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The twisted (h, q) -Genocchi numbers and polynomials with weight α and q -bernstein polynomials with weight α

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Hannam University, Daejeon 306-791, Korea**Abstract**

In this article, we give some identities on the twisted (h, q) -Genocchi numbers and polynomials and q -Bernstein polynomials with weighted α .

Keywords: Genocchi numbers and polynomials, twisted (h, q) -Genocchi numbers and polynomials with weight α , q -Bernstein polynomials

1 Introduction

Let p be a fixed odd prime number. The symbol, \mathbb{Z}_p , \mathbb{Q}_p , and \mathbb{C}_p denote the ring of p -adic integers, the field of p -adic rational numbers and the completion of algebraic closure of \mathbb{Q}_p , respectively. Let \mathbb{N} be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. As well known definition, the p -adic absolute value is given by $|x|_p = p^{-r}$, where $x = p^r \frac{t}{s}$ with $(t, p) = (s, p) = (t, s) = 1$. When one talks of q -extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a p -adic number $q \in \mathbb{C}_p$. In this article, we assume that $q \in \mathbb{C}_p$ with $|1 - q|_p < 1$.

For $f \in UD(\mathbb{Z}_p) = \{f/f : \mathbb{Z}_p \rightarrow \mathbb{C}_p \text{ is uniformly differentiable function}\}$, Kim defined the fermionic p -adic q -integral on \mathbb{Z}_p as follows:

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N \rightarrow \infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x) (-q)^x. \quad (1.1)$$

For $n \in \mathbb{N}$, let $f_n(x) = f(x + n)$ be translation. As well known equation, by (1.1), we have

$$q^n I_{-q}(f_n) = (-1)^n I_{-q}(f) + [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l f(l). \quad (1.2)$$

Throughout this article we use the notation:

$$[x]_q = \frac{1 - q^x}{1 - q}.$$

$\lim_{q \rightarrow 1} |x|_q = x$ for any x with $|x|_p \leq 1$ in the present p -adic case. To investigate relation of the twisted (h, q) -Genocchi numbers and polynomials with weight α and the q -Bernstein polynomials with weight α , we will use useful property for $[x]_{q^\alpha}$ as following;

$$\begin{aligned} [x]_{q^\alpha} &= 1 - [1 - x]_{q^{-\alpha}} \\ [1 - x]_{q^{-\alpha}} &= 1 - [x]_{q^\alpha} \end{aligned} \tag{1.3}$$

The twisted (h, q) -Genocchi numbers and polynomials with weight α are defined by the generating function, respectively:

$$G_{n,q,w}^{(h,\alpha)} = n \int_{\mathbb{Z}_p} q^{x(h-1)} \phi_w(x) [x]_{q^\alpha}^{n-1} d\mu_{-q}(x). \tag{1.4}$$

$$G_{n,q,w}^{(h,\alpha)}(x) = n \int_{\mathbb{Z}_p} q^{y(h-1)} \phi_w(y) [y + x]_{q^\alpha}^{n-1} d\mu_{-q}(y). \tag{1.5}$$

In the special case, $x = 0$, $G_{n,q,w}^{(h,\alpha)}(0) = G_{n,q,w}^{(h,\alpha)}$ are called the n th twisted (h, q) -Genocchi numbers with weight α (see [1]).

Let $C_{p^n} = \{w | w^{p^n} = 1\}$ be the cyclic group of order p^n and let

$$T_p = \lim_{n \rightarrow \infty} C_{p^n} = \cup_{n \geq 1} C_{p^n}$$

see [1-5].

Kim defined the q -Bernstein polynomials with weight α of degree n as follows:

$$B_{k,n}^{(\alpha)}(x, q) = \binom{n}{k} [x]_{q^\alpha}^k [1 - x]_{q^{-\alpha}}^{n-k}, \text{ where } x \in [0, 1], n, k \in \mathbb{Z}_+. \tag{1.6}$$

cf [6-12].

