| + \left|e(t)\right|\right]dt.
\end{aligned} \tag{3.12}$$

Let $E_1 = \{t\in[0,T] : |x(t-\tau(t))| \le \rho\}$, $E_2 = \{t\in[0,T] : |x(t-\tau(t))| > \rho\}$, then

$$\begin{aligned}
\int_0^T \left|\beta(t)g\left(x\left(t-\tau(t)\right)\right)\right|dt &= \int_{E_1}\left|\beta(t)g\left(x\left(t-\tau(t)\right)\right)\right|dt + \int_{E_2}\left|\beta(t)g\left(x\left(t-\tau(t)\right)\right)\right|dt \\
&\le m_1m_3T + m_1T(r_2+\varepsilon)\|x\|_\infty^{p-1},
\end{aligned} \tag{3.13}$$

where

$$m_3 = \max_{|x|\le\rho}\left|g(x)\right|. \tag{3.14}$$

By (3.12) and (3.13), we get

$$\begin{aligned}
\left\|Ax'\right\|_p^p &\le \left(1+|c|\right)\|x\|_\infty \\
&\quad \times \left[m_1m_3T + m_1T(r_2+\varepsilon)\|x\|_\infty^{p-1} + kT + r_1T^{\frac{1}{p}}\left\|x'\right\|_p^{p-1} + \int_0^T \left|e(t)\right|dt\right] \\
&= \left(1+|c|\right)\left(m_1m_3T + kT + \int_0^T \left|e(t)\right|dt\right)\|x\|_\infty + \left(1+|c|\right)m_1T(r_2+\varepsilon)\|x\|_\infty^p \\
&\quad + \left(1+|c|\right)r_1T^{\frac{1}{p}}\|x\|_\infty\left\|x'\right\|_p^{p-1} \\
&\le a_1\left[b + \left(a+T^{\frac{1}{q}}\right)\right]\left\|x'\right\|_p + a_2\left[b + \left(a+T^{\frac{1}{q}}\right)\left\|x'\right\|_p\right]^p \\
&\quad + a_3\left[b + \left(a+T^{\frac{1}{q}}\right)\left\|x'\right\|_p\right]\left\|x'\right\|_p^{p-1},
\end{aligned} \tag{3.15}$$

where $a_1 = (1+|c|)(m_1m_3T + kT + \int_0^T |e(t)|\,dt)$, $a_2 = (1+|c|)m_1T(r_2+\varepsilon)$, $a_3 = (1+|c|)r_1T^{\frac{1}{p}}$.

By elementary analysis, we know that there is a constant $h > 0$ which satisfies $b - ah > 0$ such that

$$(1+u)^p \le 1 + (1+p)u, \quad \forall u\in(0,h]. \tag{3.16}$$

By (3.16), one has

$$\begin{aligned}\bigl[b+\bigl(a+T^{\frac{1}{q}}\bigr)\|x'\|_p\bigr]^p &= \bigl(a+T^{\frac{1}{q}}\bigr)^p\|x'\|_p^p\left(1+\frac{b}{(a+T^{\frac{1}{q}})\|x'\|_p}\right)^p\\ &\le \bigl(a+T^{\frac{1}{q}}\bigr)^p\|x'\|_p^p+b(1+p)\bigl(a+T^{\frac{1}{q}}\bigr)^{p-1}\|x'\|_p^{p-1}.\end{aligned}\tag{3.17}$$

By Lemma 2.2, together with (3.15) and (3.17), we can derive

$$\begin{aligned}&\bigl|1-|c|\bigr|^p\|x'\|_p^p\\ &\quad=\bigl|1-|c|\bigr|^p\|A^{-1}Ax'\|_p^p\le\|Ax'\|_p^p\\ &\quad\le a_1\bigl[b+\bigl(a+T^{\frac{1}{q}}\bigr)\bigr]\|x'\|_p+a_2\bigl[\bigl(a+T^{\frac{1}{q}}\bigr)^p\|x'\|_p^p+b(1+p)\bigl(a+T^{\frac{1}{q}}\bigr)^{p-1}\|x'\|_p^{p-1}\bigr]\\ &\qquad+a_3\bigl[b+\bigl(a+T^{\frac{1}{q}}\bigr)\|x'\|_p\bigr]\|x'\|_p^{p-1}\\ &\quad=(a_2+a_3)\bigl(a+T^{\frac{1}{q}}\bigr)\|x'\|_p^p+\bigl[a_2b(1+p)\bigl(a+T^{\frac{1}{q}}\bigr)^{p-1}+a_3b\bigr]\|x'\|_p^{p-1}\\ &\qquad+a_1\bigl(a+T^{\frac{1}{q}}\bigr)\|x'\|_p+a_1b.\end{aligned}\tag{3.18}$$

From (3.1) and (3.18), we know that $\|x'\|_p$ also is bounded in this case. Based on the above, we can derive the result that $\|x'\|_p$ has a bound; therefore, (3.11) holds.

