The moduli space of isometry classes of globally hyperbolic spacetimes

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Abstract

This is the last article in a series of three initiated by the second author. We elaborate on the concepts and theorems constructed in the previous articles. In particular, we prove that the GH and the GGH uniformities previously introduced on the moduli space of isometry classes of globally hyperbolic spacetimes are different, but the Cauchy sequences which give rise to well-defined limit spaces coincide. We then examine properties of the strong metric introduced earlier on each spacetime, and answer some questions concerning causality of limit spaces. Progress is made towards a general definition of causality, and it is proven that the GGH limit of a Cauchy sequence of $\mathcal{C}^{\pm}_{\alpha}$, path metric Lorentz spaces is again a $\mathcal{C}^{\pm}_{\alpha}$, path metric Lorentz space. Finally, we give a necessary and sufficient condition, similar to the one of Gromov for the Riemannian case, for a class of Lorentz spaces to be precompact.

1 Introduction

The geometry of individual spacetimes, modeled in classical general relativity and similar theories by smooth manifolds with Lorentzian metrics, is a subject that has been extensively studied for decades and is fairly well understood, both locally and globally (see, for example, Ref [1]); although specific results may differ from those obtained in Riemannian geometry [2], the field is also a well-developed one. What is not nearly as well developed is the study of the space of Lorentzian geometries, which from the mathematical point of view includes questions about its topology, metric structure, and the possibility of defining a measure on it, and from the physics point of view is crucial for addressing questions such as when a sequence of spacetimes converges to another spacetime, when two geometries are close, or how to calculate an integral over all geometries.

To summarize what is known, we start by introducing a few concepts. We will denote by $\mathcal{L}(\mathcal{M}) = \mathrm{Lor}(\mathcal{M})/\sim$ the space of all Lorentzian geometries on a manifold $\mathcal{M}$, i.e., the space $\mathrm{Lor}(\mathcal{M})$ of Lorentzian metrics on $\mathcal{M}$ modulo diffeomorphisms, and by $\mathcal{LS}$ the much larger space of all Lorentz spaces (these definitions will be made more precise below). In this paper, we will consider cases in which the underlying manifold $\mathcal{M}$ or space is compact, so we assume that to be the case from now on. This implies, if the spacetime is to be regular and free of (almost) closed timelike curves, the existence of spacelike boundaries or initial and final hypersurfaces (we will therefore refer to these spacetimes as cobordisms). Timelike boundaries may exist as well, but the spacetime can have closed spatial sections instead.

Several topologies on $\mathcal{L}(\mathcal{M})$ have been known for some time [1], but distances on this space have been proposed only relatively recently [3, 4]. The more interesting situation, however, is the more general one without a fixed $\mathcal{M}$, and it turns out that the two definitions of closeness that are known for that situation [5, 6] are also more interesting and more manageable even when used just on $\mathcal{L}(\mathcal{M})$. Of the latter two proposals, the only one so far known to give an actual distance function on $\mathcal{LS}$ is the one in Ref [6], some of whose consequences were studied in Ref [7]; this distance, and related concepts, are the tools we will use in this paper to get a better understanding of the structure of the space $\mathcal{LS}$.

More specifically, in the next section we will recall the definitions of the Gromov-Hausdorff distance d_{GH} between Lorentz spaces introduced in Ref [6], as well a similar notion of closeness, and the Riemannian distance D (the "strong metric") on each such Lorentz space $(\mathcal{M}, d)$ used in Ref [7] (where this distance was denoted $D_{\mathcal{M}}$). We will then use them to give a precise definition of Lorentz space and state the questions we will address in the rest of the paper, where the power of the strong metric will become clear.

2 Basic definitions and the moduli space

If $(\mathcal{M}, g)$ is a globally hyperbolic spacetime, the metric g induces a continuous Lorentzian distance on pairs $x, y \in \mathcal{M}$: $d_g(x, y)$ is the supremum over all lengths of future oriented causal curves from x to y, if such curves exist, and zero otherwise. Here we will generalize this situation, and take the point of view that the primary objects are pairs $(\mathcal{M}, d)$, where $\mathcal{M}$ is a set and d a Lorentzian distance $d : \mathcal{M} \times \mathcal{M} \to \mathbf{R}^+ \cup \{\infty\}$, satisfying (i) $d(x, x) = 0$, (ii) $d(x, y) > 0$ implies $d(y, x) = 0$, and (iii) the "reverse triangle inequality" $d(x, z) \geq d(x, y) + d(y, z)$ for any $x, y, z \in \mathcal{M}$ such that $d(x, y)\, d(y, z) > 0$. Once such a pair $(\mathcal{M}, d)$ is given, a partial order $\ll$ on $\mathcal{M}$, interpreted as a chronological relation between events, can be quickly recovered by defining $x \ll y$ iff $d(x, y) > 0$; when $\mathcal{M}$ is a manifold and d continuous, this structure can be recovered from a metric tensor as described above [1], but in general d need not be a Lorentzian path metric.[1]

The reason for emphasizing the use of pairs $(\mathcal{M}, d)$ to characterize spacetimes here, rather than $(\mathcal{M}, g)$, is that they allow us to define [6] a Lorentzian version

[1]For a definition of Lorentzian path metric, see Definition 7.

of the Gromov-Hausdorff distance [8] between Riemannian manifolds. Specifically, given two spacetimes $(\mathcal{M}_1, d_1)$ and $(\mathcal{M}_2, d_2)$, we define the Lorentzian Gromov-Hausdorff distance as

$$d_{\rm GH}((\mathcal{M}_1, d_1), (\mathcal{M}_2, d_2)) := \inf\{\epsilon \mid (\mathcal{M}_1, d_1) \text{ and } (\mathcal{M}_2, d_2) \text{ are } \epsilon\text{−close}\}\ , \quad (1)$$

where the two pairs are said to be ϵ-close iff there exist two mappings $\psi : \mathcal{M}_1 \to \mathcal{M}_2$ and $\zeta : \mathcal{M}_2 \to \mathcal{M}_1$ such that for all $p_1, q_1 \in \mathcal{M}_1$ and $p_2, q_2 \in \mathcal{M}_2$,

$$\big|d_2(\psi(p_1), \psi(q_1)) - d_1(p_1, q_1)\big| \leq \epsilon\ , \quad \big|d_1(\zeta(p_2), \zeta(q_2)) - d_2(p_2, q_2)\big| \leq \epsilon\ . \quad (2)$$

The function $d_{\rm GH}$ is a distance on $\mathcal{L}(\mathcal{M})$, so $d_{\rm GH}((\mathcal{M}_1, d_1), (\mathcal{M}_2, d_2)) = 0$ iff $(\mathcal{M}_1, d_1)$ and $(\mathcal{M}_2, d_2)$ are diffeomorphism-equivalent. However, well-defined limits for Cauchy sequences of Lorentzian spaces have been obtained [7] only with a tighter definition of closeness, requiring that the mappings ψ and ζ be approximate inverses of each other. We say that $(\mathcal{M}_1, d_1)$ and $(\mathcal{M}_2, d_2)$ are (ϵ, δ)-close iff there exist two mappings ψ and ζ as in (2), satisfying in addition

$$\begin{aligned} &\big|d_1(\zeta\circ\psi(p_1), q_1) + d_1(q_1, \zeta\circ\psi(p_1)) - d_1(p_1, q_1) - d_1(q_1, p_1)\big| \leq \delta\ , \\ &\big|d_2(\psi\circ\zeta(p_2), q_2) + d_2(q_2, \psi\circ\zeta(p_2)) - d_2(p_2, q_2) - d_2(q_2, p_2)\big| \leq \delta\ , \end{aligned} \quad (3)$$

for all $p_1, q_1 \in \mathcal{M}_1$ and $p_2, q_2 \in \mathcal{M}_2$. Such a definition of closeness is captured by the mathematical notion of a uniformity; we call it the Hausdorff (because it separates all points), quantitative (because of the labels (ϵ, δ)), generalized Gromov-Hausdorff uniformity (GGH).

In the proof that Cauchy sequences in the GGH sense $\{(\mathcal{M}_i, d_i)\}_{i\in\mathbf{N}}$ have well-defined limit spaces, an interesting tool emerged, a Riemannian (i.e., positive-definite) metric D, called *strong metric*, defined on each $(\mathcal{M}, d)$ by

$$D(p, q) = \max_{r\in\mathcal{M}} \big|d(p, r) + d(r, p) - d(q, r) - d(r, q)\big|\ . \quad (4)$$

The definitions and results summarized above, and in particular theorem 6 of Ref [7], strongly suggest the following definition of a Lorentz space.

Definition 1 *A Lorentz space is a pair $(\mathcal{M}, d)$, where $\mathcal{M}$ is a set and d is a Lorentz distance on $\mathcal{M}$, such that $(\mathcal{M}, D)$ is a compact metric space.*

We denote by $\aleph_{\rm c}$ the space of all such Lorentz spaces. On $\aleph_{\rm c}$, we can introduce an equivalence relation $\sim$ by defining $(\mathcal{M}_1, d_1) \sim (\mathcal{M}_2, d_2)$ iff there exists a bijection ψ such that $d_2(\psi(x), \psi(y)) = d_1(x, y)$ for all $x, y \in \mathcal{M}_1$. Such a bijection is automatically a homeomorphism, and therefore $\sim$ defines an equivalence relation.

Definition 2 *The moduli space of all isometry classes of Lorentz spaces is the space $\mathcal{LS} = \aleph_c/\sim$, equipped with the Hausdorff, quantitative, generalized Gromov-Hausdorff uniformity.*

It was shown in Ref [7] that $\mathcal{LS}$ is a complete, contractible space in which the finite spaces form a dense subset. It is easily seen that it is not locally compact.

Note: The results in section 3 of Ref [6], in particular theorem 6, imply that the obvious extension of d_{GH} from $\mathcal{L}(\mathcal{M})$ to the moduli space of isometry classes of Lorentz spaces is also a metric. However, in the above definition, we prefer to equip this space with the GGH uniformity, since $\mathcal{LS}$ is then complete.[2] •

Let us comment now a bit on why these results are so easy and completely analogous to the Riemannian case. In our personal opinion, things became a lot easier than in previous attempts at defining distances between Lorentzian spaces [3, 4, 5] because we have quit considering the causal relation, separately from the volume element, as being the fundamental object. This opened up the possibility of introducing the strong metric D, which emerged despite its strong nonlocality as a natural object, and allowed us to ask questions concerning closeness and convergence in a more direct and quantitative manner. The theorems in Ref [7] clearly bring to light the technical potential of the metric D. On the other hand, the examples in that same paper show that defining a suitable causal structure out of the chronological one might prove to be a nontrivial task in the context of general Lorentz spaces, due to the existence of degenerate regions in the limit spaces. Moreover, causal curves and causal relations (as opposed to chronological ones) in Lorentz spaces have different properties from the ones which we are used to with globally hyperbolic cobordisms. For example, geodesics between two timelike related points are not necessarily timelike curves.

The goal of this paper is to further study the properties of the moduli space of Lorentzian geometries and the structures described above. In particular, we (1) Try to find out if the Lorentzian Gromov-Hausdorff metric and the generalized Gromov-Hausdoff uniformity are equivalent in the sense that they have the same Cauchy sequences; (2) Study the question whether the strong metric determines the Lorentzian distance uniquely up to time reversal (this would be particulary interesting, since if it were true, then Lorentzian cobordisms would be a subclass of Riemannian, non-path compact metric spaces modulo $\mathbf{Z}_2$) (3) Study the definition of a suitable causal relation and causal curves on limit spaces of compact globally hyperbolic cobordisms (for example, if we knew how to define a causal relation between two points in the "degenerate area" of a limit space, then one could raise the question of the physical meaning of such "causal relationships"); (4) Deal with the moduli space and some matters of precompactness.

[2]The authors are unaware of any proof of, or counterexample to the stement that the moduli space of isometry classes equipped with d_{GH} is complete.

3 d_{GH} versus the generalized Gromov-Hausdorff uniformity

In this section, we examine the relationship between the Gromov-Hausdorff (GH) distance and the generalized Gromov-Hausdorff (GGH) uniformity for Lorentz spaces $(\mathcal{M}, d)$. Along this study, some questions raised in Ref [6] will be solved. Since the difference between GH-closeness and GGH-closeness lies in the condition that the mappings used in the definition be approximate inverses of each other, we find it useful to introduce the concepts of approximate isometry and approximately surjective mapping.