In this article, we investigate some properties for the twisted (h, q) -Genocchi numbers and polynomials with weight α . By using these properties, we give some interesting identities on the twisted (h, q) -Genocchi polynomials with weight α and q -Bernstein polynomials with weight α .

2 Twisted (h, q) -genocchi numbrs and polynomials with weight α and q -bernstein polynomials with weight α

From (1.2), we can get the following form for the twisted (h, q) -Genocchi numbers with weight α :

$$G_{0,q,w}^{(h,\alpha)} = 0, \text{ and } q^h w G_{n,q,w}^{(h,\alpha)}(1) + G_{n,q,w}^{(h,\alpha)} = \begin{cases} [2]_q, & \text{if } n = 1, \\ 0, & \text{if } n > 1, \end{cases} \tag{2.1}$$

$$G_{0,q,w}^{(h,\alpha)} = 0, \text{ and } q^h w (1 + q^\alpha G_{q,w}^{(h,\alpha)})^n + q^\alpha G_{n,q,w}^{(h,\alpha)} = \begin{cases} q^\alpha [2]_q, & \text{if } n = 1, \\ 0, & \text{if } n > 1, \end{cases} \tag{2.2}$$

$$q^{\alpha x} G_{n+1,q,w}^{(h,\alpha)}(x) = \left([x]_q^\alpha + q^{\alpha x} G_{q,w}^{(h,\alpha)} \right)^{n+1} \tag{2.3}$$

with usual convention about replacing $(G_{q,w}^{(h,\alpha)})^n$ by $G_{n,q,w}^{(h,\alpha)}$.

By (1.4), we can obtain

$$G_{n,q,w}^{(h,\alpha)}(x) = n[2]_q \left(\frac{1}{1-q^\alpha} \right)^{n-1} \sum_{l=0}^{n-1} \binom{n-1}{l} (-1)^l \frac{1}{1+wq^{\alpha l+h}} \tag{2.4}$$

By (2.4), we can get

$$\begin{aligned} G_{n,q^{-1},w^{-1}}^{(h,\alpha)}(1-x) &= n[2]_{q^{-1}} \left(\frac{1}{1-q^{-\alpha}} \right)^{n-1} \sum_{l=0}^{n-1} \binom{n-1}{l} (-1)^l (q^{-1})^{\alpha l(1-x)} \frac{1}{1+w^{-1}(q^{-1})^{\alpha l+h}} \\ &= n \frac{1}{q} [2]_q \left(\frac{1}{1-q^\alpha} \right)^{n-1} (-1)^{n-1} q^{\alpha n+\alpha} \sum_{l=0}^{n-1} \binom{n-1}{l} (-1)^l q^{\alpha l x} \frac{wq^h}{1+wq^{\alpha l+h}} \\ &= n[2]_q \left(\frac{1}{1-q^\alpha} \right)^{n-1} \sum_{l=0}^{n-1} \binom{n-1}{l} (-1)^l q^{\alpha l x} \frac{1}{1+wq^{\alpha l+h}} \frac{1}{q} q^{\alpha n-\alpha} (-1)^{n-1} wq^h \\ &= (-1)^{n-1} wq^{\alpha(n-1)+(h-1)} G_{n,q,w}^{(h,\alpha)}(x). \end{aligned}$$

So, we get the following theorem.

Theorem 1. Let $n \in \mathbb{Z}_+$. For $w \in T_p$, we have

$$G_{n,q,w}^{(h,\alpha)}(x) = (-1)^{n-1} w^{-1} q^{\alpha(1-n)+(1-h)} G_{n,q^{-1},w^{-1}}^{(h,\alpha)}(1-x).$$

