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New inequalities for hyperbolic functions and their applications

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Abstract

In this paper, we obtain some new inequalities in the exponential form for the whole of the triples about the four functions $\{1, (\sinh t)/t, \exp(t\coth t - 1), \cosh t\}$. Then we generalize some well-known inequalities for the arithmetic, geometric, logarithmic, and identric means to obtain analogous inequalities for their pth powers, where $p > 0$.

MSC: 26E60; 26D07

Keywords: hyperbolic sine; hyperbolic cosine; hyperbolic cotangent; geometric mean; logarithmic mean; identric mean; arithmetic mean; best constants

1 Introduction

Let $\sinh t$, $\cosh t$, and $\coth t$ be the hyperbolic sine, hyperbolic cosine, and hyperbolic cotangent, respectively. It is well known that (see [1–6])

$$1 < \frac{\sinh t}{t} < e^{t\coth t - 1} < \cosh t \tag{1.1}$$

holds for all $t \neq 0$.

In the recent paper [7], we have established the following Cusa-type inequalities of exponential type for the triple $\{1, (\sinh t)/t, \cosh t\}$ described as follows.

Theorem 1.1 (Cusa-type inequalities [7, Part (i) of Theorem 1.1]) *Let $p \geq 4/5$, and $t \neq 0$. Then the double inequality*

$$(1-\lambda) + \lambda(\cosh t)^p < \left(\frac{\sinh t}{t}\right)^p < (1-\eta) + \eta(\cosh t)^p \tag{1.2}$$

holds if and only if $\eta \geq 1/3$ and $\lambda \leq 0$.

On the other hand, the author of this paper [8] obtains the following inequalities of exponential type for the triple $\{1, \exp(t\coth t - 1), \cosh t\}$.

Theorem 1.2 ([8, Theorem 2]) *Let $p > 0$, and $t \neq 0$. Then*

(1) *if $0 < p \leq 6/5$, the double inequality*

$$\alpha(\cosh t)^p + (1-\alpha) < e^{p(t\coth t - 1)} < \beta(\cosh t)^p + (1-\beta) \tag{1.3}$$

holds if and only if $\alpha \leq 2/3$ and $\beta \geq (2/e)^p$;

(2) *if* $p \geq 2$, *the double inequality*

$$\alpha(\cosh t)^p + (1-\alpha) < e^{p(t\coth t-1)} < \beta(\cosh t)^p + (1-\beta) \tag{1.4}$$

holds if and only if $\alpha \leq (2/e)^p$ *and* $\beta \geq 2/3$.

Next, we do the work for each of the triples $\{(\sinh t)/t, \exp(t\coth t-1), \cosh t\}$ and $\{1, (\sinh t)/t, \exp(t\coth t-1)\}$, and obtain the following two new results.

Theorem 1.3 *Let* $0 < p \leq 8/5$, *and* $t \neq 0$. *Then*

$$\alpha(\cosh t)^p + (1-\alpha)\left(\frac{\sinh t}{t}\right)^p < e^{p(t\coth t-1)} < \beta(\cosh t)^p + (1-\beta)\left(\frac{\sinh t}{t}\right)^p \tag{1.5}$$

holds if and only if $\alpha \leq 1/2$ *and* $\beta \geq (2/e)^p$.

Theorem 1.4 *Let* $p \geq 286/693$, *and* $t \neq 0$. *Then*

$$\alpha + (1-\alpha)e^{p(t\coth t-1)} < \left(\frac{\sinh t}{t}\right)^p < \beta + (1-\beta)e^{p(t\coth t-1)} \tag{1.6}$$

holds if and only if $\beta \leq 1/2$ *and* $\alpha \geq 1$.

In this paper, we shall give the elementary proofs of Theorem 1.3 and Theorem 1.4. In the last section, we apply Theorems 1.1-1.4 to obtain some new results for four classical means.

2 Lemmas

Lemma 2.1 ([9–11]) *Let* $f, g : [a,b] \to \mathbb{R}$ *be two continuous functions which are differentiable on* (a,b). *Further, let* $g' \neq 0$ *on* (a,b). *If* f'/g' *is increasing* (*or decreasing*) *on* (a,b), *then the functions* $(f(x)-f(b^-))/(g(x)-g(b^-))$ *and* $(f(x)-f(a^+))/(g(x)-g(a^+))$ *are also increasing* (*or decreasing*) *on* (a,b).