Definition 3 *Given a Lorentz space $(\mathcal{M}, d)$ and an $\epsilon > 0$, a mapping $f : \mathcal{M} \to \mathcal{M}$ is:*

- *an ϵ-isometry iff f changes d-distances between any pair of points by no more than ϵ; i.e., for all $x, y \in \mathcal{M}$*

$$\left| d(f(x), f(y)) - d(x, y) \right| < \epsilon \; ;$$

- *an ϵ-surjection iff any point is within a D-distance ϵ of the image of some point; i.e., for all $p \in \mathcal{M}$ there exists a $q \in \mathcal{M}$ such that*

$$D(p, f(q)) < \epsilon \; ,$$

where D is the strong metric on $\mathcal{M}$ constructed using $(\mathcal{M}, d)$.

We start with the following theorem.

Theorem 1 *Let $(\mathcal{M}, g)$ be a compact, globally hyperbolic cobordism. Then, for every $\eta > 0$ there exists an $\epsilon > 0$ such that, for any ϵ-isometry f, there is an isometry h of $\mathcal{M}$ such that:*

$$D(f(x), h(x)) < \eta \quad \forall x \in \mathcal{M} \; .$$

Proof:
Suppose that the statement is false. Then, given $\eta > 0$, for each $n \in \mathbf{N}$ there exists a $\frac{1}{n}$-isometry f_n, such that for any isometry f we can find a point $x(n, f)$ in $\mathcal{M}$ such that

$$D(f_n(x(n, f)), f(x(n, f))) \geq \eta \; .$$

The proof of theorem 6 in Ref [6] reveals that the sequence $\{f_n\}_{k \in \mathbf{N}}$ has a subsequence $\{f_{n_k}\}_{k \in \mathbf{N}}$ such that $f_{n_k} \overset{k \to \infty}{\to} f$ pointwise. We show now that this convergence is uniform in the strong metric, which provides the necessary contradiction. We restrict ourselves to proving that for any interior point p and $\epsilon > 0$, there exists a $\delta > 0$ such that $q \in B_D(p, \delta)$ implies that $D(f(q), f_n(q)) < \epsilon$, for n big enough; The rest of the statement is easy (but tedious) and is left as an exercise to the courageous reader. Choose two points $s, r \in B_D(p, \frac{\epsilon}{2})$ such that $s \ll p \ll r$ with, say, $d(s, p) = d(p, r)$ as large as possible. Let $\delta = \frac{1}{3} d(p, r)$;

then for $n > \frac{1}{\delta}$ such that $f_n(r) \in B_D(f(r), \delta)$ and $f_n(s) \in B_D(f(s), \delta)$ we have that

$$f_n(B_D(p, \delta)) \subset A(f(r), f(s)) \subset B_D(f(p), \tfrac{\epsilon}{2}) ,$$

where $A(\cdot, \cdot)$ denotes the Alexandrov set between two points in $\mathcal{M}$. Hence,

$$D(f_n(q), f(q)) \leq \tfrac{\epsilon}{2} + \delta < \epsilon ,$$

for all $q \in B_D(p, \delta)$. •

This result reveals that for any $\eta > 0$, there exists an $\epsilon > 0$ such that any ϵ-isometry is an η-surjection. This, however, is a fairly weak result and we would like to know if η could be bounded by some universal function of ϵ, the timelike diameter and dimension of $\mathcal{M}$, which goes to zero when either ϵ or the timelike diameter go to zero. An example will show that such a function cannot exist, and lead us to other examples that address directly our goal for this section.

Example 1

Consider the region $\mathcal{R}_\delta = \{(x, t) \mid x \in [0, \pi],\ t \leq T(x)\}$ of the 2-dimensional flat cylinder $\mathcal{CYL} = (\mathrm{S}^1 \times [0, 1], -\mathrm{d}t^2 + \mathrm{d}\theta^2)$, where T is some differentiable function satisfying $T(x) \geq 1 - \delta$ for all $x \in [0, \pi]$, $T(0) = T(\pi) = 1$, $T(\frac{\pi}{2}) = 1 - \delta$, and $|T'(x)| < 1$ (see Fig 1). We can now show an approximate isometry ψ which is

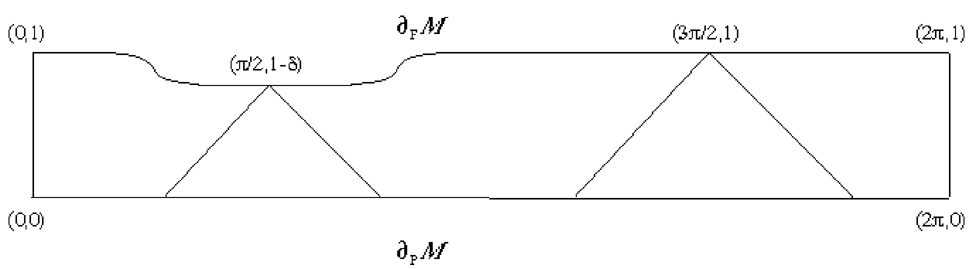

Figure 1: Illustration of example 1

far from any isometry. The mapping ψ is constructed as the composition of a rotation by π times a retraction $R_{\mathcal{R}_\delta}$ which maps a point (x, t) to the unique closest point $(x, \tilde{t}) \in \mathcal{R}_\delta$. It is not difficult to verify that ψ is a $\sqrt{2\delta}$-isometry. However, the point $p = (\frac{3\pi}{2}, 1)$ gets mapped to a point which is a strong distance $1 = tdiam(\mathcal{R}_\delta)$ away.[3] This shows that for δ arbitrarily small, one can construct spaces which allow δ-isometries to be a distance 1 apart from any isometry (our analysis is simplified by the fact that the only isometry is the identity!). •

Example 2

[3]For a Lorentz space $(\mathcal{M}, d)$ the timelike diameter is $tdiam(\mathcal{M}) := \max_{p,q \in \mathcal{M}} d(p, q)$.

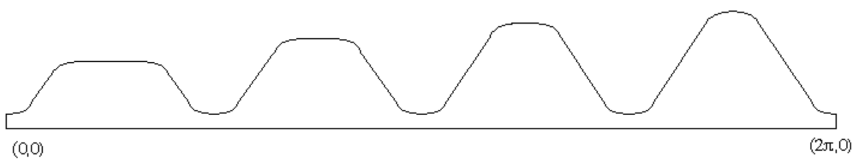

Figure 2: The carousel

In this example, we show that a near isometry can be arbitrarily far from being a surjection. The following picture shows a sequence of N "bumps" with a fixed width $L > 1$. Let $0 < \epsilon < \frac{1}{2}$ and consider the function $g_\epsilon : [-\epsilon, \epsilon] \to \mathbf{R}^+ : x \to \epsilon + x^2$. Define a sequence of functions $\Omega^i_\epsilon : [(i-1)L, iL] \to \mathbf{R}^+,\ i = 1 \ldots N$, which satisfy the following properties:

- $0 \leq \Omega^{i+1}_\epsilon(x+L) - \Omega^i_\epsilon(x) \leq \frac{L}{\sqrt{2}N}$ for $1 \leq i \leq N-1$ and $x \in [(i-1)L, iL]$.
- Ω^i_ϵ is symmetric around $x = (i - \frac{1}{2})L$.
- $\max_{x\in[(i-1)L,iL]} \Omega^i_\epsilon(x) = i\,\frac{L}{\sqrt{2}N}$.
- $\Omega^i_\epsilon(x) = g_\epsilon(x - (i-1)L)$ for all $x \in [(i-1)L, (i-1)L + \epsilon]$.
- $\left|\mathrm{d}\Omega^i_\epsilon(x)/\mathrm{d}x\right| < 1$ for all $x \in [(i-1)L, iL]$.

Let Ω_ϵ be the concatenation of all the Ω^i_ϵ. By identifying 0 and NL, we obtain that Ω is a smooth function on the circle of radius $NL/2\pi$. Define $\mathcal{A}_\epsilon$ as

$$\mathcal{A}_\epsilon = \{(x,t) \mid x \in [0, NL] \text{ and } t \in [0, \Omega_\epsilon(x)]\}\,.$$

Then $(\mathcal{A}_\epsilon, -\mathrm{d}t^2 + \mathrm{d}x^2)$ is a globally hyperbolic cobordism cut out of the cylinder universe $\mathcal{CYL} = \mathrm{S}^1 \times \mathbf{R}$ with radius $\frac{NL}{2\pi}$. Define $\psi : \mathcal{A}_\epsilon \to \mathcal{A}_\epsilon$ as the composition of a rotation to the left over an angle of $\frac{2\pi}{N}$ with a retraction $R_{\mathcal{A}_\epsilon} : \mathcal{CYL} \to \mathcal{CYL}$ which maps every point (x,t) to the closest point $(x,\tilde{t}) \in \mathcal{A}_\epsilon$. Considering the point $((N-\frac{1}{2})L, \frac{L}{\sqrt{2}})$ and suitable other points, we clearly see that ψ is a $\frac{L}{\sqrt{N}}$-isometry which is not a $\frac{(N-1)L}{\sqrt{2}N}$-surjection. The figure will be called the carousel for obvious reasons. •

Let $f, g : \mathcal{LS} \times \mathcal{LS} \times \mathbf{R}^+ \to \mathbf{R}^+$ be functions depending only upon the timelike diameters, $f(x,y,0) = g(x,y,0) = 0$ and f, g are continuous in the third element in $(x,y,0)$ for all $x, y \in \mathcal{LS}$. Do functions satisfying the above conditions exist such that if x and y are ϵ-close then they are $(f(x,y,\epsilon), g(x,y,\epsilon))$-close? It is not difficult to prove that if there exist two mappings ψ, ζ which make x and y ϵ-close such and ψ or ζ is *surjective*, then x and y are $(\epsilon, 2\epsilon)$-close. Hence if we want to find a counterexample then we have to look for mappings which are "far" from being surjections. The carousel hints at the following counterexample.

Example 3

Suppose $L = 4m$, $m \in \mathbf{N}\backslash\{0,1\}$ and let $\mathcal{P}_1^L, \mathcal{P}_2^L$ be causal sets given by the Hasse diagrams below, where it is understood that the points in each of the thicker pairs are to be identified. On a locally finite causal set $\mathcal{P}$, the maximum number of links between two timelike related points $p \ll q$ gives a Lorentz distance, and if $\mathcal{P}$ is finite it has a natural interpretation as a Lorentz space. Obviously $\mathcal{P}_1^L, \mathcal{P}_2^L$ are 1-close, as one can see using a map $\psi : \mathcal{P}_1^L \to \mathcal{P}_2^L$ which takes each column of $\mathcal{P}_1^L$ to the corresponding one in $\mathcal{P}_2^L$, and a map $\zeta : \mathcal{P}_2^L \to \mathcal{P}_1^L$ which takes the j-th column K_j^2 of $\mathcal{P}_2^L$ to the $(j+1)$-th colume K_{j+1}^1 of $\mathcal{P}_1^L$ (both with an appropriate retraction); however, in Appendix A we prove the following:

Theorem 2 *For every pair of mappings $\psi : \mathcal{P}_1^L \to \mathcal{P}_2^L$, $\zeta : \mathcal{P}_2^L \to \mathcal{P}_1^L$ which make $\mathcal{P}_1^L$ and $\mathcal{P}_2^L$ k-close, with $k < L/4$, there exists a $p \in \mathcal{P}_2^L$ such that*

$$D(p, \psi \circ \zeta(p)) = L.$$

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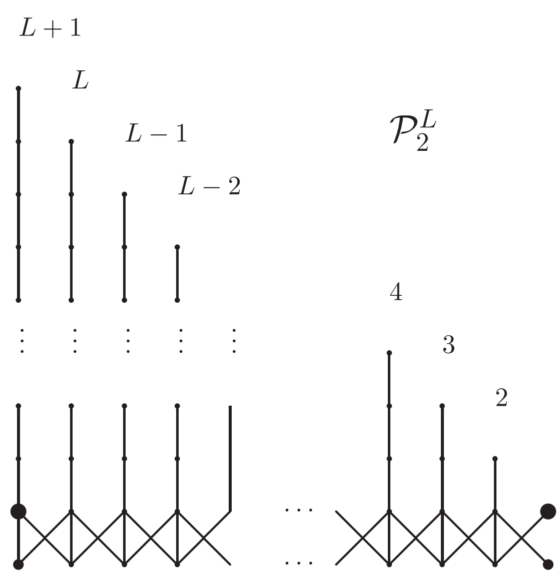