By (2.1), (2.2), and (2.3), we note that

$$\begin{aligned} G_{n,q,w}^{(h,\alpha)} &= -wq^h G_{n,q,w}^{(h,\alpha)}(1) \\ &= -wq^h (q^{-\alpha} (1 + g^\alpha G_{q,w}^{(h,\alpha)})^n) \\ &= -wq^{h-\alpha} \sum_{l=0}^n \binom{n}{l} (q^\alpha)^l G_{l,q,w}^{(h,\alpha)} \\ &= -wq^h \binom{n}{1} G_{1,q,w}^{(h,\alpha)} - wq^{h-\alpha} \sum_{l=2}^n \binom{n}{l} q^{\alpha l} (-wq^h G_{l,q,w}^{(h,\alpha)}(1)) \\ &= -wq^h \binom{n}{1} G_{1,q,w}^{(h,\alpha)} - wq^{h-\alpha} \sum_{l=2}^n \binom{n}{l} q^{\alpha l} (-wq^h q^{-\alpha} (1 + q^\alpha G_{q,w}^{(h,\alpha)})^l) \\ &= -nwq^h G_{1,q,w}^{(h,\alpha)} + w^2 q^{2h-2\alpha} \sum_{l=2}^n \binom{n}{l} q^{\alpha l} (1 + q^\alpha G_{q,w}^{(h,\alpha)})^l \\ &= -nwq^h G_{1,q,w}^{(h,\alpha)} + w^2 q^{2h-2\alpha} (1 + q^\alpha (1 + g^\alpha G_{q,w}^{(h,\alpha)}))^n - nw^2 q^{2h-2\alpha} q^\alpha (1 + q^\alpha G_{q,w}^{(h,\alpha)})^1 \\ &= -nwq^h G_{1,q,w}^{(h,\alpha)} + w^2 q^{2h-2\alpha} ([2]_{q^\alpha} + q^{2\alpha} G_{q,w}^{(h,\alpha)})^n - nw^2 q^{2h-\alpha} (q^\alpha G_{1,q,w}^{(h,\alpha)}) \\ &= -nwq^h G_{1,q,w}^{(h,\alpha)} + w^2 q^{2h-2\alpha} ([2]_{q^\alpha} + q^{2\alpha} G_{q,w}^{(h,\alpha)})^n - nw^2 q^{2h} G_{1,q,w}^{(h,\alpha)} \\ &= -nwq^h G_{1,q,w}^{(h,\alpha)} + w^2 q^{2h} G_{n,q,w}^{(h,\alpha)}(2) - nw^2 q^{2h} G_{1,q,w}^{(h,\alpha)} \end{aligned} \tag{2.5}$$

Therefore, by (2.5), we obtain the theorem below.

Theorem 2. For $n \in \mathbb{N}$ with $n > 1$, we have

$$G_{n,q,w}^{(h,\alpha)}(2) = w^{-2} q^{-2h} G_{n,q,w}^{(h,\alpha)} + w^{-1} q^{-h} \frac{n[2]_q}{1+wq^h} + \frac{n[2]_q}{1+wq^h}.$$

From Theorem 2,

$$\begin{aligned} \frac{G_{n+1,q,w}^{(h,\alpha)}(2)}{n+1} &= \frac{1}{n+1} \left(\frac{(n+1)[2]_q}{1+wq^h} + \frac{(n+1)w^{-1}q^{-h}[2]_q}{1+wq^h} \right) + w^{-2}q^{-2h} \frac{G_{n+1,q,w}^{(h,\alpha)}}{n+1} \\ &= \frac{[2]_q}{1+wq^h} + w^{-1}q^{-h} \frac{[2]_q}{1+wq^h} + w^{-2}q^{-2h} \frac{G_{n+1,q,w}^{(h,\alpha)}}{n+1} \end{aligned}$$

Therefore, we obtain the Corollary 3 by (1.5) and Theorem 2.

