# Dressed-State Approach to Population Trapping in the Jaynes-Cummings Model

D. Jonathan<sup>\*</sup>, K. Furuya<sup>†</sup>, A. Vidiella-Barranco<sup>‡</sup> Instituto de Física "Gleb Wataghin", Universidade Estadual de Campinas, CEP 13083-970, Campinas, Brazil.

## Abstract

The phenomenon of atomic population trapping in the Jaynes-Cummings Model is analysed from a dressed-state point of view. A general condition for the occurrence of partial or total trapping from an arbitrary, pure initial atom-field state is obtained in the form of a bound to the variation of the atomic inversion. More generally, it is found that in the presence of initial atomic or atom-field coherence the population dynamics is governed not by the field's initial photon distribution, but by a 'weighted dressedness' distribution characterising the joint atom-field state. In particular, individual revivals in the inversion can be analytically described to good approximation in terms of that distribution, even in the limit of large population trapping. This result is obtained through a generalisation of the Poisson Summation Formula method for analytical description of revivals developed by Fleischhauer and Schleich [Phys. Rev. A **47**, 4258 (1993)].

PACS numbers: 42.50.Ct

Typeset using REVT<sub>E</sub>X

<sup>\*</sup>Present address: Optics Section, The Blackett Laboratory, Imperial College, London SW7 2BZ, United Kingdom; e-mail: d.jonathan@ic.ac.uk

<sup>&</sup>lt;sup>†</sup>e-mail: furuya@ifi.unicamp.br

<sup>&</sup>lt;sup>‡</sup>e-mail: vidiella@ifi.unicamp.br

## I. INTRODUCTION

One of the most fundamental models of quantum optical resonance is the interaction of a single two-level atom with a single quantised mode of radiation, described by the Jaynes-Cummings Hamiltonian [1,2]. Despite being simple enough to be analytically soluble in the rotating-wave approximation, this model has been a long-lasting source of insight into the nuances of the interaction between light and matter. It has led to nontrivial predictions, such as the existence of 'collapses' and 'revivals' in the atomic excitation [3], and has also allowed a deeper understanding of the dynamical entangling and disentangling of the atomfield system in the course of time [4]. Further interest in the Jaynes-Cummings model (JCM) comes from the fact that these predictions are directly accessible to experimental verification. A JCM interaction can be experimentally realized in cavity-QED setups [5] and also, as an effective interaction, in laser-cooled trapped ions [6] (in which case the ionic harmonic motion assumes the role normally played by the field mode). For example, revivals in the atomic excitation have recently been observed in a cavity-QED experiment [7], providing direct evidence for the discreteness of photons.

In spite of these successes, a closed analytical description of the collapse-revival pattern has so far proved to be elusive; however, an elegant approximation scheme valid for a number of initial conditions has been presented by Fleischhauer and Schleich [8], improving the earlier work of Eberly and co-workers [3]. Among other things, they have demonstrated that, when the atom is initially completely excited or de-excited, and the initial photonnumber distribution of the field  $(P_n)$  is sufficiently smooth (as is the case in a coherent state) then the shape of each revival is a direct reflection of the shape of  $P_n$ . This direct relationship can be affected by the presence of initial atomic coherence. It has been noticed [9] that if the atom is initially prepared in a coherent superposition of its energy eigenstates, then the revivals can be largely suppressed, effectively freezing the value of the atomic state populations. Appropriately, these initial atomic states have been referred to as "atomic trapping states" [9,10]. This population trapping has been connected to the existence of a specific phase difference between the relative phase of the atomic superposition and the phase of the initial field state [9]. In particular, in the resonant case exact trapping has been shown to exist [11] if the field is initially prepared in a phase coherent state [12] (eigenstate of the Susskind-Glogower phase operator [13]). As has been pointed out by Cirac and Sánchez-Soto [14], this can be understood by the fact that in these cases only one state out of each pair of dressed eigenstates of the JCM is ever populated.