Secondly, we prove that there is a constant R_2 such that

$$\|x'\|_\infty\le R_2.\tag{3.19}$$

Based on (1.1), together with (3.13) and the condition $(\mathrm{A}_2)(1)$, we get

$$\begin{aligned}&\int_0^T\bigl|\bigl(\phi_p\bigl((Ax'(t))\bigr)\bigr)'\bigr|\\ &\quad\le\int_0^T\bigl[\bigl|f\bigl(x'(t)\bigr)\bigr|+\bigl|\beta(t)g\bigl(x\bigl(t-\tau(t)\bigr)\bigr)\bigr|+\bigl|e(t)\bigr|\bigr]\,dt\\ &\quad\le kT+r_1T^{\frac{1}{p}}\|x'\|_p^{p-1}+m_1m_3T+m_1T(r_2+\varepsilon)\|x\|_\infty^{p-1}+\int_0^T\bigl|e(t)\bigr|\,dt:=R_3.\end{aligned}$$

Because $A(x(0))=A(x(T))$, there exists $t_0\in[0,T]$ such that $(Ax(t_0))'=A(x'(t_0))=0$. Noting that $\phi_p(0)=0$, we have

$$\bigl|\phi_p\bigl(Ax'(t)\bigr)\bigr|=\left|\int_{t_0}^t\bigl(\phi_p\bigl(A\bigl(x'(t)\bigr)\bigr)\bigr)'\,dt\right|\le\int_0^T\bigl|\phi_p\bigl(A\bigl(x'(t)\bigr)\bigr)'\bigr|\,dt\le R_3,$$

then

$$\|Ax'\|_\infty\le\phi_q(R_3).$$

From Lemma 2.1, we derive

$$\|x'\|_\infty=\|A^{-1}Ax'\|_\infty\le\frac{\|Ax'\|_\infty}{|1-|c||}\le\frac{\phi_q(R_3)}{|1-|c||}:=R_2,$$

therefore, (3.19) is satisfied.

Let $y(t) = (Ax(t))$, then (3.2) is equivalent to the following equation:

$$\bigl(\phi_p\bigl(y'(t)\bigr)\bigr)' + \lambda f\bigl(A^{-1}y'(t)\bigr) + \lambda g\bigl(A^{-1}y\bigl(t-\tau(t)\bigr)\bigr) = \lambda e(t). \tag{3.20}$$

Then $x = A^{-1}y$ is a T-periodic solution of (3.2) if y is a T-periodic solution of (3.20). Let

$$F\bigl(y(t)\bigr) = e(t) - f\bigl(\bigl(A^{-1}y(t)\bigr)'\bigr) - g\bigl(A^{-1}\bigl(y\bigl(t-\tau(t)\bigr)\bigr)\bigr), \tag{3.21}$$

since f, g are continuous and A has a continuous inverse, the mapping $F: C_T^1 \longrightarrow C_T$ in (3.21) is continuous and takes bounded sets into bounded sets.

In addition, (3.20) can be represented as

$$\bigl(\phi_p\bigl(y'(t)\bigr)\bigr)' = \lambda F\bigl(y(t)\bigr). \tag{3.22}$$

Let $R = M\max\{R_1, R_2, \rho\}$; $M > 1 + |c|$ is a constant, $\Omega = \{y(t) \in C_T^1, \|y\|_\infty < R, \|y'\|_\infty < R\}$, then (3.22) has no solution on $\partial\Omega$ for $\lambda \in (0,1)$. In fact, if $y = Ax$ is a solution to (3.22) on $\partial\Omega$, then $\|y\|_\infty = R$ or $\|y'\|_\infty = R$. If $\|y\|_\infty = R$, then $\|y\|_\infty = \|Ax\|_\infty = \|x(t) - cx(t-\sigma)\|_\infty \leq (1+|c|)\|x\|_\infty$. That is to say, $\|x\|_\infty \geq \frac{\|y\|_\infty}{1+|c|} > R_1$. This is a contradiction with (3.11). Similarly, $\|y'\|_\infty \neq R$. Then (3.22) satisfies the condition (i) of Lemma 2.3.