Corollary 3. For $n \in \mathbb{N}$, we have

$$\int_{\mathbb{Z}_p} q^{y(h-1)} \phi_w(y) [\gamma + 2]_{q^\alpha}^n q \mu_{-q}(y) = \frac{[2]_q}{1+wq^h} + w^{-1}q^{-h} \frac{[2]_q}{1+wq^h} + w^{-2}q^{-2h} \frac{G_{n+1,q,w}^{(h,\alpha)}}{n+1}$$

By Theorems 1, 2 and fermionic integral on \mathbb{Z}_p , we note that

$$\begin{aligned} \int_{\mathbb{Z}_p} q^{x(h-1)} \phi_w(x) [1-x]_{q^{-\alpha}}^n q \mu_{-q}(x) &= (-1)^n q^{\alpha n} \int_{\mathbb{Z}_p} q^{x(h-1)} \phi_w(x) [x-1]_{q^\alpha}^n d\mu_{-q}(x) \\ &= (-1)^n q^{\alpha n} \frac{G_{n+1,q,w}^{(h,\alpha)}(-1)}{n+1} \\ &= w^{-1}q^{1-h} \frac{G_{n+1,q^{-1},w^{-1}}^{(h,\alpha)}(2)}{n+1} \\ &= w^{-1}q^{1-h} \left(\frac{[2]_{q^{-1}}}{1+q^{-h}w^{-1}} + wq^h \frac{[2]_{q^{-1}}}{1+q^{-1}w^{-1}} + w^2q^{2h} \frac{G_{n+1,q^{-1},w^{-1}}^{(h,\alpha)}}{n+1} \right) \tag{2.6} \\ &= \frac{[2]_q}{1+wq^h} + wq^h \frac{[2]_q}{1+wq^h} + wq^{h+1} \frac{G_{n+1,q^{-1},w^{-1}}^{(h,\alpha)}}{n+1} \\ &= [2]_q + wq^{h+1} \frac{G_{n+1,q^{-1},w^{-1}}^{(h,\alpha)}}{n+1}. \end{aligned}$$

Hence, we get the following theorem.

Theorem 4. for $n \in \mathbb{N}$ with $n > 1$, we have

$$\int_{\mathbb{Z}_p} q^{x(h-1)} \phi_w(x) [1-x]_{q^{-\alpha}}^n d\mu_{-q}(x) = [2]_q + wq^{h+1} \frac{G_{n+1,q^{-1},w^{-1}}^{(h,\alpha)}}{n+1}. \tag{2.7}$$

Corollary 5.

From (1.3) and Theorem 4, we take the fermionic p -adic invariant integral on \mathbb{Z}_p for q -Bernstein polynomials as follows:

$$\begin{aligned} \int_{\mathbb{Z}_p} q^{x(h-1)} \phi_w(x) B_{k,n}(x, q) d\mu_{-q}(x) &= \int_{\mathbb{Z}_p} q^{x(h-1)} \phi_w(x) \binom{n}{k} [x]_{q^\alpha}^k [1-x]_{q^{-\alpha}}^{n-k} d\mu_{-q}(x) \\ &= \binom{n}{k} \int_{\mathbb{Z}_p} q^{x(h-1)} \phi_w(x) [x]_{q^\alpha}^k (1-[x]_{q^\alpha})^{n-k} d\mu_{-q}(x) \\ &= \binom{n}{k} \int_{\mathbb{Z}_p} q^{x(h-1)} \phi_w(x) [x]_{q^\alpha}^k (1-[x]_{q^\alpha})^{n-k} d\mu_{-q}(x) \tag{2.8} \\ &= \binom{n}{k} \sum_{l=0}^{n-k} \binom{n}{n-k} (-1)^l \int_{\mathbb{Z}_p} q^{x(h-1)} \phi_w(x) [x]_{q^\alpha}^{k+l} d\mu_{-q}(x) \\ &= \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \frac{G_{k+l+1,q,w}^{(h,\alpha)}}{k+l+1} \end{aligned}$$