We now note that, since the atomic excitation dynamics can be modified merely by altering the initial atomic state, then it is apparent that the initial photon distribution cannot be the sole responsible for the existence and shape of revivals. In fact, as we demonstrate in this paper, the key to understand the collapse-revival pattern under more general initial conditions is to consider not only the initial coherence of the atom and field by themselves, but also their joint properties as a single quantum system. This is so even if the system is initially disentangled. Building on Cirac and Sánchez-Soto's observation, we find that the revival structure depends essentially on the *relative weight* of each dressed eigenstate in the initial atom-field state. We are able to identify an atom-field variable which plays, in the general case, the same role as that of the photon distribution when the atom is initially excited or de-excited. This allows us to estimate for any initial condition the degree of suppression of the revivals (that is, the amount of trapping which occurs). In addition, in some particular cases we are able to describe analytically to a good approximation the shape of the revivals, even when they are partially suppressed. To illustrate the dressedstate distribution dependence of the revival patterns, we show that essentially identical examples can be found both in the presence of initial atomic coherence, and in cases where this coherence is not only atomic, but where atom and field are already entangled.

Our work is organised as follows: in section II, we rewrite the dynamics of the Jaynes-Cummings model from a point of view based on the entangled dressed-state eigenbasis. With the help of an appropriate coordinate system, the expression for the atomic inversion assumes a simpler form; as a consequence, we are able to calculate bounds for the evolution of this quantity for arbitrary initial conditions. In order to illustrate our technique, specific cases are treated in section III. In section IV, we generalise Fleischhauer and Schleich's scheme to include the cases where trapping occurs. Our conclusions are summarised in section V.

## II. DRESSED-STATE COORDINATES FOR THE JCM

#### A. The Model

The Jaynes-Cummings Hamiltonian, on resonance and in the rotating-wave approximation is given by [1,2]:

$$H = \frac{1}{2}\hbar\omega\hat{\sigma}_z + \hbar\omega\hat{a}^{\dagger}\hat{a} + \hbar\lambda(\hat{\sigma}_+\hat{a} + \hat{a}^{\dagger}\hat{\sigma}_-)$$
(1)

where  $\omega$  is the atomic transition frequency,  $\lambda$  is the atom-field dipole coupling constant,  $\hat{a}^{\dagger}(\hat{a})$  is the field photon creation (annihilation) operator and  $\hat{\sigma}_z$ ,  $\hat{\sigma}_{\pm}$  are the atomic inversion, rising and lowering operators. The first two terms in this expression describe the internal energy levels of the spin-1/2-like 2-level atom and the harmonic oscillator-like field mode, and the interaction term can be straightforwardly understood as a one-excitation exchange between these two systems.

An important feature of this fully-quantised matter-radiation interaction is that its steady states (known as 'dressed states') are *entangled*. Switching to an interaction-picture representation for convenience, it can be shown that these states and their corresponding energy levels are [15]:

$$|n\pm\rangle = \frac{1}{\sqrt{2}} \left[|e,n\rangle \pm |g,n+1\rangle\right]$$
 (2a)

$$E_{n\pm} = \pm \hbar \lambda \sqrt{n+1} \equiv \pm \frac{\hbar}{2} \Omega_n \tag{2b}$$

 $(\Omega_n \text{ is the Rabi frequency})$ . It is apparent that in all these cases the atomic state is completely undetermined, being an equal mixture of the excited and ground states. Thus, if we assume that at t = 0 the atom and field are independently prepared in the states

$$\left|\psi\left(0\right)\right\rangle_{A} = p\left|e\right\rangle + q\left|g\right\rangle \tag{3}$$

$$\left|\Phi\left(0\right)\right\rangle_{F} = \sum_{n=0}^{\infty} c_{n} \left|n\right\rangle \tag{4}$$

then the initial state  $|\Psi(0)\rangle_{AF} = |\psi(0)\rangle_A \otimes |\Phi(0)\rangle_F$  will evolve into

$$\begin{split} |\Psi(t)\rangle_{AF} &= qc_0 |g, 0\rangle + \\ &+ \sum_{n=0}^{\infty} \left[ \begin{array}{c} pc_n \cos\left(\lambda t \sqrt{n+1}\right) - \\ -iqc_{n+1} \sin\left(\lambda t \sqrt{n+1}\right) \end{array} \right] |e, n\rangle + \\ &+ \sum_{n=0}^{\infty} \left[ \begin{array}{c} qc_{n+1} \cos\left(\lambda t \sqrt{n+1}\right) - \\ -ipc_n \sin\left(\lambda t \sqrt{n+1}\right) \end{array} \right] |g, n+1\rangle \end{split}$$
(5)