Figure 3: Example 3

Now choose $\alpha > 0$ and let $\epsilon_m = \frac{4\alpha}{4m+1}$, $L_m = 4m$, $m \in \mathbf{N} \setminus \{0,1\}$. Define for $i = 1, 2$ the discrete Lorentz spaces $\mathcal{P}_i^{\epsilon_m, L_m}$ by the same Hasse diagrams, but now suppose that every link has length ϵ_m. Arguments analogous to the ones around theorem 2 tell us that $\mathcal{P}_1^{\epsilon_m, L_m}$ and $\mathcal{P}_2^{\epsilon_m, L_m}$ are ϵ_m-close, but for every pair of mappings $\psi_m : \mathcal{P}_1^{\epsilon_m, L_m} \to \mathcal{P}_2^{\epsilon_m, L_m}$, $\zeta_m : \mathcal{P}_2^{\epsilon_m, L_m} \to \mathcal{P}_1^{\epsilon_m, L_m}$ which make $\mathcal{P}_1^{\epsilon_M, L_m}, \mathcal{P}_2^{\epsilon_m, L_m}$ ϵ- close, with $\epsilon < \frac{4m\alpha}{4m+1}$, there exists a $p \in \mathcal{P}_2^{\epsilon_m, L_m}$ such that

$$D_m(p, \psi_m \circ \zeta_m(p)) = \frac{16m\alpha}{4m+1} \, .$$

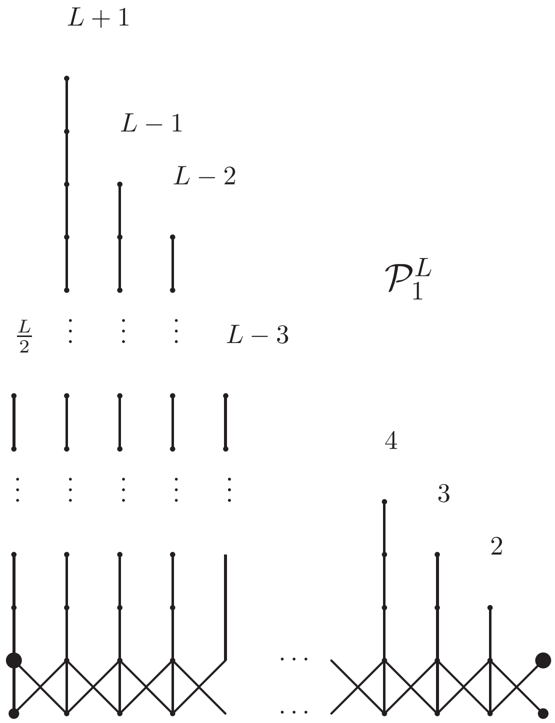

Figure 4: Example 3

However, $tdiam(\mathcal{P}_i^{\epsilon_m,L_m}) = 4\alpha$ for $i = 1, 2$ and $m > 1$. •

In other words, $\mathcal{P}_1^{\epsilon_m,L_m}$ and $\mathcal{P}_2^{\epsilon_m,L_m}$ are ϵ-close with $\epsilon \to 0$ as $m \to \infty$, but for every pair of maps ψ and ζ that realize the closeness condition, there exists a $p \in \mathcal{P}_2^{\epsilon_m,L_m}$ such that $D_m(p, \psi_m \circ \zeta_m(p)) \to 4\alpha$. This proves our claim that the qualitative uniformities defined by GH and GGH are inequivalent.

However, this does not prove yet that there exist GH Cauchy sequences which are not GGH. In fact, we show now that if a Lorentz space $(\mathcal{M}, d)$ is a limit space of a GH Cauchy sequence $\{(\mathcal{M}_i, d_i)\}_{i\in\mathbf{N}}$, then this sequence is GGH Cauchy and converges to the same limit (up to isometry). An intermediate result is the following.

Theorem 3 *Any isometry ψ on a Lorentz space $(\mathcal{M}, d)$ is a bijection.*

Proof:
We only have to show that ψ is a surjection since, obviously, it is an injection. Evidently, $D(\psi(p), \psi(q)) \geq D(p, q)$ for all $p, q \in \mathcal{M}$. Suppose we can find an open ball $B_D(r, \epsilon)$ which is not in $\psi(\mathcal{M})$; then $\psi^k(r) \notin B_D(\psi^l(r), \epsilon)$ for all $k > l$. Since $\mathcal{M}$ is compact, we may, by passing to a subsequence if necessary, assume that $\psi^l(r) \stackrel{l\to\infty}{\to} \psi^\infty(r)$. Hence, we arrive at the contradiction that $\psi^\infty(r) \notin B_D(\psi^\infty(r), \epsilon)$. This implies that all isolated points belong to $\psi(\mathcal{M})$. But then we have that

$$D(\psi(p), \psi(q)) = \sup_{r\in\mathcal{M}} \left| d(\psi(r), \psi(p)) + d(\psi(p), \psi(r)) - d(\psi(r), \psi(q)) - d(\psi(q), \psi(r)) \right| .$$

The rhs of this equation equals $D(p,q)$. This shows that ψ is surjective since all isometries of compact metric spaces are. $\bullet$

Before we prove the main result we still need the following.

Theorem 4 *Let $\{\psi_i \mid i \in \mathbf{N}_0\}$ be a set of $\frac{1}{i}$-isometries on $\mathcal{M}$. Then there exists a subsequence $\{\psi_{i_n}\}_{n\in\mathbf{N}}$ which uniformly converges in the strong sense to an isometry ψ.*

Proof:
As usual, let $\mathcal{C}$ be a countable dense subset of $\mathcal{M}$ and let $\{\psi_{i_n}\}_{n\in\mathbf{N}}$ be a subsequence such that $\psi_{i_n}(p) \stackrel{n\to\infty}{\to} \psi(p)$ for all $p \in \mathcal{C}$. It is easy to see that ψ has a unique extension to a D-isometry (and d isometry) using Theorem 3. The proof of Theorem 3 also implies that $\psi(\mathcal{C})$ is dense in $\mathcal{M}$. As a consequence, we have that for any $\epsilon > 0$ there exists a $k(\epsilon) > 0$ such that $\psi_{i_k}(\mathcal{C})$ is ϵ-dense in $\mathcal{M}$ for $k > k(\epsilon)$. Hence, $|D(\psi_{i_k}(p), \psi_{i_k}(q)) - D(p,q)| < \epsilon + \frac{2}{i_k}$ for $k > k(\epsilon)$ and for all $p, q \in \mathcal{M}$. This implies that

$$D(\psi(r), \psi_{i_k}(r)) \leq D(\psi(p), \psi_{i_k}(p)) + 2D(p,r) + \frac{2}{i_k} + \epsilon \ ,$$

for $k > k(\epsilon)$ and $p \in \mathcal{C}$. Since ϵ and p can be independently chosen arbitrarily close to 0 and r respectively, the result follows. $\bullet$

We are now in a position to prove the main result.

Theorem 5 *Let $\{(\mathcal{M}_i, d_i)\}_{i\in\mathbf{N}}$ be a GH Cauchy sequence of Lorentz spaces converging to a Lorentz space $(\mathcal{M}, d)$; then this sequence is GGH Cauchy and converges to the same limit space.*

Proof:
Choose $\delta > 0$; then Theorem 4 implies that there exists a $\gamma > 0$, such that if f is a γ-isometry, then there exists an isometry g such that

$$D(f(x), g(x)) < \tfrac{\delta}{2} \qquad \forall x \in \mathcal{M}.$$

Let $\psi_i : \mathcal{M}_i \to \mathcal{M}$ and $\zeta_i : \mathcal{M} \to \mathcal{M}_i$ be mappings which make $(\mathcal{M}_i, d_i)$ and $(\mathcal{M}, d)$ ϵ_i-close, where $\epsilon_i \stackrel{i\to\infty}{\to} 0$. Then the previous remark implies that for i large enough that $2\epsilon_i < \min\left\{\gamma, \frac{\delta}{2}\right\}$, there exists an isometry β_i such that

$$D(\beta_i(x), \psi_i \circ \zeta_i(x)) < \tfrac{\delta}{2} \qquad \forall x \in \mathcal{M} \ ,$$

or

$$D(x, \psi_i \circ \zeta_i \circ \beta_i^{-1}(x)) < \tfrac{\delta}{2} \qquad \forall x \in \mathcal{M} \ .$$

Hence

$$D_i(p_i, \zeta_i \circ \beta_i^{-1} \circ \psi_i(p_i)) \leq 2\epsilon_i + D(\psi_i(p_i), \psi_i \circ \zeta_i \circ \beta_i^{-1} \circ \psi_i(p_i)) \ ,$$

which implies that

$$D_i(p_i, \zeta_i \circ \beta_i^{-1} \circ \psi_i(p_i)) \leq 2\epsilon_i + \tfrac{\delta}{2} < \delta \; .$$

Hence, for i sufficiently large, ψ_i and $\zeta_i \circ \beta_i^{-1}$ make $(\mathcal{M}_i, d_i)$ and $(\mathcal{M}, d)$, (ϵ_i, δ)-close. •

This does not show that every GH Cauchy sequence is GGH, but one which is not GGH either has no sensible limit, or its limit is not "spatially compact". This last theorem also has as a consequence that the trivial extension of d_{GH} to the moduli space of isometry classes of Lorentz spaces is a metric.

4 Some properties of the strong metric

In this section, we study some properties of the strong metric (4) introduced in Ref [6], with the aim of understanding better its relationship with the Lorentz distance. It was remarked in Ref [6] that the topologies induced by these two distances are equivalent on globally hyperbolic spacetimes. A natural question to ask therefore is whether in fact the distances are equivalent, in the sense that the strong metric determines the Lorentz metric, up to time reversal (obviously, applying a "time reversal" $d'(x,y) = d(y,x)$ to a Lorentz distance produces a different Lorentz distance with the same associated strong metric). We shall only begin to deal with this question here, by giving an example which shows that the answer in general is no, but as a beginning in the study of this relationship, most of this section will be devoted to the simpler question, What is the shape of open ball $B_D(p, \epsilon)$ of radius ϵ around a point p in the strong metric?"

To start with, we "split" the strong metric D into two pseudodistances $D^{\pm}$ which will be useful later on, by defining

$$D^+(p,q) = \max_{r\in\mathcal{M}} \big| d_g(p,r) - d_g(q,r) \big|$$

and

$$D^-(p,q) = \max_{r\in\mathcal{M}} \big| d_g(r,p) - d_g(r,q) \big| \; ,$$

in terms of which D can be recovered as [7]

$$D(p,q) = \max \big\{ D^+(p,q), D^-(p,q) \big\} \; .$$

Notice that, although individually D^+ and D^- are *pseudo* metrics ($D^{\pm}(p,q) = 0$ does not necessarily imply that $p = q$), this limitation of D^+ (D^-) arises only for p and q both belonging to the future (past) boundary of $\mathcal{M}$. For example, clearly $D^+(p,q) = 0$ for all $p, q \in \partial_{\mathrm{F}}\mathcal{M}$, but if $p \notin \partial_{\mathrm{F}}\mathcal{M}$ and $q \in \partial_{\mathrm{F}}\mathcal{M}$, any $r \in I^+(p)$ gives $d_g(p,r) = |d_g(p,r) - d_g(q,r)| > 0$, and if both $p, q \notin \partial_{\mathrm{F}}\mathcal{M}$, the same is true with any $r \in I^+(p) \setminus I^+(q)$ (assuming $p \notin J^+(q)$, otherwise just switch p and q).

These remarks show that both $D^{\pm}$ are true distances on the interior of $\mathcal{M}$, and they also motivate us to try to locate the "distance-maximizing points", i.e., the points r which realize the maximum in the definition of those functions for given p and q. Such points will then allow us to control the non-locality in $D^{\pm}$ and D, which is what makes the study of their detailed properties more difficult than those of a distance arising from a positive-definite metric tensor.

Property *Given any two points p and q not both belonging to $\partial_{\rm F}\mathcal{M}$, any point r such that $D^+(p,q) = |d_g(p,r) - d_g(q,r)|$ is an element of $I^+(p) \triangle I^+(q)$,*[4] *and $I^+(r) \subset I^+(p) \cap I^+(q)$. A dual property holds for D^-.*

Proof:
Obviously, $r \in I^+(p) \cup I^+(q)$. Suppose $r \in I^+(p) \cap I^+(q)$ and, without loss of generality, assume that $d_g(p,r) > d_g(q,r)$. Let γ be a distance-maximizing geodesic from p to r; then γ cuts $E^+(q)$ in a point s. But then, the reverse triangle inequality implies that

$$\begin{aligned} d_g(p,r) - d_g(q,r) &= d_g(p,s) + d_g(s,r) - d_g(q,r) \\ &< d_g(p,s) \;=\; d_g(p,s) - d_g(q,s)\;, \end{aligned}$$

so r cannot be distance-maximizing. The second statement says that r belongs to the boundary of the light cone of p or q. Again, without loss of generality assume that $r \in I^+(p) \setminus I^+(q)$ (in this case, $p \notin \partial_{\rm F}\mathcal{M}$). Then if the statement is false we can extend the distance-maximizing geodesic from p to r to a new point r' such that $d_g(p,r') > d_g(p,r)$ but still $d_g(q,r') = 0$, contrary to the assumption. $\bullet$

Now, if $(\mathcal{M},g)$ contains no cut points, then r must belong to $\partial_{\rm F}\mathcal{M}$. Since suppose $r \in I^+(p) \setminus I^+(q)$, but does not belong to $\partial_{\rm F}\mathcal{M}$; then $I^+(r) \subset I^+(p) \cap I^+(q)$ implies that r belongs to $E^+(q)$. Let γ be the unique null geodesic from q to r, then moving r to the future along this null geodesic up to $\partial_{\rm F}\mathcal{M}$ keeps r out of $I^+(q)$, otherwhise the geodesic would have a cut point, which is contrary to the assumption. The following theorem is also valid when there are cut points, but then the statement can be made sharper.