And we get the following formula;

$$\begin{aligned}
 & \int_{\mathbb{Z}_p} q^{x(h-1)} \phi_w(x) B_{k,n}(x, q) d\mu_{-q}(x) \\
 &= \int_{\mathbb{Z}_p} q^{x(h-1)} \phi_w(x) \binom{n}{k} [x]_{q^\alpha}^{n-k} [1-x]_{q^{-\alpha}}^k d\mu_{-q}(x) \\
 &= \int_{\mathbb{Z}_p} q^{x(h-1)} \phi_w(x) \binom{n}{k} [1-x]_{q^{-\alpha}}^k (1 - [1-x]_{q^{-\alpha}})^{n-k} d\mu_{-q}(x) \\
 &= \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^{n-k-l} [1-x]_{q^{-\alpha}}^{n-k-l} \int_{\mathbb{Z}_p} q^{x(h-1)} \phi_w(x) [1-x]_{q^{-\alpha}}^k d\mu_{-q}(x) \\
 &= \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^{n-k-l} \int_{\mathbb{Z}_p} q^{x(h-1)} \phi_w(x) [1-x]_{q^{-\alpha}}^{n-l} d\mu_{-q}(x) \\
 &= \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^{n-k-l} \left([2]_q + wq^{1+h} \frac{G_{n-l+1, q^{-1}, w^{-1}}^{(h, \alpha)}}{n-l+1} \right)
 \end{aligned} \tag{2.9}$$

Hence, we can get the following theorem by (2.8) and (2.9).

Theorem 5. for $n \in \mathbb{N}$ with $n > 1$, we have

$$\begin{aligned}
 & \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \frac{G_{k+l+1, q, w}^{(h, \alpha)}}{k+l+1} \\
 &= \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^{n-k-l} \left([2] + wq^{1+h} \frac{G_{n-l+1, q^{-1}, w^{-1}}^{(h, \alpha)}}{n-l+1} \right) \\
 &= \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^{n-k-l} \left(w^{-1} q^{1-h} \frac{G_{n-l+1, q^{-1}, w^{-1}}^{(h, \alpha)}(2)}{n-l+1} \right)
 \end{aligned} \tag{2.10}$$

Also, we can see that

$$\begin{aligned}
 & \int_{\mathbb{Z}_p} q^{x(h-1)} \phi_w(x) B_{k,n}(x, q) d\mu_{-q}(x) \\
 &= \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \frac{G_{k+l+1, q, w}^{(h, \alpha)}}{k+l+1} \\
 &= \binom{n}{k} \int_{\mathbb{Z}_p} q^{x(h-1)} \phi_w(x) [1-x]_{q^\alpha}^{n-k} [x]_{q^\alpha}^k d\mu_{-q}(x) \\
 &= \binom{n}{k} \int_{\mathbb{Z}_p} q^{x(h-1)} \phi_w(x) [1-x]_{q^\alpha}^{n-k} (1 - [1-x]_{q^\alpha})^k d\mu_{-q}(x) \\
 &= \binom{n}{k} \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} \int_{\mathbb{Z}_p} q^{x(h-1)} \phi_w(x) [1-x]_{q^\alpha}^{n-l} d\mu_{-q}(x) \\
 &= \binom{n}{k} \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} \left([2]_q + wq^{1+h} \frac{G_{n-l+1, q^{-1}, w^{-1}}^{(h, \alpha)}}{n-l+1} \right).
 \end{aligned} \tag{2.11}$$

Therefore, we have the theorem below.

Theorem 6. For $n, k \in \mathbb{Z}_+$ with $n > k + 1$, we have

$$\begin{aligned} & \int_{\mathbb{Z}_p} q^{x(h-1)} \phi_w(x) B_{k,n}(x, q) d\mu_{-q}(x) \\ &= \binom{n}{k} \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} \left([2]_q + wq^{1+h} \frac{G_{n-l+1, q^{-1}, w^{-1}}^{(h, \alpha)}}{n-l+1} \right). \end{aligned} \tag{2.12}$$

By (2.7) and Theorem 6, we can get the theorem below.