Lemma 2.2 *Let* $t \in (0,+\infty)$. *Then the inequality*

$$D(t) \triangleq t\sinh^5 t + 2t\sinh^3 t + t^4\cosh t - \sinh^4 t\cosh t - t^3\sinh^3 t - 2t^3\sinh t > 0$$

holds.

Proof Using the power series expansions of the functions $\sinh^5 t$, $\sinh^3 t$, $\cosh t$, $\sinh^4 t \times \cosh t$, and $\sinh t$, we have

$$\begin{aligned} D(t) &= \frac{1}{16}t(\sinh 5t - 5\sinh 3t + 10\sinh t) + \frac{1}{2}t(\sinh 3t - 3\sinh t) + t^4\cosh t \\ &\quad - \frac{1}{16}(\cosh 5t - 3\cosh 3t + 2\cosh t) - \frac{1}{4}t^3(\sinh 3t - 3\sinh t) - 2t^3\sinh t \\ &= \frac{1}{16}\sum_{n=0}^{\infty}\frac{5^{2n+1} - 5\cdot 3^{2n+1} + 10}{(2n+2)!}t^{2n+1} + \frac{1}{2}\sum_{n=0}^{\infty}\frac{3^{2n+1}-3}{(2n+1)!}t^{2n+2} + \sum_{n=0}^{\infty}\frac{1}{(2n)!}t^{2n+4} \end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{16}\sum_{n=0}^{\infty}\frac{5^{2n}-3\cdot 3^{2n}+2}{(2n)!}t^{2n}-\frac{1}{4}\sum_{n=0}^{\infty}\frac{3^{2n+1}-3}{(2n+1)!}t^{2n+4}-2\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}t^{2n+4}\\
&=\frac{1}{16}\sum_{n=3}^{\infty}\frac{l_n}{(2n+4)!}t^{2n+4},
\end{aligned}
$$

where

$$
\begin{aligned}
l_n &= (2n+4)\bigl(5^{2n+3}-5\cdot 3^{2n+3}+10\bigr)+8(2n+4)\bigl(3^{2n+3}-3\bigr)\\
&\quad+16(2n+4)(2n+3)(2n+2)(2n+1)-\bigl(5^{2n+4}-3\cdot 3^{2n+4}+2\bigr)\\
&\quad-4(2n+4)(2n+3)(2n+2)\bigl(3^{2n+1}-3\bigr)-32(2n+4)(2n+3)(2n+2)\\
&=(250n-125)25^n+\bigl(279-462n-432n^2-96n^3\bigr)9^n\\
&\quad+256n^4+1{,}120n^3+1{,}520n^2+532n-154,\quad n=3,4,\ldots.
\end{aligned}
$$

Using a basic differential method, we can easily prove

$$
\begin{aligned}
f(x) &\triangleq (250x-125)25^x+\bigl(279-462x-432x^2-96x^3\bigr)9^x\\
&\quad+256x^4+1{,}120x^3+1{,}520x^2+532x-154>0
\end{aligned}
$$

on $[3,\infty)$. This leads to $l_n>0$ for $n=3,4,\ldots$, and $D(t)>0$. So, the proof of Lemma 2.2 is complete. □

3 Proof of Theorem 1.3

Let

$$
F(t)\equiv\frac{(\frac{t}{\sinh t}e^{t\coth t-1})^p-1}{(t\coth t)^p-1}=\frac{f_1(t)-f_1(0^+)}{g_1(t)-g_1(0^+)},
$$

where $f_1(t)=(\frac{t}{\sinh t}e^{t\coth t-1})^p$ and $g_1(t)=(t\coth t)^p$. Then

$$
k_1(t)\triangleq\frac{f_1'(t)}{g_1'(t)}=\frac{e^{p(t\coth t-1)}}{(\cosh t)^{p-1}}\cdot\frac{\sinh^2 t-t^2}{\sinh t(\sinh t\cosh t-t)}.
$$