Despite being straightforwardly solvable in this way, the JCM is well-known for the fact that the time-evolution of most expectation values is usually expressible only in series form. For instance, assume that at time t = 0 we have:

$$|\Psi(0)\rangle = (p|e\rangle + q|g\rangle) \otimes \left(\sum_{n=0}^{\infty} c_n |n\rangle\right)$$
$$= qc_0 |g,0\rangle + \sum_{n=0}^{\infty} pc_n |e,n\rangle + qc_{n+1} |g,n+1\rangle$$
(6)

The time evolution of the atomic population inversion is then given by:

$$\langle \hat{\sigma}_{z} \rangle (t) = 2 \sum_{n=0}^{\infty} \left| pc_{n} \cos \left( \lambda t \sqrt{n+1} \right) - iqc_{n+1} \sin \left( \lambda t \sqrt{n+1} \right) \right|^{2} - 1$$

$$= - \left| qc_{0} \right|^{2} + \sum_{n=0}^{\infty} \left( \left| pc_{n} \right|^{2} - \left| qc_{n+1} \right|^{2} \right) \cos \left( 2\lambda t \sqrt{n+1} \right) +$$

$$+ 2Im \left( (pc_{n})^{*} qc_{n+1} \right) \sin \left( 2\lambda t \sqrt{n+1} \right)$$

$$(7)$$

where

$$\sum_{n=0}^{\infty} |pc_n|^2 + |qc_{n+1}|^2 = 1 - |qc_0|^2$$
(8)

In particular, if at t = 0 the atom is completely excited (q = 0) or in the ground state (p = 0), and admitting for simplicity that  $c_0$  is negligible, then this expression reduces to the simpler form

$$\langle \sigma_z \rangle (t) = \pm \sum_{n=0}^{\infty} P_n \cos\left(2\lambda\sqrt{n+1t}\right)$$
 (9)

where  $P_n = |c_n|^2$  is the initial photon distribution of the field. Due to the  $\sqrt{n}$  scaling of the various Rabi frequencies in this sum, no closed form for it is known. Nevertheless, it is apparent that features of this time evolution, such as collapse-revival phenomena, must be directly related to the characteristics of  $P_n$ . The Fleischhauer-Schleich approximation scheme, which we shall discuss in section IV, allows this relationship to be explicitly described in certain situations, such as when  $P_n$  is sufficiently smooth.

Apart from the simple case represented above, in general there is no way to relate the inversion dynamics exclusively to the initial field state. As can be seen from expression (7), it will usually depend on atomic and field variables in an apparently complicated way. In the next section, we show how this evolution can be recast in a simple form, reminiscent of expression (9), independently of the initial conditions.

#### **B.** Dressed-State Coordinates

Our discussion will be based on the relative contribution of each dressed state for the global atom-field state. For simplicity, we restrict ourselves to *pure* initial conditions. Therefore, we may expand an arbitrary initial state in terms of  $|n\pm\rangle$ :

$$|\Psi(0)\rangle = w_{-1}|g;0\rangle + \sum_{n=0}^{\infty} w_n e^{i\chi_n} |\psi_n\rangle,$$
 (10)

where

$$|\psi_n\rangle = \left[\cos\left(\frac{\theta_n}{2}\right)|n+\rangle + e^{-i\phi_n}\sin\left(\frac{\theta_n}{2}\right)|n-\rangle\right],\tag{11}$$

with the normalisation condition  $\sum_{n=-1}^{\infty} w_n^2 = 1$ . The parameters  $w_n \in [0, 1]$ ,  $\theta_n \in [0, \pi]$  and  $\chi_n, \phi_n \in [0, 2\pi]$ , will henceforth be referred to as *dressed-state coordinates*.