Theorem 7. Let $n, k \in \mathbb{Z}_+$ with $n > k + 1$. Then we have

$$\begin{aligned} & \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \frac{G_{k+l+1, q, w}^{(h, \alpha)}}{k+l+1} \\ &= \sum_{l=0}^k \binom{k}{l} (-1)^{k-1} \left([2]_q + wq^{1+h} \frac{G_{n-l+1, q^{-1}, w^{-1}}^{(h, \alpha)}}{n-l+1} \right). \end{aligned}$$

Let $n_1, n_2, k \in \mathbb{Z}_+$ with $n_1 + n_2 > 2k + 1$. Then we get

$$\begin{aligned} & \int_{\mathbb{Z}_p} q^{x(h-1)} \phi_w(x) B_{k, n_1}^{(\alpha)}(x, q) B_{k, n_2}^{(\alpha)}(x, q) d\mu_{-q}(x) \\ &= \int_{\mathbb{Z}_p} q^{x(h-1)} \phi_w(x) \binom{n_1}{k} [x]_{q^\alpha}^k [1-x]_{q^{-\alpha}}^{n_1-k} \binom{n_2}{k} [x]_{q^\alpha}^k [1-x]_{q^{-\alpha}}^{n_2-k} d\mu_{-q}(x) \\ &= \binom{n_1}{k} \binom{n_2}{k} \int_{\mathbb{Z}_p} q^{x(h-1)} \phi_w(x) [x]_{q^\alpha}^{2k} [1-x]_{q^{-\alpha}}^{n_1+n_2-k} d\mu_{-q}(x) \\ &= \binom{n_1}{k} \binom{n_2}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{2k-l} \int_{\mathbb{Z}_p} q^{x(h-1)} \phi_w(x) [1-x]_{q^{-\alpha}}^{n_1+n_2-l} d\mu_{-q}(x) \\ &= \binom{n_1}{k} \binom{n_2}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{2k-l} \left([2]_q + wq^{1+h} \frac{G_{n_1+n_2-l+1, q^{-1}, w^{-1}}^{(h, \alpha)}}{n_1+n_2-l+1} \right). \end{aligned}$$

Therefore, we obtain the theorem below.

Theorem 8. For $n_1, n_2, k \in \mathbb{Z}_+$, we have

$$\begin{aligned} & \int_{\mathbb{Z}_p} q^{x(h-1)} \phi_w(x) B_{k, n_1}^{(\alpha)}(x, q) B_{k, n_2}^{(\alpha)}(x, q) d\mu_{-q}(x) \\ &= \binom{n_1}{k} \binom{n_2}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{2k-l} \left([2]_q + wq^{1+h} \frac{G_{n_1+n_2-l+1, q^{-1}, w^{-1}}^{(h, \alpha)}}{n_1+n_2-l+1} \right) \\ &= \begin{cases} \left([2]_q + wq^{1+h} \frac{G_{n_1+n_2-l+1, q^{-1}, w^{-1}}^{(h, \alpha)}}{n_1+n_2-l+1} \right), & \text{if } k = 0, \\ wq^{1+h} \binom{n_1}{k} \binom{n_2}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{2k-l} \frac{G_{n_1+n_2-l+1, q^{-1}, w^{-1}}^{(h, \alpha)}}{n_1+n_2-l+1}, & \text{if } k > 0, \end{cases} \end{aligned} \tag{2.13}$$