We compute

$$
k_1'(t)=\frac{e^{p(t\coth t-1)}}{(\cosh t)^p}\cdot\frac{u_1(t)}{(\sinh t)^3(\sinh t\cosh t-t)^2},
$$

where

$$
\begin{aligned}
u_1(t) &= 2t^2\sinh^4 t\cosh t+\sinh^4 t\cosh t-4t\sinh^5 t\\
&\quad-3t\sinh^3 t+3t^2\sinh^2 t\cosh t-t^3\sinh t\\
&\quad-p\bigl(t\sinh^5 t+2t\sinh^3 t+t^4\cosh t-\sinh^4 t\cosh t-t^3\sinh^3 t-2t^3\sinh t\bigr)\\
&=2t^2\sinh^4 t\cosh t+\sinh^4 t\cosh t-4t\sinh^5 t\\
&\quad-3t\sinh^3 t+3t^2\sinh^2 t\cosh t-t^3\sinh t-pD(t).
\end{aligned}
$$

If $0 < p \leq 8/5$, by Lemma 2.2 we have

$$\begin{aligned} 5u_1(t) &\geq 10t^2 \sinh^4 t \cosh t + 13 \sinh^4 t \cosh t - 28t \sinh^5 t \\ &\quad - 46t \sinh^3 t + 30t^2 \sinh^2 t \cosh t + 6t^3 \sinh t - 8t^4 \cosh t + 8t^3 \sinh^3 t \\ &= \sum_{n=3}^{\infty} \frac{h_n}{16(2n+4)!} t^{2n+4}, \end{aligned}$$

where

$$\begin{aligned} h_n &= 10(2n+4)(2n+3)\left(5^{2n+2} - 3 \cdot 3^{2n+2} + 2\right) + 13\left(5^{2n+4} - 3 \cdot 3^{2n+4} + 2\right) \\ &\quad - 28(2n+4)\left(5^{2n+3} - 5 \cdot 3^{2n+3} + 10\right) - 184(2n+4)\left(3^{2n+3} - 3\right) \\ &\quad + 120(2n+4)(2n+3)\left(3^{2n+2} - 1\right) + 96(2n+4)(2n+3)(2n+2)(2n+1)2n \\ &\quad - 128(2n+4)(2n+3)(2n+2)(2n+1) + 96(2n+4)(2n+3)(2n+2)\left(3^{2n} - 1\right) \\ &= \left(1{,}000n^2 - 3{,}500n - 2{,}875\right)25^n + \left(768n^3 + 6{,}696n^2 + 13{,}956n + 4{,}113\right)9^n \\ &\quad + (2n+4)(2n+3)(2n+2)(2n+1)(192n - 128) \\ &\quad - 96(2n+4)(2n+3)(2n+2) - 100(2n+4)(2n+3) + 272(2n+4) + 26 \\ &> 0 \end{aligned}$$

for $n = 3, 4, \ldots$.

We have $u_1(t) > 0$ for $0 < p \leq 8/5$. So, $k_1'(t) > 0$ for $t > 0$, and $f_1'(t)/g_1'(t) = k_1(t)$ is increasing on $(0, +\infty)$. Hence, $F(t)$ is increasing on $(0, +\infty)$ by Lemma 2.1. At the same time, $\lim_{t \to 0^+} F(t) = 1/2$ and $\lim_{t \to +\infty} F(t) = (2/e)^p$. So, the proof of Theorem 1.3 is complete.

4 Proof of Theorem 1.4

Let

$$S(t) \equiv \frac{(\frac{\sinh t}{t} e^{1 - t \coth t})^p - 1}{e^{p(1 - t \coth t)} - 1} = \frac{f_2(t) - f_2(0^+)}{g_2(t) - g_2(0^+)},$$

where $f_2(t) = (\frac{\sinh t}{t} e^{1 - t \coth t})^p$ and $g_2(t) = e^{p(1 - t \coth t)}$. Then

$$k_2(t) \triangleq \frac{f_2'(t)}{g_2'(t)} = \left(\frac{\sinh t}{t}\right)^{p-1} \frac{(\sinh t)^3 - t^2 \sinh t}{t^2(\sinh t \cosh t - t)},$$

and

$$k_2'(t) = \left(\frac{\sinh t}{t}\right)^{p-2} \frac{u_2(t)}{t^4(\sinh t \cosh t - t)^2},$$