If $y \in \partial\Omega \cap \mathbb{R}$, y is a constant and $|y| = \|y\|_\infty = R$, then $f((A^{-1}y)') = 0$ and $|y| \leq (1+|c|)|x|$, $|x| \geq \frac{|y|}{1+|c|} > \rho > d$. By ($A_4$), we obtain $g(A^{-1}(y(t-\tau(t)))) = g(A^{-1}y) = g(x) \neq 0$. Therefore

$$\mathcal{F}(y) = \frac{1}{T}\int_0^T F(y)\,dt = -g\bigl(A^{-1}\bigl(y\bigl(t-\tau(t)\bigr)\bigr)\bigr) = -g(x) \neq 0 \tag{3.23}$$

on $\partial\Omega \cap \mathbb{R}$. This indicates that (3.22) satisfies the condition (ii) of Lemma 2.3.

We know $\partial(\Omega \cap \mathbb{R}) = \{-R, R\}$, then for $\forall y \in \partial(\Omega \cap \mathbb{R})$, we have $y = R > d$ or $y = -R < -d$. By (3.23) and the condition (A_4), we conclude that $\mathcal{F}(y) \neq \mu\mathcal{F}(-y)$, $\mu \in [0,1]$, $y \in \partial(\Omega \cap \mathbb{R})$. Based on Lemma 2.4, we get $\deg\{\mathcal{F}, \Omega \cap \mathbb{R}, 0\} \neq 0$.

Based on the above, (3.22) satisfies all the conditions of Lemma 2.3. So does (3.20). By Lemma 2.3, (3.20) has at least one T-periodic solution, then (1.1) has also at least a periodic solution. □

4 Example

Consider the following equation:

$$\left(\phi_4\left(x'(t) - \frac{1}{10}x'\left(t - \frac{1}{2}\right)\right)\right)' + f\bigl(x'(t)\bigr) + \beta(t)g\left(x\left(t - \frac{1}{2}\sin t\right)\right) = \frac{1}{400}\cos t, \tag{4.1}$$

where $p = 4$, $c = \frac{1}{10}$, $\sigma = \frac{1}{2}$, $\tau(t) = \frac{1}{2}\sin t$, $e(t) = \frac{1}{400}\cos t$, $T = 2\pi$. Obviously we get $\tau'(t) < 1$, $m_2 = \frac{2}{3}$.

If we take $f(x) = \begin{cases} \frac{1}{100}x, & |x| \leq 1, \\ \frac{1}{100}x^3, & |x| > 1, \end{cases}$ $g(x) = \frac{x}{100} + \frac{x^3}{10}$, $\beta(t) = \frac{1}{10(1+\sin^2 t)}$, then the condition ($A_4$) of Theorem 3.1 is satisfied and

$$\bigl|f(x)\bigr| \leq \frac{1}{100} + \frac{1}{100}\bigl|x^3\bigr|, \qquad \lim_{|x|\longrightarrow\infty}\frac{|g(x)|}{|x|^3} \leq \frac{1}{10}, \qquad m_0 = \frac{1}{20}, \qquad m_1 = \frac{1}{10}. \tag{4.2}$$

By (4.2), we obtain $k = \frac{1}{100}$, $r_1 = \frac{1}{100}$, $r_2 = \frac{1}{10}$.

If we choose $\varepsilon = 0.01$, $\rho > 1$, then when $|x| > \rho$, we have

$$\frac{|g(x)|}{|x|^3} \geq r_2 - \varepsilon, \qquad \frac{|g(x)|}{|x|^3} \leq r_2 + \varepsilon.$$

We calculate easily that

$$a = 0.843, \quad \left|1 + |c|\right|\left(a + T^{\frac{1}{q}}\right)\left[m_1 T(r_2 + \varepsilon) + r_1 T^{\frac{1}{p}}\right] = 0.4496 < \left|1 - |c|\right|^4 = 0.6561.$$

Based on the above, we know that (4.1) satisfies all conditions included in Theorem 3.1; therefore, (4.1) has at least one T-periodic solution.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
ZMH performed all the steps of proof in this research and also wrote the paper. JHS suggested many good ideas that made this paper possible and helped to improve the manuscript. All authors read and approved the final manuscript.

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