And we can easily have that

$$\begin{aligned}
 & \int_{\mathbb{Z}_p} q^{x(h-1)} \phi_\omega(x) B_{k,n_1}^{(\alpha)}(x, q) B_{k,n_2}^{(\alpha)}(x, q) d\mu_{-q}(x) \\
 &= \int_{\mathbb{Z}_p} q^{x(h-1)} \phi_\omega(x) \binom{n_1}{k} [x]_{q^\alpha}^k [1-x]_{q^{-\alpha}}^{n_1-k} \binom{n_2}{k} [x]_{q^\alpha}^k [1-x]_{q^{-\alpha}}^{n_2-k} d\mu_{-q}(x) \\
 &= \binom{n_1}{k} \binom{n_2}{k} \int_{\mathbb{Z}_p} q^{x(h-1)} \phi_w(x) [x]_{q^\alpha}^{2k} [1-x]_{q^{-\alpha}}^{n_1+n_2-k} d\mu_{-q}(x) \\
 &= \binom{n_1}{k} \binom{n_2}{k} \int_{\mathbb{Z}_p} q^{x(h-1)} \phi_w(x) [x]_{q^\alpha}^{2k} (1-[x]_{q^\alpha})^{n_1+n_2-2k} d\mu_{-q}(x) \tag{2.14} \\
 &= \binom{n_1}{k} \binom{n_2}{k} \int_{\mathbb{Z}_p} q^{x(h-1)} \phi_w(x) [x]_{q^\alpha}^{2k} \sum_{l=0}^{n_1+n_2-2k} \binom{n_1+n_2-2k}{l} (-1)^l [x]_{q^\alpha}^l d\mu_{-q}(x) \\
 &= \binom{n_1}{k} \binom{n_2}{k} \sum_{l=0}^{n_1+n_2-2k} (-1)^l \binom{n_1+n_2-2k}{l} \int_{\mathbb{Z}_p} q^{x(h-1)} \phi_w(x) [x]_{q^\alpha}^{2k+l} d\mu_{-q}(x) \\
 &= \binom{n_1}{k} \binom{n_2}{k} \sum_{l=0}^{n_1+n_2-2k} (-1)^l \binom{n_1+n_2-2k}{l} \frac{G_{2k+l+1, q, w}^{(h, \alpha)}}{2k+l+1}, \text{ where } n_1, n_2, k \in \mathbb{Z}_+.
 \end{aligned}$$

Therefore, by (2.14) and Theorem 8, we obtain the theorem below.

Theorem 9. Let $n_1, n_2, k \in \mathbb{Z}_+$ with $n_1 + n_2 > 2k + 1$. Then we have

$$\begin{aligned}
 & \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{2k-l} \left([2]_q + wq^{1+h} \frac{G_{n_1+n_2-l+1, q^{-1}, w^{-1}}^{(h, \alpha)}}{n_1+n_2-l+1} \right) \\
 &= \sum_{l=0}^{n_1+n_2-2k} (-1)^l \binom{n_1+n_2-2k}{l} \frac{G_{2k+l+1, q, w}^{(h, \alpha)}}{2k+l+1}.
 \end{aligned}$$

For $n_1, n_2, \dots, n_s, k \in \mathbb{Z}_+, n_1 + n_2 + \dots + n_s > sk + 1$, then by the symmetry of q -Bernstein polynomials with weight α , we see that

$$\begin{aligned}
 & \int_{\mathbb{Z}_p} q^{x(h-1)} \phi_w(x) \prod_{i=1}^s B_{k,n_i}^{(\alpha)}(x, q) d\mu_{-q}(x) \\
 &= \prod_{i=1}^s \binom{n_i}{k} \int_{\mathbb{Z}_p} q^{x(h-1)} \phi_w(x) [x]_{q^\alpha}^{sk} [1-x]_{q^{-\alpha}}^{n_1+n_2+\dots+n_s-sk} d\mu_{-q}(x) \\
 &= \prod_{i=1}^s \binom{n_i}{k} \int_{\mathbb{Z}_p} q^{x(h-1)} \phi_w(x) (1-[x]_{q^{-\alpha}})^{sk} [1-x]_{q^{-\alpha}}^{n_1+n_2+\dots+n_s-sk} d\mu_{-q}(x) \\
 &= \prod_{i=1}^s \binom{n_i}{k} \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{sk-l} \int_{\mathbb{Z}_p} q^{x(h-1)} \phi_w(x) [1-x]_{q^{-\alpha}}^{n_1+n_2+\dots+n_s-l} d\mu_{-q}(x) \\
 &= \prod_{i=1}^s \binom{n_i}{k} \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{sk-l} \left([2]_q + wq^{1+h} \frac{G_{n_1+n_2+\dots+n_s-l+1, q^{-1}, w^{-1}}^{(h, \alpha)}}{n_1+n_2+\dots+n_s-l+1} \right).
 \end{aligned}$$

Therefore, we have the theorem below.