The use of this coordinate system allows us to obtain a simple geometrical picture of the manifold of pure atom-field states. For each value of  $n \ge 0$  one can imagine a Bloch-type spherical shell corresponding to the two-level system formed by  $|n\pm\rangle$  (Fig. 1), which are parametrised by the spherical coordinates  $\theta_n$  and  $\phi_n$ . The poles  $(\theta_n = 0, \pi)$  are associated to the dressed states  $|n\pm\rangle$ , and points on the equator  $(\theta_n = \frac{\pi}{2})$  to the states of the form  $\frac{1}{\sqrt{2}} \left[ |n+\rangle + e^{-i\phi_n} |n-\rangle \right]$ , which include the particular product states  $|e, n\rangle$  and  $|g, n + 1\rangle$  (corresponding to  $\phi_n = 0, \pi$ ). Therefore the coordinate  $\theta_n$  gives a measure of what we shall call the 'dressedness' (or degree of proximity to the nearest dressed state) of the components  $|\psi_n\rangle$  of  $|\Psi(0)\rangle$ . We note that, while this property is related to the entanglement of  $|\psi_n\rangle$  (since the dressed states are in fact maximally entangled), they are two different concepts: states with the same dressedness, such as  $|e, n\rangle$  and  $\cos \mu |e, n\rangle - i \sin \mu |g, n + 1\rangle$  can have different amounts of entanglement (as measured by the von Neumann entropy of their subsystems). Meanwhile, the weight factors  $w_n$  measure the relative importance of each of these components. In physical terms, their squares  $w_n^2$  correspond to the probability distribution for measurements of the *total excitation number* operator

$$\hat{X}_{tot} = \frac{1}{2}\hat{\sigma}_z + \hat{N}.$$
(12)

on the initial state  $|\Psi(0)\rangle$  ( $w_n^2$  corresponds to the probability for n + 1 excitations). We note that, in the special case where the atom is prepared in one of the states  $|e\rangle$  or  $|g\rangle$ , then  $\theta_n \equiv \frac{\pi}{2}$  and  $\phi_n \equiv 0 (\pi)$  for all n, while  $w_n^2$  reduces to the photon number distribution:

$$w_n^2 \to \begin{cases} P_n = \left| \langle n | \Phi(0) \rangle_F \right|^2 & \text{(if } |e\rangle) \\ P_{n+1} = \left| \langle n+1 | \Phi(0) \rangle_F \right|^2 & \text{(if } |g\rangle) \end{cases}$$
(13)

We now show that the dressed-state coordinate system allows a simple description of the time evolution of any initial state. From equations (2a,b), the evolution of state (10) is given by

$$|\Psi(t)\rangle = w_{-1}|g,0\rangle + \sum_{n=0}^{\infty} w_n e^{\frac{i}{2}(2\chi_n - \Omega_n t)} \left[ \cos\left(\frac{\theta_n}{2}\right)|n+\rangle + e^{-i(\phi_n - \Omega_n t)} \sin\left(\frac{\theta_n}{2}\right)|n-\rangle \right].$$
(14)

We can re-express this directly in terms of the dressed-state coordinates:

$$w_n(t) = w_n(0)$$
  

$$\theta_n(t) = \theta_n(0)$$
  

$$\chi_n(t) = \chi_n(0) - \frac{1}{2}\Omega_n t$$
  

$$\phi_n(t) = \phi_n(0) - \Omega_n t$$
  
(15)

Thus,  $w_n$  and  $\theta_n$  are both constants of motion, while the angles  $\chi_n$  and  $\phi_n$  precess with a constant angular velocity proportional to the Rabi frequency  $\Omega_n$ . A way of visualising this is by means of a geometrical image as in Fig. 1. The simplicity of this picture reflects the symmetries of the resonant JCM: the conservation of  $\hat{X}_{tot}$  and the fact that, in the interaction picture, the energies in each 2-dimensional eigensubspace generated by  $|n\pm\rangle$  are always symmetric with respect to zero.

#### 1. Population Inversion in Dressed-State Coordinates

Let us now apply this coordinate system to describe the evolution of atomic variables. In the basis  $\{|e\rangle, |g\rangle\}$ , the reduced atomic density operator can be expressed as:

$$\rho_A(t) = \begin{bmatrix} \rho_{ee}(t) & \rho_{eg}(t) \\ \rho_{eg}^*(t) & \rho_{gg}(t) \end{bmatrix}$$
(16)

where

$$\rho_{ee}(t) = \frac{1}{2} \left( 1 - w_{-1}^2 \right) + \frac{1}{2} \sum_{n=0}^{\infty} w_n^2 \sin(\theta_n) \cos(\phi_n(t)); \qquad \rho_{gg}(t) = 1 - \rho_{ee}(t)$$
(17a)  