As was remarked earlier [6], the ϵ-balls $B_D(p,\epsilon)$ are causally convex, in the sense that if $x, y \in B_D(p,\epsilon)$, then $A(x,y) \subset B_D(p,\epsilon)$. We now wish to find out more about those sets. To begin with, notice that

$$B_D(p,\epsilon) = B_{D^+}(p,\epsilon) \cap B_{D^-}(p,\epsilon)\;.$$

Then we have:

[4] For any two sets A and B, $A \triangle B$ stands for the symmetric difference $(A \setminus B) \cup (B \setminus A)$.

Theorem 6 *Let $(\mathcal{M}, g)$ be a spacetime with no cut points and choose a point $p \in \mathcal{M} \setminus \partial_{\mathrm{F}}\mathcal{M}$ and an $\epsilon > 0$ such that $K^+(p,\epsilon) \neq \emptyset$. Then the open "sphere" $B_{D^+}(p,\epsilon)$ of radius ϵ centred at p with respect to the pseudometric D^+ satisfies*

$$B_{D^+}(p,\epsilon) \subseteq \left[\bigcap_{x\in\mathcal{H}^+(p)} \big(\mathcal{O}^-(x,\epsilon)\big)^{\mathrm{c}}\right] \bigcap \left[\bigcap_{x\in\mathcal{F}^+(p,\epsilon)} I^-(x)\right],$$

where

- $K^+(x,\epsilon) = \{y \in \mathcal{M} \mid d_g(x,y) = \epsilon\}$*, the future ϵ-sphere centred at x,*
- $\mathcal{O}^-(x,\epsilon) = \{y \in \mathcal{M} \mid d_g(y,x) \geq \epsilon\}$*, the closed outer past ϵ-ball around x,*
- $\mathcal{H}^+(p) = E^+(p) \cap \partial_{\mathrm{F}}\mathcal{M}$*,*
- $\mathcal{F}^+(p,\epsilon) = K^+(p,\epsilon) \cap \partial_{\mathrm{F}}\mathcal{M}$*.*

The open sphere $B_{D^-}(p,\epsilon)$ defined with respect to the pseudometric D^- satisfies a similar inclusion property with all pasts and futures interchanged.

Proof:
Let $x \in B_{D^+}(p,\epsilon)$; then x must be chronologically connected to all points in $\mathcal{F}^+(p,\epsilon)$. For, suppose there exists a point $y \in \mathcal{F}^+(p,\epsilon)$ such that $x \notin I^-(y)$; then $d_g(p,y) - d_g(x,y) = \epsilon$, which is a contradiction. On the other hand, x cannot belong to $\mathcal{O}^-(z,\epsilon)$ for any $z \in \mathcal{H}^+(p)$, since otherwise

$$d_g(x,z) - d_g(p,z) \geq \epsilon\ ,$$

which is impossible. •

The following figure 2 shows that the above inclusion can be an equality. The universe is $(S^1 \times [0,1]\,, -\mathrm{d}t^2 + \mathrm{d}\theta^2)$ and the shaded area represents $B_{D^+}(p,\epsilon)$ for ϵ sufficiently small.

Obviously, a dual statement holds for $B_{D^-}(p,\epsilon)$. Putting together these two bounds on $B_{D^\pm}$, we also notice that for B_D we can write down a simpler, but weaker bound

$$B_D(p,\epsilon) \subseteq \bigcap_{x\in\mathcal{F}^-(p,\epsilon),\, y\in\mathcal{F}^+(p,\epsilon)} A(x,y)\ ,$$

which says that $B_D(p,\epsilon)$ is contained in all of the "long skinny" (for small ϵ) Alexandrov neighborhoods defined by pairs of points on the past and future boundaries of M which are "almost null related to p".

Concerning the second question posed at the beginning of this section, we notice that the strong metric D does not determine the Lorentz distance d up to time reversal for discrete Lorentz spaces.

Example 4

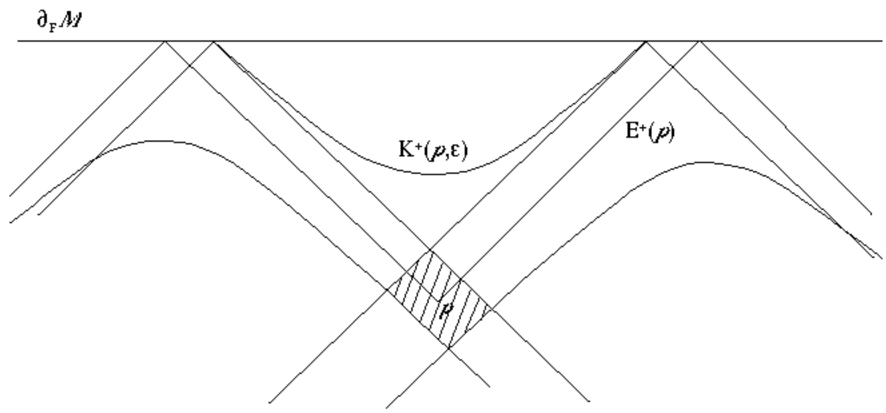

Figure 5: Illustration of theorem 2

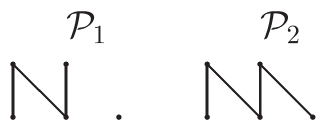

Figure 6: Hasse diagrams

Consider the causal sets $\mathcal{P}_1$ and $\mathcal{P}_2$ defined by the Hasse diagrams below. Clearly $d_{GH}((\mathcal{P}_1, d_1), (\mathcal{P}_2, d_2)) = 1$, while $d_{GH}((\mathcal{P}_1, D_1), (\mathcal{P}_2, D_2)) = 0$ since in both posets we have $D_i(p, q) = 1$ for all distinct p and q. Moreover, $D_1^+ \neq D_2^+$ while $D_1^- = D_2^-$. This clearly shows that convergence in the strong metric is not sufficient to guarantee convergence of the Lorentz metrics at least as far as Lorentz spaces are concerned. $\bullet$

5 Causal properties of the limit space

In this section, we study further causal properties of the limit spaces of sequences of Lorentzian globally hyperbolic spacetimes that were introduced in Ref [7]. The most daunting problem is how to define a causal relation on the degenerate regions. First of all, one might ask whether this is a physically meaningful question, in the sense of whether any relation we may define can give rise to sensible causal curves along which "particles" travel, given the fact that this causal structure may be very non-local and depend on what the limit space is like both to the future and to the past of the degenerate region. We cannot stress the word *global* enough; for example, even a small change in the future lightcones in a "distant" region according to one set of observers could be spotted by the strong metric in a "tiny" Alexandrov neighborhood which is "boost equivalent" to a neighborhood that is more localized for a set of observers used to realize the bound in the definition of the strong distance.[5] Hence, it is not entirely clear whether defining a causal relation is physically meaningful or not. However, it is for sure a quite interesting mathematical problem and it shall be treated as such in the rest of this section.

In the sequel we will call a proposal for the causal relation *good* if, roughly speaking, it coincides on limit spaces of arbitrary GGH Cauchy sequences of conformally equivalent spacetimes with the causal relations defined by the elements of these sequences. To start with, we give an example which shows that the closure of the space of discrete Lorentz spaces contains spaces whose causal behaviour differs significantly from the kind of limit spaces already considered before [7].[6]

Example 5

Suppose that $L > 2$ and let $(\mathcal{P}_n^L, d_n)$ be a discrete Lorentz space defined by the elements $\left\{\frac{iL}{n} \mid i = 0 \dots n\right\}$ and $d_n(p, q) = \max\{0, q - p - 1\}$. It is easy to check that d_n defines a Lorentz distance. Moreover $(\mathcal{P}_n^L, d_n) \overset{n \to \infty}{\to} ([0, L], d)$ where, evidently, $d(p, q) = \max\{0, q - p - 1\}$. Obviously, we want the causal relation to be the ordinary order relation on $[0, L]$. Hence, any pair of timelike related points in the limit space can *only* be connected by a causal curve γ which is nowhere timelike in the sense that for any t on γ, $0 < s - t < 1$ implies that $d(t, s) = 0$. The strong metric $D(t, s)$ between points $t < s$ equals $s - t$ unless $0 < t < 1$ and $L - 1 < s < L$, in which case it equals $\max\{L - t, s\} - 1$. Hence, locally, D is the path metric defined by the standard line element $\mathrm{d}t^2$. Conclusion: although the limit space has a manifold structure, the Lorentzian distance is far from being derivable from a tensor. •

In this example, we defined the causal relation using our intuition. Since we are looking for a general prescription, we might postulate something like $p \leq q$ iff $I^+(q) \subset I^+(p)$ and $I^-(p) \subset I^-(q)$. However, as mentioned in Ref [7], this is not sufficient. Therefore, let us start by defining the causal relation on the subset of $\mathcal{M}$ that we are most familiar with. As announced at the end of [7], a good candidate for $\leq$ on the closure $\overline{\mathcal{TCON}}$ of the timelike continuum is the K^+ causal relation defined by Sorkin and Woolgar [9], i.e.,

Definition 4 *K^+ is the smallest topologically closed partial order in $\mathcal{M} \times \mathcal{M}$ containing I.*

We need to remark here that, as mentioned in [9], K^+ can be defined as the relationship $\prec$ built by transfinite induction with the following procedure:

- $\prec^0 = I^+$;

[5] Notice that if p is a point of the past boundary, then $x \to D^-(p, x)$ is a time function.
[6] A similar kind of limit space was communicated to the second author by Rafael Sorkin.

- $\prec^{\alpha} = \bigcup_{\beta<\alpha} \prec^{\beta}$ if α is a limit ordinal;
- $\prec^{\beta+1}$ is constructed from $\prec^{\beta}$ by adding pairs which are implied either by transitivity or by closure.

Since $\mathcal{M} \times \mathcal{M}$ has at most $2^{\aleph_0}$ elements, the procedure must terminate at an ordinal[7] with cardinality less than or equal to $2^{\aleph_0}$. The following example illustrates that it can run up to an ordinal of cardinality $\aleph_0$.

Example 6

Let $\mathbf{N} = \aleph_0$, $\mathcal{P} = \left\{ w^i,\, x^j_k,\, y^j_k,\, z^j \mid i \in \mathbf{N}+1 \text{ and } j,k \in \mathbf{N} \right\}$, and construct the discrete Lorentz space $(\mathcal{P}, d)$ by defining

$$\begin{aligned} d(w^i, z^j) = d(x^i_k, z^j) &= \frac{1}{(j+1)^2} \;, \\ d(y^i_k, z^{j+1}) &= \frac{1}{(j+2)^2} \;, \qquad d(x^i_j, y^i_j) = \frac{1}{(j+1)(i+1)^2} \;, \end{aligned} \tag{5}$$

for all k and $i \leq j$ in $\mathbf{N}$; all other distances are calculated from these values by taking the maximum over all "timelike" chains. From these data, it is easy to calculate the strong distance:

$$\begin{aligned} D(w^i, x^i_j) = D(w^{i+1}, y^i_j) &= \frac{1}{(j+1)(i+1)^2} \\ D(x^i_j, y^i_j) &= \frac{1}{(i+1)^2} \;. \end{aligned} \tag{6}$$

We now start our program: $\prec^1$ is the closure of I^+. Obviously, the new relations induced by this procedure are $w^i \prec^1 w^{i+1}$. $\prec^2$ is constructed from $\prec^1$ by adding pairs which are implied by transitivity and closure: this results in $w^i \prec^2 w^{i+2}$ for all $i \in \mathbf{N}$ and $w^{\mathbf{N}} \prec^2 w^{\mathbf{N}}$. The reader may easily check that at stage $n > 2$, $\prec^n \,=\, \prec^{n-1} \cup \left\{ (w^i, w^j) \mid i + 2^{n-1} < j \leq i + 2^n \in \mathbf{N} \right\}$. Hence,

$$\prec^{\mathbf{N}} = I^+ \cup \left\{ (w^i, w^j) \mid i < j \in \mathbf{N} \right\} \cup \left\{ (w^{\mathbf{N}}, w^{\mathbf{N}}) \right\} .$$

But this relation is not closed yet, and

$$\prec^{\mathbf{N}+1} = I^+ \cup \left\{ (w^i, w^j) \mid i < j \in \mathbf{N}+1 \right\} \cup \left\{ (w^{\mathbf{N}}, w^{\mathbf{N}}) \right\} .$$

So the procedure stops at the $(\mathbf{N}+1)$-st step and the cardinality of $\mathbf{N}+1$ is $\aleph_0$. •

Since K gives in general more information than I,[8] we might hope that adding

[7] For more information about ordinals and transfinite induction, see [10].