where

$$\begin{aligned} &u_2(t) \\ &\quad = \left[t \sinh^6 t + 2t \sinh^4 t - \sinh^5 t \cosh t - t^3 \sinh^4 t - 2t^3 \sinh^2 t + \frac{t^4}{2} \sinh 2t\right](p-1) \end{aligned}$$

$$+ \left(t\sinh^6 t + 5t\sinh^4 t + t^3\sinh^4 t - t^3\sinh^2 t - 3t^2\sinh^3 t\cosh t + t^4\sinh t\cosh t - 2\sinh^5 t\cosh t\right)$$

$$= \sum_{n=3}^{\infty}\left[c_n(p-1) + d_n\right]t^{2n+5} = \sum_{n=3}^{\infty} c_n\left[p - \left(1 - \frac{d_n}{c_n}\right)\right]t^{2n+5} = \sum_{n=3}^{\infty} c_n[p - e_n]t^{2n+5},$$

where $e_n = 1 - (d_n/c_n)$ and

$$c_n = \frac{1}{2^5}\frac{6^{2n+4} - 6\cdot 4^{2n+4} + 15\cdot 2^{2n+4}}{(2n+4)!} + \frac{1}{2^2}\frac{4^{2n+4} - 4\cdot 2^{2n+4}}{(2n+4)!} - \frac{1}{2^3}\frac{4^{2n+2} - 4\cdot 2^{2n+2}}{(2n+2)!}$$
$$- \frac{1}{2^5}\frac{6^{2n+5} - 4\cdot 4^{2n+5} + 5\cdot 2^{2n+5}}{(2n+5)!} - \frac{2^{2n+2}}{(2n+2)!} + \frac{1}{2}\frac{2^{2n+1}}{(2n+1)!} > 0, \quad n = 3, 4, \ldots,$$

$$d_n = \frac{1}{2^5}\frac{6^{2n+4} - 6\cdot 4^{2n+4} + 15\cdot 2^{2n+4}}{(2n+4)!} + \frac{5}{2^3}\frac{4^{2n+4} - 4\cdot 2^{2n+4}}{(2n+4)!} + \frac{1}{2^3}\frac{4^{2n+2} - 4\cdot 2^{2n+2}}{(2n+2)!}$$
$$- \frac{1}{2^4}\frac{6^{2n+5} - 4\cdot 4^{2n+5} + 5\cdot 2^{2n+5}}{(2n+5)!} - \frac{3}{2^3}\frac{4^{2n+3} - 2\cdot 2^{2n+3}}{(2n+3)!} + \frac{1}{2}\frac{2^{2n+1}}{(2n+1)!}$$
$$- \frac{1}{2}\frac{2^{2n+2}}{(2n+2)!}, \quad n = 3, 4, \ldots.$$

Let

$$j(n) = -12(2n+5)\left(4^{2n+4} - 4\cdot 2^{2n+4}\right) + \left(6^{2n+5} - 4\cdot 4^{2n+5} + 5\cdot 2^{2n+5}\right)$$
$$- 8(2n+5)(2n+4)(2n+3)\left(4^{2n+2} - 4\cdot 2^{2n+2}\right)$$
$$- 16(2n+5)(2n+4)(2n+3)2^{2n+2} + 12(2n+5)(2n+4)\left(4^{2n+3} - 2\cdot 2^{2n+3}\right)$$
$$= 7{,}776\cdot 36^n + \left[768(2n+5)(2n+4) - 128(2n+5)(2n+4)(2n+3) - 3{,}072(2n+5) - 4{,}096\right]16^n$$
$$+ \left[64(2n+5)(2n+4)(2n+3) - 192(2n+5)(2n+4) + 768(2n+5) + 160\right]4^n,$$

$$i(n) = (2n+5)\left(6^{2n+4} - 6\cdot 4^{2n+4} + 15\cdot 2^{2n+4}\right) + 8(2n+5)\left(4^{2n+4} - 4\cdot 2^{2n+4}\right)$$
$$- \left(6^{2n+5} + 4\cdot 4^{2n+5} - 5\cdot 2^{2n+5}\right) - 16(2n+5)(2n+4)(2n+3)\left(4^{2n+1} - 2^{2n+2}\right)$$
$$- 32(2n+5)(2n+4)(2n+3)2^{2n+2} + 32(2n+5)(2n+4)(2n+3)(2n+2)2^{2n}$$
$$= (2{,}592n - 1{,}296)\cdot 36^n + \left[512(2n+5) + 4{,}096 - 64(2n+5)(2n+4)(2n+3)\right]16^n$$
$$+ \left[32(2n+5)(2n+4)(2n+3)(2n+2) - 64(2n+5)(2n+4)(2n+3) - 272(2n+5) - 160\right]4^n.$$