Theorem 10. For $n_1, n_2, n_3, \dots, n_s, k \in \mathbb{Z}_+$ with $n_1 + n_2 + \dots + n_s > sk + 1$, we have

$$\begin{aligned} & \int_{\mathbb{Z}_p} q^{x(h-1)} \phi_w(x) \prod_{i=1}^s B_{k, n_i}^{(\alpha)}(x, q) d\mu_{-q}(x) \\ &= \prod_{i=1}^s \binom{n_i}{k} \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{sk-l} \left([2]_q + wq^{1+h} \frac{G_{n_1+n_2+\dots+n_s-l+1, q^{-1}, w^{-1}}^{(h, \alpha)}}{n_1 + n_2 + \dots + n_s - l + 1} \right). \end{aligned}$$

In the same manner as in (2.11), we can get the following relation:

$$\begin{aligned} & \int_{\mathbb{Z}_p} q^{x(h-1)} \phi_w(x) \prod_{i=1}^s B_{k, n_i}^{(\alpha)}(x, q) d\mu_{-q}(x) \\ &= \prod_{i=1}^s \binom{n_i}{k} \int_{\mathbb{Z}_p} q^{x(h-1)} \phi_w(x) [x]_{q^s}^{sk} (1 - [x]_{q^s})^{n_1+n_2+\dots+n_s-sk} d\mu_{-q}(x) \\ &= \prod_{i=1}^s \binom{n_i}{k} \int_{\mathbb{Z}_p} q^{x(h-1)} \phi_w(x) [x]_{q^s}^{sk} \sum_{l=0}^{n_1+n_2+\dots+n_s-sk} (-1)^l \binom{n_1+n_2+\dots+n_s-sk}{l} (-1)^l [x]_{q^s}^l d\mu_{-q}(x) \\ &= \prod_{i=1}^s \binom{n_i}{k} \sum_{l=0}^{n_1+n_2+\dots+n_s-sk} (-1)^l \binom{n_1+n_2+\dots+n_s-sk}{l} \int_{\mathbb{Z}_p} q^{x(h-1)} \phi_w(x) [x]_{q^s}^{sk+l} d\mu_{-q}(x) \\ &= \prod_{i=1}^s \binom{n_i}{k} \sum_{l=0}^{n_1+n_2+\dots+n_s-sk} (-1)^l \binom{n_1+n_2+\dots+n_s-sk}{l} \frac{G_{sk+l+1, q, w}^{(h, \alpha)}}{sk+l+1}, \end{aligned}$$

where $n_1, n_2, \dots, n_s, k \in \mathbb{Z}_+$ with $n_1 + n_2 + \dots + n_s > sk + 1$.

By Theorem 11 and (2.9), we have the following corollary.

Corollary 11. Let $m \in \mathbb{N}$. For $n_1, n_2, \dots, n_s, k \in \mathbb{Z}_+$ with $n_1 + \dots + n_s > sk + 1$, we have

$$\begin{aligned} & \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{sk-l} \left([2]_q + wq^{1+h} \frac{G_{n_1+n_2+\dots+n_s-l+1, q^{-1}, w^{-1}}^{(h, \alpha)}}{n_1 + n_2 + \dots + n_s - l + 1} \right) \\ &= \sum_{l=0}^{n_1+n_2+\dots+n_s-sk} (-1)^l \binom{n_1+n_2+\dots+n_s-sk}{l} \frac{G_{sk+l+1, q, w}^{(h, \alpha)}}{sk+l+1}, \end{aligned}$$

Authors' contributions

All authors contributed equally to the manuscript and read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

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