[8] One can construct Lorentz spaces where $K^+(q) \subseteq K^+(p)$ and $K^-(p) \subseteq K^-(q)$ do *not* imply that $I^+(q) \subseteq I^+(p)$ and $I^-(p) \subseteq I^-(q)$ and vice versa. However, $K^+(q) \subseteq K^+(p)$ and $K^-(p) \subseteq K^-(q)$ do imply that $I^+(q) \subseteq I^+(p)$ and $I^-(p) \subseteq I^-(q)$ for Lorentz spaces $(\mathcal{M}, d)$ satisfying the following division property:

$$\forall p \ll q,\; \exists r :\; p \ll r \ll q \;.$$

the conditions $K^+(q) \subset K^+(p)$ and $K^-(p) \subset K^-(q)$ in order for $p \leq q$ would lead to a satisfying definition. Unfortunately it does not, as illustrated by the following example.

Example 7

To simplify the notation in this discussion, define a candidate relation $\mathcal{R}$ between p, $q \in \mathcal{M} \setminus \overline{\mathcal{TCON}}$ as follows:

$$\begin{aligned} p\,\mathcal{R}\,q \quad \Leftrightarrow \quad & K^+(q) \subseteq K^+(p)\ ,\ K^-(p) \subseteq K^-(q)\ , \\ & I^+(q) \subseteq I^+(p) \text{ and } I^-(p) \subseteq I^-(q)\ . \end{aligned}$$

Figure 7 shows a Lorentz space which is part of the cylinder, with degenerate regions indicated by the shaded areas. The points p and q are such that $p\,\mathcal{R}\,q$, while clearly we do not want that $p \leq q$. However, there exists no curve γ between p and q satisfying the condition that $\gamma(t)\,\mathcal{R}\,\gamma(s)$ for all $t \leq s$. •

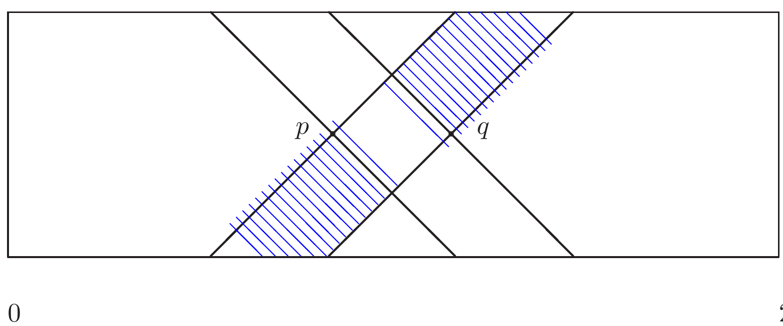

Figure 7: Example 6, cylinder universe with degenerate regions.

The above example is very unfortunate, in the sense that it shows that using only relations between points derived from the chronological partial order on zero-dimensional objects is not sufficient for obtaining a satisfactory causal relation. However, it also suggests that the following definition might be more successful.

Definition 5 *Define a partial order $\mathcal{P}$ as $p\,\mathcal{P}\,q$ iff there exists a continuous curve $\gamma : [0,1] \to \mathcal{M}$ from p to q such that $\gamma(t)\,\mathcal{R}\,\gamma(s)$ for all $0 \leq t \leq s \leq 1$. Finally, define $\leq_d$ on $\mathcal{M} \setminus \overline{\mathcal{TCON}}$ as the smallest topologically closed transitive relationship extending $\mathcal{P}$ and I^+. It is easy to see that $\leq_d$ is compatible with K^+, i.e., $p \leq_d q$ and $q \in K^-(r)$ imply that $p \in K^-(r)$ and vice versa.*

The following example in three dimensions shows that this definition also has its limitations. However, in two dimensions it does work, as is proven in Theorem 7.

Example 8

Consider the three-dimensional cylinder universe $(\mathrm{S}^2 \times [-1,1], -\mathrm{d}t^2 + \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)$, and the spacelike geodesic in it defined by $\gamma : [0, \frac{1}{4}] \to \mathrm{S}^2 \times [-1,1] : s \to \gamma(s) = (\theta_0 + s, \phi_0, 0)$. Take the limit $(\mathrm{S}^2 \times [-1,1], d)$ over a suitable sequence of conformally equivalent metrics, with conformal factors which converge to zero on thin specific (see figure 8) open neighborhoods of $J^+(\gamma(s)) \setminus J^+(\gamma(t))$ and $J^-(\gamma(t)) \setminus J^-(\gamma(s))$ which are subsets of $J^+(\gamma(t))^{\mathrm{c}}$ and $J^-(\gamma(s))^{\mathrm{c}}$, respectively, for all $t < s$.[9] It is not difficult to see that for any point q belonging to the

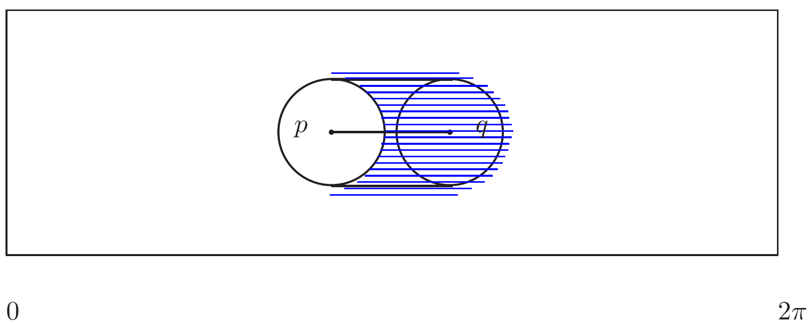

Figure 8: Example 7, lightcones at fixed time $t > 0$ drawn on a Euclidean coordinate patch. The shading indicates the degenerate regions.

degenerate region, either $I_d^+(q) \cap I_d^+(\gamma(0)) \neq \emptyset$ or $I_d^-(q) \cap I_d^-(\gamma(1/4)) \neq \emptyset$. Using this, it is easy to see that for any two points p and q belonging to the degenerate region, we have that $I_d^-(p) \neq I_d^-(q)$ or $I_d^+(p) \neq I_d^+(q)$. •

Theorem 7 *Let* $(\mathcal{M}, g)$ *be a* two-dimensional *globally hyperbolic interpolating spacetime wich is isometrically embeddable in the interior of an interpolating spacetime* without cut points *and suppose* Ω_i *is a sequence of positive* C^∞ *functions on* $\mathcal{M}$ *such that* $\left| d_{\Omega_i^2 g}(p,q) - d_{\Omega_j^2 g}(p,q) \right| < \frac{1}{i}$ *for all* $j > i > 0$ *and* $p, q \in \mathcal{M}$. *Denote by* $(\mathcal{N}, d)$ *the GGH limit space* and *suppose that* $\mathcal{M} = \mathcal{N}$, *i.e., no points get identified. Then, one has that* $p \leq_g q$ *iff* $p \leq_d q$ *for all* $p, q \in \mathcal{M}$.

[9] The relation J^+ denotes here the usual causal relation defined by the line element $-\mathrm{d}t^2 + \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$.

Note: It is easy to see that if $(\mathcal{M}, g)$ were allowed to have cut points, then the theorem would not be valid anymore. Readers should convince themselves of this by making a drawing on the two dimensional cylinder universe.

Proof:
First notice that the strong topology defined by D coincides with the manifold topology.
$\Rightarrow$) We show that *any* g-causal curve γ is an $\mathcal{R}$-causal curve. Clearly, $I^+_d(\gamma(s)) \subseteq I^+_d(\gamma(t))$ and $I^-_d(\gamma(t)) \subseteq I^-_d(\gamma(s))$ for all $t < s$, which proves the basis of induction. Let $\alpha = \beta + 1$ and suppose that $t < s$ implies that $\gamma(s) \prec^{\beta} y \Rightarrow \gamma(t) \prec^{\beta} y$. Obviously, if $\gamma(s) \prec^{\beta} y \prec^{\beta} z$ then $\gamma(t) \prec^{\beta} y \prec^{\beta} z$. So, suppose that there exist sequences $\{q_n\}_{n \in \mathbf{N}}$ and $\{y_n\}_{n \in \mathbf{N}}$, converging to $\gamma(s)$ and y respectively, such that $q_n \prec^{\beta} y_n$ for any n; then, since $(\mathcal{M}, g)$ has no cut points, there exists a sequence $\{p_n\}_{n \in \mathbf{N}}$ converging to $\gamma(t)$ with $p_n \prec_g q_n$. The induction hypothesis then implies that $p_n \prec^{\beta} y_n$ for all n, which proves the claim.
$\Leftarrow$) We have to show that g-spacelike events cannot be connected by an $\mathcal{R}$-causal curve. Suppose that p and q are such events, and that $\gamma : [0, 1] \rightarrow \mathcal{M}$ is an $\mathcal{R}$-causal curve between them. Without loss of generality, we may assume that γ is spacelike related to p and q, in the sense that $\gamma(t)$, p and q are g-spacelike events for all $t \in (0, 1)$.[10] Moreover, we may assume that γ is a subset of a convex open neighborhood $\mathcal{U}$ on which g is conformally flat. By using the nonexistence of cut points, one can deduce that the set $\mathcal{S} \subset \mathcal{M}$ bounded by the two right g null geodesics containing γ is entirely degenerate, as shown in figure 9.[11] Any two points in $\mathcal{S} \cap \mathcal{U}$ belonging to any left g null geodesic have the same chronological relations. •

The results of Example 8 and Theorem 7 are quite discouraging, since any good definition seems to depend upon some notion of *dimension* of the Lorentz space. One could try to make the definition more restrictive so that it would be possible to reproduce a result analogous to Theorem 7 in all dimensions. It seems to us that "local"[12] ideas won't work; we are working on other ideas concerning a

[10] Note that γ cannot intersect $E^-(p)$ nor $E^+(q)$, because this would violate the assumtion that the limit space equals $\mathcal{M}$. By continuity, there exists a maximal t and a minimal $s > t$ such that $\gamma(t) \in E^+(p)$, but $\gamma(u) \notin E^+(p)$ for all $u > t$ and s is the minimal number larger than t such that $\gamma(s) \in E^-(q)$.

[11] Choose $\gamma(t), t \in (0, 1)$. Then there exists an open neighborhood $\mathcal{O}$ of $\gamma(t)$ such that for all $r \in \mathcal{O}$ which are g spacelike to the left or in the g chronological past of $\gamma(t)$, we have that the left, future oriented, null geodesic starting at r does *not* intersect the future oriented, right null geodesic starting at $\gamma(t)$ otherwise $\gamma(t)$ would have a cut point in any extension of $(\mathcal{M}, g)$. Hence for all $s < t$ such that $\gamma(s) \in \mathcal{O}$ is such point r, we have that any point in $J^+(\gamma(t))$ to the right of the right null geodesic emanating from $\gamma(s)$ belongs to the degenerate area. A similar argument is valid for the past with left and right switched. Using this for all t leads to picture 9.

[12] Local in the sense that one uses properties of local congruences of curves between neighborhoods of points. One such idea would be to construct the following kind of definition: define the relation $p \, \mathcal{Q} \, q$ iff there exist neighborhoods $\mathcal{U}$, $\mathcal{V}$ of p and q, respectively, and a mapping $\psi : \mathcal{U} \times [0, 1] \rightarrow \mathcal{M}$ such that

- $\psi_t : \mathcal{U} \rightarrow \mathcal{M} : r \rightarrow \psi(r, t)$ is a homeomorphism for any t and $\psi_1(\mathcal{U}) = \mathcal{V}$.

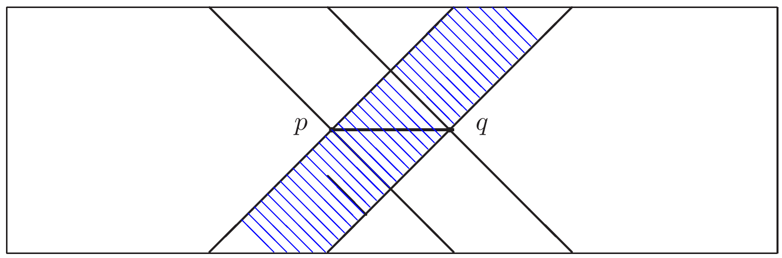

Figure 9: Proof of theorem 7, conflict with the T_0 property.

promising definition, but we do not have any proof yet.