Then

$$e_n = 1 - \frac{d_n}{c_n} = \frac{j(n)}{i(n)}.$$

Let $\Delta(n) = 286i(n) - 693j(n)$. Then

$$\Delta(n) = (741{,}313n - 5{,}759{,}424)36^n + 16^n\left[2{,}275{,}328(2n+5) + 4{,}009{,}984 + 70{,}400(2n+5)(2n+4)(2n+3) - 532{,}224(2n+5)(2n+4)\right]$$

$$+ 4^n\big[9{,}152(2n+5)(2n+4)(2n+3)(2n+2) - 62{,}656(2n+5)(2n+4)(2n+3) + 133{,}056(2n+5)(2n+4) - 610{,}016(2n+5) - 156{,}640\big].$$

First, we check that $\Delta(n) > 0$ for $n = 3, 4, 5, 6, 7$; second, we can easily obtain that $\Delta(n) > 0$ for $n \geq 8$. So, we have that $\Delta(n) > 0$ for $n = 3, 4, \ldots$.

So, we have $u_2(t) > 0$ for $p \geq 286/693$. So, $k_2'(t) > 0$ for $t > 0$, and $f_2'(t)/g_2'(t) = k_2(t)$ is increasing on $(0, +\infty)$. Hence, $S(t)$ is increasing on $(0, +\infty)$ by Lemma 2.1 when $p \geq 286/693$. At the same time, $\lim_{t\to 0^+} S(t) = 1/2$ and $\lim_{t\to+\infty} S(t) = 1$. So, the proof of Theorem 1.4 is complete.

5 Applications of theorems

In this section, we assume that x and y are two different positive numbers. Let $A(x,y)$, $G(x,y)$, $L(x,y)$, and $I(x,y)$ be the arithmetic, geometric, logarithmic, and identric means, respectively. Without loss of generality, we set $0 < x < y$. By the transformation $t = (\log(y/x))/2$, we can compute and obtain

$$\frac{L(x,y)}{G(x,y)} = \frac{\sinh t}{t},$$
$$\frac{I(x,y)}{G(x,y)} = e^{t\coth t - 1},$$
$$\frac{A(x,y)}{G(x,y)} = \cosh t,$$

where $t > 0$.

Now, the four results in Section 1 are equivalent to the following ones for four classical means.

Theorem 5.1 *Let $p \geq 4/5$, and x and y be positive real numbers with $x \neq y$. Then*

$$\alpha A^p(x,y) + (1-\alpha)G^p(x,y) < L^p(x,y) < \beta A^p(x,y) + (1-\beta)G^p(x,y) \tag{5.1}$$

holds if and only if $\alpha \leq 0$ and $\beta \geq 1/3$.

Theorem 5.1 can deduce the following one, which is from Zhu [8].

Corollary 5.2 ([8, Theorem 1]) *Let $p \geq 1$, and x and y be positive real numbers with $x \neq y$. Then*

$$\alpha A^p(x,y) + (1-\alpha)G^p(x,y) < L^p(x,y) < \beta A^p(x,y) + (1-\beta)G^p(x,y) \tag{5.2}$$

holds if and only if $\alpha \leq 0$ and $\beta \geq 1/3$.

When letting $p = 1$ in Theorem 5.1, one can obtain the result (see [12–14], [15, Theorem 1]).

Corollary 5.3 *Let x and y be positive real numbers with $x \neq y$. Then*

$$\alpha A(x,y) + (1-\alpha)G(x,y) < L(x,y) < \beta A(x,y) + (1-\beta)G(x,y) \tag{5.3}$$

holds if and only if $\alpha \leq 0$ and $\beta \geq 1/3$.