$$\rho_{eg}(t) = \frac{w_{-1} w_0}{\sqrt{2}} e^{i(\chi_0(t))} \left( \cos\left(\frac{\theta_0}{2}\right) + e^{-i\phi_0(t)} \sin\left(\frac{\theta_0}{2}\right) \right) + \frac{1}{2} \sum_{n=0}^{\infty} w_n w_{n+1} e^{i(\chi_{n+1}(t) - \chi_n(t))} \left( \cos\left(\frac{\theta_{n+1}}{2}\right) + e^{-i\phi_{n+1}(t)} \sin\left(\frac{\theta_{n+1}}{2}\right) \right) \times \left( \cos\left(\frac{\theta_n}{2}\right) - e^{i\phi_n(t)} \sin\left(\frac{\theta_n}{2}\right) \right)$$
(17b)

We thus see that the evolution of the atomic level populations, and therefore of the population inversion, can be expressed in a simple form for any initial condition:

$$\langle \hat{\sigma}_{z}(t) \rangle = 2\rho_{ee}(t) - 1 = -w_{-1}^{2} + \sum_{n=0}^{\infty} w_{n}^{2} \sin(\theta_{n}) \cos(\phi_{n}(t)).$$
 (18)

This should be contrasted with the much more complicated expression (7) for this same quantity, written in terms of separate atomic and field coordinates. In fact, we can see that the present expression is a direct generalisation of eq. (9), obtained by phase-shifting the  $n^{th}$  term in the sum by  $\phi_n(0)$  and by replacing the photon number distribution  $P_n$  with

$$D_n = w_n^2 \sin\left(\theta_n\right). \tag{19}$$

It is not difficult to understand the reason for this substitution. First of all, as was noted by Cirac and Sánchez-Soto [14], dressed states have no population inversion. Therefore, if a component  $|\psi_n\rangle$  of a given state  $|\Psi(0)\rangle$  has a large 'dressedness'  $(\sin(\theta_n) \to 0)$ , it will contribute very little to the overall inversion. Moreover, even if  $|\psi_n\rangle$  does have a large inversion, this may still be of little consequence to the total average  $\langle \hat{\sigma}_z(t) \rangle$  if its relative importance  $w_n^2$  is small. Thus, the product  $D_n$ , which we shall refer to as the 'weighted dressedness distribution' gives the appropriate magnitude of the contribution of  $|\psi_n\rangle$  to the inversion. In other words, for general initial conditions it is this distribution, not  $P_n$ , that will control the evolution of the inversion. For instance, we can expect the existence of collapse-revival structures if  $D_n$  is 'narrow', in the sense that few values of n (and therefore few Rabi frequencies  $\Omega_n$ ) feature significantly in the sum.

## 2. Population Trapping

The main difference between the general evolutions allowed by equation (18) and those obtained in the special case (9) lies in the fact that the distribution  $P_n$  is normalised, while  $D_n$  is generally *not*. This implies that, in general, there will be an upper bound for the allowed range of variation of the population inversion, given by:

$$\left|\left\langle\hat{\sigma}_{z}\left(t\right)\right\rangle+w_{-1}^{2}\right|\leq\sum_{n=0}^{\infty}D_{n}\equiv M\leq1$$
(20)

As we can see, this bound is specified essentially by the *average dressedness* of the components of  $|\Psi(0)\rangle$ . For example: if the most important components (those with largest weights  $w_n^2$ ) are highly dressed (sin  $(\theta_n)$  is small), then the amplitude of variation of the population inversion will also be small  $(M \sim 0)$ . In other words, these are the conditions for the existence of *population trapping* in the (resonant) JCM. The steady-state value of the inversion is given simply by  $-w_{-1}^2$ , i.e., by the fixed population in the ground state. We emphasise that, given any initial condition, the bound M can be immediately calculated from the constants of the motion  $w_n$ ,  $\theta_n$ , thus giving an estimate of the amount of trapping that can be expected from that state's evolution [18].