The rest of this section is devoted to proving that the limit space $(\mathcal{M}, d)$ of a $\mathcal{C}^+_\alpha$ and $\mathcal{C}^-_\alpha$ sequence $\{(\mathcal{M}_i, d_i)\}_{i\in\mathbf{N}}$ of path metric[13] Lorentz spaces is a path metric Lorentz space. Strictly speaking, we should still define the $\mathcal{C}^\pm_\alpha$ properties for general Lorentz spaces $(\mathcal{M}, d)$. Looking at the definition in the Intermezzo of section 3 of Ref [7], the reader can see that this boils down to defining the future and past boundaries of $(\mathcal{M}, d)$. Obviously, $\partial_{\mathrm{P}}\mathcal{M}$ is the set of points p such that $I^-(p) = \emptyset$, and $\partial_{\mathrm{F}}\mathcal{M}$ is defined dually.

Property: The $\mathcal{C}^+_\alpha$ property implies that the interior of $\partial_{\mathrm{P}}\mathcal{M}$ is empty and, likewise, the $\mathcal{C}^-_\alpha$ property implies that the interior of $\partial_{\mathrm{F}}\mathcal{M}$ is empty.

Proof:
We will only prove the first claim. Notice that $\partial_{\mathrm{P}}\mathcal{M}\cap\partial_{\mathrm{F}}\mathcal{M}$ contains at most one point. Let $p \in \partial_{\mathrm{P}}\mathcal{M} \setminus \partial_{\mathrm{F}}\mathcal{M}$ and $\epsilon > 0$ be such that $B_D(p, \epsilon) \subset \partial_{\mathrm{P}} > \mathcal{M} \setminus \partial_{\mathrm{F}}\mathcal{M}$. Then $d(p, r) = 0$ for all $r \in B_D(p, \epsilon)$, which is impossible by the $\mathcal{C}^+_\alpha$ property. •

As a consequence, we have that for a Lorentz space $(\mathcal{M}, d)$ satisfying the $\mathcal{C}^+_\alpha$ and $\mathcal{C}^-_\alpha$ property, $\overline{\mathcal{TCON}} \cup (\partial_{\mathrm{P}}\mathcal{M} \cap \partial_{\mathrm{F}}\mathcal{M}) = \mathcal{M}$.[14] Note that the second term on the left-hand side of this equality only needs to be accounted for iff $\partial_P\mathcal{M}\cap\partial_{\mathrm{F}}\mathcal{M}$

- $\psi_r : [0, 1] \to \mathcal{M} : t \to \psi(r, t)$ defines a $\mathcal{R}$-causal curve for any $r \in \mathcal{U}$.

$\leq_d$ is then defined as the smallest topologically closed transitive relation encompassing $\mathcal{Q}$ and I^+. Again it is not difficult to construct a counterexample similar to example 7.

[13] For a precise definition of a path metric Lorentz space, see Definition 7.

[14] For example, let $p \in \partial_{\mathrm{P}}\mathcal{M}\setminus\partial_{\mathrm{F}}\mathcal{M}$; then for $\epsilon > 0$ sufficiently small, we have that $B_D(p, \epsilon)\cap \partial_{\mathrm{F}}\mathcal{M} = \emptyset$. By the $\mathcal{C}^+_\alpha$ property, there exists an $r \in \overline{B_D(p, \frac{\epsilon}{2})}$ such that $d(p, r) = \alpha(\frac{\epsilon}{2})$. Hence,

$$D(r, \partial_{\mathrm{P}}\mathcal{M}), D(r, \partial_{\mathrm{F}}\mathcal{M}) \geq \alpha\,(\epsilon/2)\ ,$$

which implies, by the $\mathcal{C}^+_\alpha$ and $\mathcal{C}^-_\alpha$ properties, that $r \in \mathcal{TCON}$.

is an *isolated* point. Hence, the causal relation on such space is the K^+ relation.

First, we give an example of spacetime with the $\mathcal{C}^{\pm}_{x^2/2}$ property.

Example 9

Consider again the cylinder universe $\mathcal{CYL} = (\mathrm{S}^1 \times [0,1], -\mathrm{d}t^2 + \mathrm{d}\theta^2)$. We argue that $\mathcal{CYL}$ belongs to the category defined by $\alpha : \mathbf{R}^+ \to \mathbf{R}^+ : x \to \frac{1}{2}x^2$. Since SO(2) is the isometry group of d, it is sufficient to prove the assertion for points with a fixed spatial coordinate, say $\theta = \pi$. Let $1 \geq \tilde{t} > t \geq 0$; then it is easy to prove that

$$D((\pi, t), (\pi, \tilde{t})) = \sqrt{\tilde{t} - t}\, \max\left\{\sqrt{\tilde{t} + t}, \sqrt{2 - (t + \tilde{t})}\right\},$$

where D is the strong metric on $\mathcal{CYL}$. Hence

$$d((\pi, t), (\pi, \tilde{t})) \leq D((\pi, t), (\pi, \tilde{t}))^2 \leq 2\, d((\pi, t), (\pi, \tilde{t})) \ ,$$

which proves the assertion. •

Before we proceed, we should define causal curves γ and lengths thereof.[15]

Definition 6 *Let $(\mathcal{M}, d)$ be a Lorentz space, assume that $a < b$ and let $\gamma : [a, b] \to \mathcal{M}$ be a continuous mapping (with respect to the strong topology) such that for all $a \leq t < s \leq b$, $\gamma(t) \leq \gamma(s)$ ($\gamma(t) \ll \gamma(s)$); then γ is a basic causal (timelike) curve. Now let $a < b$, $c < d$ and $\gamma_1 : [a, b] \to \mathcal{M}$, $\gamma_2 : [c, d] \to \mathcal{M}$ be causal curves such that $\gamma_2(c) = \gamma_1(b)$; we define the concatenation $\gamma_2 \circ \gamma_1$ of γ_2 with γ_1 as the basic causal curve $\gamma_2 \circ \gamma_1 : [a, b + d - c] \to \mathcal{M}$ such that*

$$\gamma_2 \circ \gamma_1(t) = \begin{cases} \gamma_1(t) & \text{if } a \leq t \leq b \\ \gamma_2(t + c - b) & \text{if } b \leq t \leq b + d - c. \end{cases}$$

A (countably infinite) concatenation of basic causal curves is a causal curve.

The length $L(\gamma)$ of a basic causal curve $\gamma : [a, b] \to \mathcal{M}$ is defined as

$$L(\gamma) = \inf_{\Delta} \sum_{i=0}^{|\Delta|-1} d(\gamma(t_i), \gamma(t_{i+1})) \ ,$$

where $\Delta = \{t_i \mid a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b\}$ is a partition of $[a, b]$. Obviously,

$$L(\gamma_2 \circ \gamma_1) = L(\gamma_1) + L(\gamma_2) \ .$$

Now, we are able to give the definition of a path metric Lorentz space.

Definition 7 *$(\mathcal{M}, d)$ is a path metric lorentz space iff for any $p \leq q$ there exists a causal curve γ from p to q such that $L(\gamma) = d(p, q)$.*

[15] The reader can find similar definitions in Ref [11].

In case the causal relation coincides with the K^+ relation, we can prove that $(\mathcal{M}, d)$ is a path metric space iff for any $p \ll q$, there exists a distance-realising (K^+) causal curve from p to q. We need to introduce the Vietoris topology on the set $2^{(\mathcal{M},D)}$ of all closed, non-empty subsets of $(\mathcal{M}, D)$ for which a *sub-basis* is given by the sets $\mathcal{B}(\mathcal{M}, \mathcal{O})$ and $\mathcal{B}(\mathcal{O}, \mathcal{M})$. The former are sets with as members closed sets which meet the open set $\mathcal{O}$, the latter consists of the closed subsets of $\mathcal{O}$. It is known that $2^{(\mathcal{M},D)}$ equipped with the Vietoris topology is compact [9]; in that reference, it is also proven using topological arguments only that the Vietoris limit of a sequence of K^+-causal curves is a K^+-causal curve.

Theorem 8 *Let $(\mathcal{M}, d)$ be a Lorentz space; then $(\mathcal{M}, d)$ is a path metric space with respect to the K^+ relation iff for any $p \ll q$, there exists a distance-realising K^+-causal curve from p to q.*

Proof:
We only have to prove that the latter implies the former, the other way around being obvious. We shall once more proceed by transfinite induction, using as induction hypothesis H_α the statement that $p \prec^\alpha q$ implies that there exists a distance-realising K^+-causal curve from p to q. The basis of induction is nothing else but our assumption. Hence, let $\alpha = \beta + 1$ and assume H_β is valid. If $p \prec^\beta q \prec^\beta r$, then there exists a K^+-causal curve from p to r, by the induction hypothesis and concatenation, which is obviously distance-maximising. So assume that there exist sequences $\{p_n\}_{n\in\mathbf{N}}$ and $\{q_n\}_{n\in\mathbf{N}}$ converging to p and q, respectively, and that $p_n \prec^\beta q_n$ for all $n \in \mathbf{N}$. Then, H_β implies that there exist K^+-causal curves γ_n from p_n to q_n. We may assume that, by passing to a subsequence if necessary, $\{\gamma_n\}_{n\in\mathbf{N}}$ converges in the Vietoris topology to a (distance-maximising) K^+-causal curve connecting p with q. •

Before we turn to the study of properties of the space of causal curves between two points, it is necessary to look at some properties of causal curves with respect to the strong metric D.

Example 10

We show that the D-length of a compact, basic causal curve is in general infinity. Obviously, the way to define the D-length, $DL(\gamma)$, of a basic causal curve $\gamma : [a, b] \to \mathcal{M}$ is

$$DL(\gamma) = \sup_{\Delta} \sum_{i=0}^{|\Delta|-1} D(\gamma(t_i), \gamma(t_{i+1})) \ ,$$

where, as before, $\Delta = \{t_i \mid a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b\}$ is a partition of $[a, b]$. Returning to Example 1, to prove that the length of the interval $\{(\pi, t) \mid 0 \leq t \leq 1\}$ equals ∞ we choose $\Delta_n = \left\{1 - \frac{1}{k} \mid k = 1 \ldots n\right\} \cup \{1\}$; for $k > 2$

$$D((\pi, t_k), (\pi, t_{k+1})) = D^-((\pi, t_k), (\pi, t_{k+1})) = \frac{\sqrt{2k^2 - 1}}{k\,(k+1)} > \frac{1}{k}$$

and the sum diverges, which proves the claim. $\bullet$

Since the above example shows that the D-length of a causal curve is a meaningless concept, we have to come up with some other way to divide a causal curve into smaller pieces. As a starter, we mention the following result.

Theorem 9 *Let $\gamma : [a, b] \to \mathcal{M}$ be any basic causal curve in an interpolating spacetime $(\mathcal{M}, g)$. Then there exists no $t \in (a, b)$ such that $D(\gamma(a), \gamma(t)) = D(\gamma(t), \gamma(b)) = \frac{1}{2} D(\gamma(a), \gamma(b))$, i.e., γ has no D-midpoint.*

Proof:
We show that for all $t \in (a, b)$: $D(\gamma(a), \gamma(b)) < D(\gamma(a), \gamma(t)) + D(\gamma(t), \gamma(b))$. Assume that the point r, which realizes $D(\gamma(a), \gamma(b))$, belongs to $I^+(\gamma(a)) \setminus I^+(\gamma(b))$; the case where r belongs to $I^-(\gamma(b)) \setminus I^-(\gamma(a))$ is identical and is left as an exercise to the reader. Hence,

$$D(\gamma(a), \gamma(b)) = d(\gamma(a), r) = (d(\gamma(a), r) - d(\gamma(t), r)) + d(\gamma(t), r) \; .$$

Both terms on the rhs are non-negative, and bounded by $D(\gamma(a), \gamma(t))$ and $D(\gamma(t), \gamma(b))$ respectively. The first term can only realize $D(\gamma(a), \gamma(t))$ if $r \in I^+(\gamma(a)) \setminus I^+(\gamma(t))$. But in that case $d(\gamma(t), r) = 0$, which concludes the proof. $\bullet$

Hence, we define the following concept of division of a causal curve.