When letting $\beta = 1/3$ in the right-hand inequality of (5.3), one can obtain the well-known inequality by Carlson [16]

$$L(x,y) < \frac{1}{3}A(x,y) + \frac{2}{3}G(x,y). \tag{5.4}$$

Theorem 5.4 *Let $p > 0$. Then*

(1) *if $0 < p \leq 6/5$, the double inequality*

$$\alpha A^p(x,y) + (1-\alpha)G^p(x,y) < I^p(x,y) < \beta A^p(x,y) + (1-\beta)G^p(x,y) \tag{5.5}$$

holds if and only if $\alpha \leq 2/3$ and $\beta \geq (2/e)^p$;

(2) *if $p \geq 2$, the double inequality*

$$\alpha A^p(x,y) + (1-\alpha)G^p(x,y) < I^p(x,y) < \beta A^p(x,y) + (1-\beta)G^p(x,y) \tag{5.6}$$

holds if and only if $\alpha \leq (2/e)^p$ and $\beta \geq 2/3$.

The part (2) of Theorem 5.4 is a result of Trif [17].

When letting $p = 2$ and $\beta = 2/3$ in the right-hand inequality of (5.6), one can obtain the following result, which is from Sándor and Trif [18].

$$I^2(x,y) < \frac{2}{3}A^2(x,y) + \frac{1}{3}G^2(x,y). \tag{5.7}$$

When letting $p = 1$ in the double inequality (5.5), one can obtain the following result (see [12], [15, Theorem 2]).

Corollary 5.5 *Let x and y be positive real numbers with $x \neq y$. Then*

$$\alpha A(x,y) + (1-\alpha)G(x,y) < I(x,y) < \beta A(x,y) + (1-\beta)G(x,y) \tag{5.8}$$

holds if and only if $\alpha \leq 2/3$ and $\beta \geq 2/e$.

When letting $\alpha = 2/3$ in the left-hand inequality in (5.8), one can obtain the following result, which is from Sándor [19].

$$\frac{2}{3}A(x,y) + \frac{1}{3}G(x,y) < I(x,y). \tag{5.9}$$

Theorem 5.6 *Let $0 < p \leq 8/5$, x and y be positive real numbers with $x \neq y$. Then*

$$\alpha A^p(x,y) + (1-\alpha)L^p(x,y) < I^p(x,y) < \beta A^p(x,y) + (1-\beta)L^p(x,y) \tag{5.10}$$

holds if and only if $\alpha \leq 1/2$ and $\beta \geq (2/e)^p$.

Theorem 5.6 can deduce the following result (see Zhu [15]).

Corollary 5.7 ([15, Theorem 3]) *Let x and y be positive real numbers with $x \neq y$. Then*

$$\alpha A(x,y) + (1-\alpha)L(x,y) < I(x,y) < \beta A(x,y) + (1-\beta)L(x,y) \tag{5.11}$$

holds if and only if $\alpha \leq 1/2$ and $\beta \geq 2/e$.

When letting $\alpha = 1/2$ in the left-hand inequality of (5.11), one can obtain the following result, which is from Sándor [4, 19].

$$I(x,y) > \frac{A(x,y) + L(x,y)}{2}. \tag{5.12}$$

Finally, we give the bounds for $L^p(x,y)$ in terms of $G^p(x,y)$ and $I^p(x,y)$, and obtain the following new result.

Theorem 5.8 *Let x and y be positive real numbers with $x \neq y$, and $p \geq 286/693$. Then*

$$\alpha G^p(x,y) + (1-\alpha)I^p(x,y) < L^p(x,y) < \beta G^p(x,y) + (1-\beta)I^p(x,y) \tag{5.13}$$

holds if and only if $\beta \leq 1/2$ and $\alpha \geq 1$.

Theorem 5.8 can deduce a result of Zhu [15]:

Corollary 5.9 ([15, Theorem 4]) *Let x and y be positive real numbers with $x \neq y$. Then*

$$\alpha G(x,y) + (1-\alpha)I(x,y) < L(x,y) < \beta G(x,y) + (1-\beta)I(x,y) \tag{5.14}$$

holds if and only if $\beta \leq 1/2$ and $\alpha \geq 1$.

Obviously, the right-hand side of (5.14) is an extension of the following inequality:

$$L(x,y) < \frac{1}{2}\big(G(x,y) + I(x,y)\big), \tag{5.15}$$

which was given by Alzer [5].

Competing interests
The author declares that they have no competing interests.

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