### **III. EXAMPLES**

In order to illustrate our technique, we analyse three classes of initial atom-field states for which population trapping occurs:

## A. Perfect Trapping States

In [11], Cirac and Sánchez-Soto found a class of factorised initial conditions which exhibit *perfect* trapping, that is, for which the atomic populations are strictly constant over time. These were of the form:

$$|\Psi(0)\rangle = \sqrt{1 - |z^2|} (z |e\rangle + |g\rangle) \otimes \frac{1}{\sqrt{1 + |z^2|}} \sum_{n=0}^{\infty} z^n |n\rangle$$
 (21)

where |z| < 1. In this case, the field is prepared in an eigenstate of the Susskind-Glogower phase operator  $\hat{V} = \sum_{n=0}^{\infty} |n\rangle \langle n+1|$  [13] ('phase-coherent state' [12]), while the atomic superposition is chosen so that the phase of the atomic dipole matches that of the eigenvalue z of the field state [19]. The resulting population trapping can thus be attributed to a suitable matching of atomic and field parameters. However, as was later noticed by the same authors [14], a clearer explanation can be found by using a dressed-state point of view: Rewriting this state in terms of the dressed-state basis, we find

$$|\Psi(0)\rangle = \sqrt{\frac{1 - |z^2|}{1 + |z^2|}} \left[ |g, 0\rangle + \sum_{n=0}^{\infty} \sqrt{2} z^n |n+\rangle \right].$$
 (22)

The reason for the existence of trapping now becomes immediately clear: all the components  $|\psi_n\rangle$  of this state are completely dressed (sin  $(\theta_n) \equiv 0$ ), and thus do not contribute to the inversion.

Cirac and Sánchez-Soto also stated [11] that these are the *only* factorised states exhibiting perfect trapping. We note that this is not strictly true: the same result is still achieved for all states of the form

$$|\Psi(0)\rangle = \sqrt{1 - |z^2|} (z |e\rangle + |g\rangle) \otimes \frac{1}{\sqrt{1 + |z^2|}} \sum_{n=0}^{\infty} j(n) z^n |n\rangle$$
(23)

where j(n) can be  $\pm 1$  and |z| < 1 (in this case, the field state is not in general an eigenstate of  $\hat{V}$ ). Once again, this can be seen by rewriting the state in the dressed-state basis. In this case, we obtain

$$|\Psi(0)\rangle = \sqrt{\frac{1 - |z^2|}{1 + |z^2|}} \left[ j(0) |g, 0\rangle + \sum_{n=0}^{\infty} \sqrt{2} j(n) z^n |n\pm\rangle \right]$$
(24)

where for each *n* there is a single dressed state  $|n\pm\rangle$  present, whose sign is the same as that of the ratio  $\frac{j(n+1)}{j(n)}$ . The converse statement is also easy to show: suppose  $|\Psi(0)\rangle$  is a perfect trapping state, of the form:

$$|\Psi(0)\rangle = k (z |e\rangle + |g\rangle) \otimes \sum_{n=0}^{\infty} a_n |n\rangle$$
  
=  $ka_0 |g, 0\rangle + \sum_{n=0}^{\infty} za_n |e, n\rangle + a_{n+1} |g, n+1\rangle$  (25)

where k is a constant and  $a_n$ , z are arbitrary. Then for each  $|\psi_n\rangle$  component of this state to be perfectly dressed it is necessary that:

$$a_{n+1} = \pm z a_n \tag{26}$$

and hence

$$a_n \propto \pm z^n. \tag{27}$$

We recover expression (23) by noting that |z| must be less than 1 for  $|\Psi(0)\rangle$  to be normalisable. Thus, for the resonant JCM there can be no population trapping with positive population inversion.

#### B. Zaheer-Zubairy 'atomic trapping states'

While the initial conditions considered above lead to perfect trapping, preparing the appropriate field states requires elaborate state-engineering techniques. It has been shown (numerically) by Zaheer and Zubairy, however, that *approximate* population trapping is also possible if the cavity field is initially in a coherent state  $|\alpha\rangle$ , which are easily accessible in experiment [9]. This is portrayed in Fig. 2: if the atom is initially completely excited, the inversion evolves according to the familiar collapse-revival pattern [3]; however, by suitably rotating the atomic state, the revival peaks become more and more reduced, practically disappearing when the atom is prepared in state

$$\left|\psi\left(0\right)\right\rangle_{A} = \frac{1}{\sqrt{2}}\left(\left|e\right\rangle + e^{-i\nu_{a}}\left|g\right\rangle\right) \tag{28}$$

(where  $\nu_{\alpha}$  is the phase of the field's coherent amplitude  $\alpha$ ). Close inspection also reveals that, in this limit, the revival peaks become split into doublets.