Definition 8 *Let $\gamma : [a, b] \to \mathcal{M}$ be any basic causal curve, and denote $\delta = D(\gamma(a), \gamma(b))$. The division of γ is the set of points $\gamma^{1/2} = \{p_i \mid i = 0 \ldots k\}$ such that $\gamma(a) = p_0 \leq p_1 \prec \ldots \prec p_{k-1} \prec p_k = \gamma(b)$, $D(p_i, p_{i+1}) = \frac{\delta}{2}$, $\forall i : 0 \ldots k-2$, $\frac{\delta}{4} \leq D(p_{k-1}, p_k) < \frac{3\delta}{4}$ and there exists no point q such that $p_{k-1} \prec q \prec \gamma(b)$ with $D(p_{k-1}, q) = \frac{\delta}{2}$ and $\frac{\delta}{4} \leq D(q, \gamma(b)) < \frac{3\delta}{4}$. Such finite number $2 \leq k = |\gamma^{1/2}| - 1$ exists, since γ is continuous with respect to the strong topology.*

This new concept facilitates the proof of the final theorem.

Theorem 10 *The limit space $(\mathcal{M}, d)$ of a GGH $\mathcal{C}^+_\alpha$ and $\mathcal{C}^-_\alpha$ Cauchy sequence $\{(\mathcal{M}_i, d_i)\}_{i \in \mathbf{N}}$ of path metric Lorentz spaces, is a path metric Lorentz space.*

Proof: According to Theorem 8 and the arguments preceding it, we only have to prove that $p \ll q$, $p, q \in \mathcal{M}$, implies that there exists a K^+ causal curve connecting p with q. Let $\psi_i : \mathcal{M}_i \to \mathcal{M}$ and $\zeta_i : \mathcal{M} \to \mathcal{M}_i$ be mappings under which $(\mathcal{M}_i, d_i)$ and $(\mathcal{M}, d)$ are (ϵ_i, ϵ_i)-close, where $\epsilon_i \stackrel{i \to \infty}{\to} 0$. Choose $p, q \in \mathcal{M}$ such that $d(p, q) > 0$. Choose $\epsilon < \frac{1}{8} d(p, q)$ and choose i sufficiently large, such that $\epsilon_i < \alpha(\epsilon)$. Hence, $|d_i(\zeta_i(p), \zeta_i(q)) - d(p, q)| < \epsilon_i$ and $|D_i(\zeta_i(p), \zeta_i(q)) - D(p, q)| < 4\,\epsilon_i$. Let γ_i, be a geodesic from $\zeta_i(p)$ to $\zeta_i(q)$ and consider $\gamma_i^{1/2} = \left\{p^i_s \mid s = 0 \ldots k_i\right\}$. Assume that s^i is the largest number such that $d_i(p^i_{s^i+1}, \zeta_i(q)) > \frac{1}{2} d_i(\zeta_i(p), \zeta_i(q))$. Then, for all $s \leq s^i$, choose r^i_{s+1} such that $\zeta_i(q) \gg_i r^i_{s+1} \gg_i p^i_{s+1}$, $D_i(p^i_{s+1}, r^i_{s+1}) \leq \epsilon$ and $d_i(p^i_{s+1}, r^i_{s+1}) = \alpha(\epsilon)$. This is possible, since the $\mathcal{C}^+_\alpha$ property is valid and since $\frac{1}{2} d_i(\zeta_i(p), \zeta_i(q)) >$

$\frac{1}{2}\left[d(p,q)-\alpha(\epsilon)\right] > \frac{7}{16}\,d(p,q)$. If $d_i(p^i_{s^i+1},p^i_{s^i+2}) < \alpha(\epsilon)$, then construct in a similar way $r^i_{s^i+2}$; this is possible since $\frac{5}{16}\,d(p,q) > \epsilon$. For all $s > s^i+1$, define $t^i_s \ll_i p^i_s$ such that $D_i(t^i_s,p^i_{s+1}) \leq \epsilon$ and $d_i(t^i_s,p^i_{s+1}) = \alpha(\epsilon)$. Obviously, $d_i(\zeta_i(p),t^i_s), d_i(r^i_s,\zeta_i(q)) > \frac{3}{16}\,d(p,q)$ and $d_i(p^i_s,r^i_{s+1}), d_i(t^i_s,p^i_{s+1}) > \alpha\,(\epsilon)$. Hence, one can uniquely define sequences of the types

$$\left\{\zeta_i(p),r^i_1,p^i_1,r^i_2,p^i_2,\ldots,p^i_{s^i},r^i_{s^i+1},p^i_{s^i+1},r^i_{s^i+2},t^i_{s^i+2},p^i_{s^i+3},t^i_{s^i+3},\ldots,t^i_{k_i-1},\zeta_i(q)\right\}$$

and

$$\left\{\zeta_i(p),r^i_1,p^i_1,r^i_2,p^i_2,\ldots,p^i_{s^i},r^i_{s^i+1},p^i_{s^i+1},p^i_{s^i+2},t^i_{s^i+2},p^i_{s^i+3},t^i_{s^i+3},\ldots,t^i_{k_i-1},\zeta_i(q)\right\}$$

depending on whether $d_i(p^i_{s^i+1},p^i_{s^i+2}) < \alpha(\epsilon)$ or $d_i(p^i_{s^i+1},p^i_{s^i+2}) \geq \alpha(\epsilon)$, respectively. In general, we have constructed a sequence of the form $\{z^i_s\}^{2k_i-1}_{s=0}$ with the following useful properties:

- $z^i_0 = \zeta_i(p)$ and $z^i_{2k_i-1} = \zeta_i(q)$;
- $\frac{1}{2}\,D_i(\zeta_i(p),\zeta_i(q)) \leq D_i(z^i_{2s},z^i_{2s+1}) \leq \frac{1}{2}\,D_i(\zeta_i(p),\zeta_i(q)) + \epsilon$ for $s \leq k^i-2$, $\frac{1}{4}\,D_i(\zeta_i(p),\zeta_i(q)) \leq D_i(z^i_{2k^i-2},z^i_{2k^i-1}) < \frac{3}{4}\,D_i(\zeta_i(p),\zeta_i(q)) + \epsilon$ and $d_i(z^i_{2s},z^i_{2s+1}) \geq \alpha(\epsilon)$ for all $s \leq k^i-1$;
- $D_i(z^i_{2s-1},z^i_{2s}) < 2\epsilon$.

Hence, the sequence $\{\psi_i(z^i_s)\}^{2k_i-1}_{s=0}$ satisfies:

- $D(\psi_i(z^i_0),p), D(\psi_i(z^i_{2k_i-1}),q) < \alpha(\epsilon)$;
- $\frac{1}{2}\,D(p,q) - 7\epsilon \leq D(\psi_i(z^i_{2s}),\psi_i(z^i_{2s+1})) \leq \frac{1}{2}\,D(p,q) + 8\epsilon$ for $s \leq k^i-2$, $\frac{1}{4}\,D(p,q) - 6\epsilon < D(\psi_i(z^i_{2k^i-2}),\psi_i(z^i_{2k^i-1})) < \frac{3}{4}\,D(p,q) + 10\epsilon$ and $d_i(\psi_i(z^i_{2s}),\psi_i(z^i_{2s+1})) > 0$ for all $s \leq k^i-1$;
- $D(\psi_i(z^i_{2s-1}),\psi_i(z^i_{2s})) < 6\epsilon$.

Hence, for every n such that $\epsilon_n < \alpha(d(p,q)/8))$ we can find a sequence $\{\alpha^n_s\}^{2k_n-1}_{s=0}$ in $\mathcal{M}$ satisfying the above properties.[16] By using a diagonalisation argument, we can find a subsequence (which we label with the same index) such that $k_{n+1} \geq k_n$ for all $n \in \mathbf{N}$ and a sequence $\{\alpha_s\}^{2\sup_n k_n-1}_{s=0}$ such that $\alpha^n_s \overset{n\to\infty}{\to} \alpha_s$ for all $s \leq 2\sup_n k_n - 1$. Obviously $\sup_n k_n$ must be finite, since otherwise we would have found an infinite sequence of points which are all a distance greater or equal than $\frac{1}{2}\,D(p,q)$ apart, which is impossible by compactness.[17] Hence, we have found a finite sequence of points $\beta_s \leq \beta_{s+1}$, $s = 0\ldots k$, such that:

- $\beta_0 = p$ and $\beta_k = q$;

[16] In the sequel, the reader should keep in mind that these finite sequences can be extended to infinite ones by setting every element after $2k_n-1$ equal to q.

[17] To see this, the reader should use the fact that the strong distance is increasing along causal paths.

- $\sum_{s=0}^{k-1} d(\beta_s, \beta_{s+1}) = d(p,q)$;
- $D(\beta_s, \beta_{s+1}) = \frac{1}{2} D(p,q)$, $s \leq k-2$ and $\frac{1}{4} D(p,q) \leq D(\beta_{k-1}, q) \leq \frac{3}{4} D(p,q)$.

It is possible that for some s, $d(\beta_s, \beta_{s+1}) = 0$, but these are limits of timelike intervals as follows from the construction. Subdividing each of these approximating timelike intervals and using a compactness argument together with the continuity of K in the strong topology, one obtains that every two timelike related points are connected by a causal geodesic. $\bullet$

This result is, in the authors' viewpoint, very encouraging since it shows that all concepts fit nicely together. Notice also that the proof is considerably more difficult than the one in the metric case where it suffices to use the existence of a midpoint for path metrics.

6 Compactness of classes of Lorentz spaces

In this section, we give some criteria for a collection of Lorentz spaces to be precompact with respect to the generalized Gromov-Hausdorff uniformity. The motivation for such criteria is that they give metric-type conditions under which a class of Lorentz spaces is "bounded" in some sense, which is desirable because it makes the class more controllable; an example would be the possibility of defining summations over classes of discrete Lorentz spaces which are known to be bounded when formulating a finite quantum dynamics for causal sets. The ideas presented here can be traced back to Gromov, and proofs of the results at hand can be found in Petersen [2].

Let $(\mathcal{M}, d)$ be a Lorentz space, and define (as in Gromov [8])

- $\mathrm{Cap}_{\mathcal{M}}(\epsilon) =$ maximum number of disjoint $\frac{\epsilon}{2}$-balls in $(\mathcal{M}, D)$.
- $\mathrm{Cov}_{\mathcal{M}}(\epsilon) =$ minimum number of ϵ-balls needed to cover $\mathcal{M}$.

Clearly, $\mathrm{Cov}_{\mathcal{M}}(\epsilon) \leq \mathrm{Cap}_{\mathcal{M}}(\epsilon)$ and both are decreasing functions of ϵ. What do these definitions mean? $\mathrm{Cov}_{\mathcal{M}}(\epsilon)$ tells us that one can choose $\mathrm{Cov}_{\mathcal{M}}(\epsilon)$ points p_i in $\mathcal{M}$ such that the pair $(\{p_i \mid i = 1 \ldots \mathrm{Cov}_{\mathcal{M}}(\epsilon)\}, d)$ is $(2\epsilon, \epsilon)$-close in the Gromov-Hausdorff sense to $(\mathcal{M}, d)$. On the other hand, suppose that $(\mathcal{M}_1, d_1)$ and $(\mathcal{M}_2, d_2)$ are (ϵ, δ) Gromov-Hausdorff close; then we know from theorem 2 in [7] that

$$d_{\mathrm{GH}}((\mathcal{M}_1, D_1), (\mathcal{M}_2, D_2)) \leq \epsilon + \frac{3\delta}{2} ,$$

and therefore one obtains from the triangle inequality that

$$\mathrm{Cov}_{\mathcal{M}_1}(\gamma + 2\epsilon + 3\delta) \leq \mathrm{Cov}_{\mathcal{M}_2}(\gamma)$$

and

$$\mathrm{Cap}_{\mathcal{M}_1}(\gamma) \geq \mathrm{Cap}_{\mathcal{M}_2}(\gamma + 4\epsilon + 6\delta)$$

for all $\gamma > 0$. Since we have a quantitative Hausdorff uniformity on $\mathcal{LS}$ with a countable basis around every point, the following two criteria for compactness are equivalent:

- Every open cover has a finite subcover.
- Every sequence has a subsequence which converges to a limit point.

Theorem 11 *For a class $\mathcal{C} \subset \mathcal{LS}$, the following statements are equivalent:*

1. *$\mathcal{C}$ is precompact in $\mathcal{LS}$, i.e., every sequence in $\mathcal{C}$ has a subsequence that is convergent in $\mathcal{LS}$;*
2. *There is a function $N(\epsilon) : (0, \alpha) \to (0, \infty)$ such that $Cap_{\mathcal{M}}(\epsilon) \leq N(\epsilon)$ for all $(\mathcal{M}, d) \in \mathcal{C}$;*
3. *There is a function $N(\epsilon) : (0, \alpha) \to (0, \infty)$ such that $Cov_{\mathcal{M}}(\epsilon) \leq N(\epsilon)$ for all $(\mathcal{M}, d) \in \mathcal{C}$.*

Proof:
$1 \Rightarrow 2$): If $\mathcal{C}$ is precompact, then for any $\epsilon > 0$ there exist points $(\mathcal{M}_1, d_1)$, ..., $(\mathcal{M}_k, d_k) \in \mathcal{C}$ such that any $(\mathcal{M}, d)$ is $(\frac{\epsilon}{16}, \frac{\epsilon}{24})$ close to some $(\mathcal{M}_i, d_i)$. Hence, $\mathrm{Cap}_{\mathcal{M}}(\epsilon) \leq \mathrm{Cap}_{\mathcal{M}_i}(\frac{\epsilon}{2}) \leq \max_j \mathrm{Cap}_{\mathcal{M}_j}(\frac{\epsilon}{2})$, which clearly proves a bound for $\mathrm{Cap}_{\mathcal{M}}(\epsilon)$ for any $\epsilon > 0$.

$2 \Rightarrow 3$) is obvious.

$3 \Rightarrow 1$): Because of the generalized triangle inequality, it suffices to show that for any $\epsilon > 0$ there exists a finite collection $\mathcal{A}$ of spaces in $\mathcal{LS}$ such that any pair $(\mathcal{M}, d) \in \mathcal{C}$ is (ϵ, ϵ)-close to one of the elements in $\mathcal{A}$. Observe that for any $(\mathcal{M}, d)$ and $\delta > 0$: $tdiam(\mathcal{M}) \leq 2\delta \, \mathrm{Cov}_{\mathcal{M}}(\delta)$ since $D_{\mathcal{M}}(p, q) \geq d(p, q)$ for all $p, q \in \mathcal{M}$. The hypothesis implies the existence of a function $N(\epsilon)$ such that $\mathrm{Cov}_{\mathcal{M}}(\frac{\epsilon}{8}) \leq N(\frac{\epsilon}{8})$. Hence every space in $\mathcal{C}$ is $(\frac{\epsilon}{4}, \frac{\epsilon}{8})$-close to a finite space with $N(\frac{\epsilon}{8})$ elements, such that the timelike distance between any two points does not exceed the value $\frac{\epsilon}{4} N(\frac{\epsilon}{8})$. The Lorentz metric on such a finite space consists of a square matrix $(d_{ij})_{1 \leq i,j \leq N(\epsilon/8)}$ such that $0 \leq d_{ij} \leq \frac{\epsilon}{4} N(\frac{\epsilon}{8})$. Obviously, one can find a finite collection $\mathcal{A}$ of Lorentzian metric spaces with $N(\frac{\epsilon}{8})$ elements such that any of the $(d_{ij})_{1 \leq i,j \leq N(\epsilon/8)}$ is $(\frac{\epsilon}{4}, 0)$-close to some element of $\mathcal{A}$. Hence, all spaces $(\mathcal{M}, d) \in \mathcal{C}$ are $(\frac{\epsilon}{2}, \frac{5\epsilon}{8})$-close to some element of $\mathcal{A}$ which concludes the proof. •

We show that the covering property with covering function N is stable under generalized Gromov-Hausdorff convergence provided that N is continuous (cf. the $\mathcal{C}^{\pm}_{\alpha}$ properties in Ref [7]).

Theorem 12 *Let $\mathcal{C}(N(\epsilon))$ be the collection of pairs $(\mathcal{M}, d) \in \mathcal{LS}$ such that $Cov_{\mathcal{M}}(\epsilon) \leq N(\epsilon)$ for all $\epsilon > 0$; suppose N is continuous. Then $\mathcal{C}(N(\epsilon))$ is compact.*

Proof:
We already know that $\mathcal{C}(N(\epsilon))$ is precompact, hence suppose $(\mathcal{M}_i, d_i) \stackrel{i\to\infty}{\to} (\mathcal{M}, d)$ in the generalized Gromov-Hausdorff uniformity, then with $\alpha_i \stackrel{i\to\infty}{\to} 0$ such that $(\mathcal{M}_i, d_i)$ and $(\mathcal{M}, d)$ are (α_i, α_i)-close, we obtain that

$$\mathrm{Cov}_{\mathcal{M}}(\epsilon) \leq \mathrm{Cov}_{\mathcal{M}_i}(\epsilon - 5\alpha_i) \leq N(\epsilon - 5\alpha_i) \; .$$

The continuity of N concludes the proof. •

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Appendix A

In this Appendix, we prove Theorem 2. First, we introduce some notational conventions. Denote by $\alpha^i_{j,k}$ the k-th element of the j-th column K^i_j in the causal set $\mathcal{P}^L_i$. The labeling of elements in a column starts from zero. For example: the maximal element in K^1_2 is $\alpha^1_{2,L+1}$. For notational simplicity, we agree that $\alpha^i_{j,0} \equiv b^i_j$ and the top element of the column K^i_j is denoted by t^i_j. In $\mathcal{P}^L_2$, this results in $t^2_j = \alpha^2_{j,L+2-j}$. The idea of the proof is to determine how the bottom and top elements shift under the maps ψ and ζ. The following Lemma is crucial.

Lemma 1 *Let $r < \frac{L}{4}+5$ ($1 < r < \frac{L}{4}+5$), and suppose $\zeta(t^2_r) \in K^1_s$ ($\psi(t^1_r) \in K^2_s$); then $\zeta(b^2_{r+j}) \in \left\{b^1_{s-1}, b^1_s, b^1_{s+1}\right\}$ ($\psi(b^1_r) \in \left\{b^2_{s-1}, b^2_s, b^2_{s+1}\right\}$), where $j = 0, 1$ and all the indices have to be taken modulo $L+1$.*

Proof:
Remark first that $\zeta(t^2_r) \in K^1_s$ ($\psi(t^1_r) \in K^2_s$) with s a natural number between 2 (1) and $r+k$ if $r < \frac{L}{2}+1-k$ ($r \geq \frac{L}{2}+1-k$). Obviously, $\zeta(t^2_r) \geq \alpha^1_{s,L+2-r-k}$ where $\geq$ means "in the causal future of". Suppose $\zeta(b^2_{r+j}) \notin \left\{b^1_{s-1}, b^1_s, b^1_{s+1}\right\}$ for some $j = 0, 1$; then $\zeta(b^2_{r+j}) = \alpha^1_{r,q}$ with $q \geq 1$ since

$$d_2(b^2_{r+j}, t^2_r) - k \geq L + 2 - (\tfrac{L}{4}+4) - (\tfrac{L}{4}-1) = \tfrac{L}{2} - 1 > 0 \; .$$

But in this case $\zeta(t^2_{r+1}) \geq \alpha^1_{s,L+1-k-r+q}$. The above calculation reveals that $d_2(b^2_{r+2}, t^2_{r+1}) - k \geq \frac{L}{2} - 2 > 0$ and since moreover $d_2(b^2_{r+j}, b^2_{r+2}) = 0$, we obtain that $\zeta(b^2_{r+2}) \leq \alpha^1_{s,q+j}$. Hence

$$d_1(\zeta(b^2_{r+2}), \zeta(t^2_r)) \geq L + 2 - k - r + q - (k+q) \geq \frac{L}{4} \; ,$$

which is impossible since $d_2(b^2_{r+2}, t^2_r) = 0$. The result for ψ is obvious. $\bullet$

We shall further construct ζ and state similar properties of ψ later on.

Lemma 2 *$\zeta(b^2_r) = b^1_s$ if $\zeta(t^2_r) \in K^1_s$ with r between 1 and $\frac{L}{4} + 3$.*

Proof:
According to Lemma 1, we have that $\zeta(b^2_r) \in \left\{b^1_{s-1}, b^1_s, b^1_{s+1}\right\}$. Suppose that $\zeta(b^2_r) = b^1_{s+1}$; then we show that $\zeta(b^2_{r+1}) \notin \left\{b^1_{s-1}, b^1_s, b^1_{s+1}\right\}$ which is impossible by the same Lemma. The arguments for $\zeta(b^2_r) = b^1_{s-1}$ are identical. Suppose $\zeta(b^2_{r+1}) = b^1_{s+1}$; then $d_1(\zeta(b^2_r), \zeta(t^2_{r+2})) \geq L + 2 - (r+2) - k \geq \frac{L}{2} - 2 \geq \frac{L}{4}$ which is impossible since $d_2(b^2_r, t^2_{r+2}) = 0$. Hence suppose that $\zeta(b^2_{r+1}) = b^1_{s-1}$, then $\zeta(t^2_{r+1}) \in K^1_s$. Hence $\zeta(b^2_{r+2}) \in \left\{b^1_{s-1}, b^1_s, b^1_{s+1}\right\}$ which is impossible since then $d_1(\zeta(b^2_{r+2}), \zeta(t^2_r) \geq L + 2 - r - k \geq \frac{L}{2}$. So, we are only left with $\zeta(b^2_{r+1}) = b^1_s$. Obviously $\zeta(t^2_{r+1}) \in K^1_{s+1}$, since otherwise $\zeta(t^2_{r+1}) \in K^1_s$ which was proven impossible before. Hence, $\zeta(b^2_{r+2}) \in \left\{b^1_s, b^1_{s+1}, b^1_{s+2}\right\}$. Previous arguments show that $\zeta(b^2_{r+2}) = b^1_{s+2}$, but then $\zeta(t^2_{r+2}) \geq \alpha^1_{s+1, L-r-k}$ which implies that $d_1(\zeta(b^2_r), \zeta(t^2_{r+2})) \geq \frac{L}{2} - 2 \geq \frac{L}{4}$. $\bullet$

Obviously, the same theorem applies to ψ for $1 < r < \frac{L}{4} + 4$. The following Lemma almost gives the necessary result.

Lemma 3 *If $\zeta(t^2_1) \in K^1_s$ with s ranging between 2 and $k+1 \leq \frac{L}{4}$ then $\zeta(b^2_i) = b^1_{s+i-1}$ and $\zeta(t^2_i) \geq \alpha^1_{s+i-1, L+2-i-k}$ for $i \leq \frac{L}{4} + 3$.*

Proof:
As a consequence of Lemma 2 we have only two possibilities. Either $\zeta(b^2_i) = b^1_{s+i-1}$ and $\zeta(t^2_i) \geq \alpha^1_{s+i-1, L+2-i-k}$ or $\zeta(b^2_i) = b^1_{s-i-1}$ and $\zeta(t^2_i) \geq \alpha^1_{s-i-1, L+2-i-k}$ for $i \leq \frac{L}{4}+3$. If the latter were true then $\zeta(t^2_{s+1}) \geq \alpha^1_{l, L+2-s-k}$ since $s+1 \leq \frac{L}{4}+1$ which is impossible since $L + 2 - s - k > 2$. $\bullet$

First of all, it is easy to see that if $\psi(t^1_2) \in K^2_{\tilde{s}}$ with $\tilde{s}$ between 2 and $k+1$. Since suppose $\psi(t^1_2) \in K^2_1$ then $\psi(b^1_1) = b^2_L$ since otherwise or $\psi(b^1_1) \geq \alpha^2_{1,1}$ or $\psi(b^1_1) \in \left\{b^2_1, b^2_2\right\}$. The former is impossible since then $d_2(\psi(b^1_3), \psi(t^1_1)) \geq \frac{L}{2} + 2 - k > \frac{L}{4}$. The latter would imply that $\psi(b^1_1), \psi(t^1_3)) \geq L - 1 - k$ which is also impossible. But then $\psi(t^1_1) \in K^2_{L-1} \cup K^2_L \cup K^2_1$, which is impossible for $L \geq 8$ (which was the assumption). By a reasoning analogous to the one in Lemma 3, we obtain that $\psi(b^1_i) = b^2_{\tilde{s}+i-2}$ and $\psi(t^1_i) \geq \alpha^2_{\tilde{s}+i-2, L+2-i-k}$ for $1 < i \leq \frac{L}{4} + 3$. Moreover, $\psi(b^1_1) = b^2_{\tilde{s}-1}$.

We finish the proof by remarking that $d_1(b^1_{s+j}, \zeta(t^2_1)) \geq L+1-k$ for $j = -1, 0, 1$. Since $1 \geq s - 1 < s + 1 \leq \frac{L}{4} + 1$, we have that $\psi(b^1_{s+j}) = b^2_{s+\tilde{s}-2+j}$. Since $L+1-2k \geq \frac{L}{2}+3$ we obtain that $\psi \circ \zeta(t^2_1) \in K^2_{s+\tilde{s}-2}$. Hence $D_2(t^2_1, \psi \circ \zeta(t^2_1)) = L$ which finishes the proof. $\bullet$

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