In their work, Zaheer and Zubairy gave no quantitative explanation for this quenching of the revivals, attributing it qualitatively to a "destructive interference between the atomic dipole and the cavity eigenmode". Later, Cirac and Sánchez-Soto pointed out that the existence of approximate trapping in this case should be expected due to similarities between the properties of coherent states and of the phase coherent field states [11].

With the help of our dressed-state coordinate system, we can now give a quantitative description of this effect. Let us suppose that the atom-field system is prepared in the state:

$$|\Psi_{ZZ}(\alpha,\gamma,\xi)\rangle = \left[\cos\gamma |e\rangle + e^{-i\xi}\sin\gamma |g\rangle\right] \otimes |\alpha\rangle, \qquad (29)$$

where the atom is in a coherent superposition of ground and excited states and the field is in a coherent state ( $\alpha = |\alpha|e^{i\nu_{\alpha}}$ ). Expanding this state in terms of the dressed-state basis, we find that its dressed-state coordinates have the following values (we have only listed those relevant for the atomic inversion):

$$w_{n}^{2}(\alpha,\gamma,\xi) = \frac{|\alpha|^{2n}e^{-|\alpha|^{2}}}{(n+1)!} \left[ (n+1)\cos^{2}(\gamma) + |\alpha|^{2}\sin^{2}(\gamma) \right] , n \ge -1$$

$$\sin \theta_{n}(\alpha,\gamma,\xi) = \frac{|(n+1)\cos^{2}(\gamma) - |\alpha|^{2}e^{i2(\nu_{\alpha}-\xi)}\sin^{2}(\gamma)|}{(n+1)\cos^{2}(\gamma) + |\alpha|^{2}\sin^{2}(\gamma)} , \quad (\theta_{n}\in[0,\pi]) \quad (30)$$

$$\cos \phi_{n}(\alpha,\gamma,\xi) = \frac{(n+1)\cos^{2}(\gamma) - |\alpha|^{2}e^{i2(\nu_{\alpha}-\xi)}\sin^{2}(\gamma)|}{|(n+1)\cos^{2}(\gamma) - |\alpha|^{2}e^{i2(\nu_{\alpha}-\xi)}\sin^{2}(\gamma)|}$$

$$\sin \phi_{n}(\alpha,\gamma,\xi) = \frac{|\alpha|\sqrt{(n+1)}\sin(2\gamma)\sin(\nu_{\alpha}-\xi)}{|(n+1)\cos^{2}(\gamma) - |\alpha|^{2}e^{i2(\nu_{\alpha}-\xi)}\sin^{2}(\gamma)|} , \quad (\phi_{n}\in[0,2\pi]) .$$

The weight factors  $w_n^2$  follow a Poisson distribution when  $\gamma$  is equal to zero (they reduce to the photon number distribution of the coherent field) and have small deviations from that distribution for any value of  $\gamma$ . For moderately large  $|\alpha|$ , therefore,  $w_n^2$  is maximised around the integer  $n_{\text{max}}$  closest to  $|\alpha|^2$  or  $|\alpha|^2 - 1$ . Meanwhile, it is not difficult to show that the

 $\sin \theta_n$  distribution has an absolute minimum, located around  $n_{\min} \approx |\alpha|^2 \tan^2 \gamma - 1$ , with minimum value given by

$$\sin\left(\theta_{n_{\min}}\right) \simeq \sin\left(\left|\nu_{\alpha} - \xi\right|\right) \tag{31}$$

This indicates the component of  $|\Psi(\alpha, \gamma, \xi)\rangle$  having the largest dressedness.

Following the procedure outlined in section IIB2, we can now see that to attain the highest possible degree of population trapping we must maximise the largest dressedness, and match it with the largest weight factor:

$$n_{\rm max} = n_{\rm min} \to \tan \gamma \simeq 1 \tag{32a}$$

$$\sin\left(\theta_{n_{\min}}\right) = 0 \to \nu_{\alpha} = \xi \tag{32b}$$

We thus explain the "atomic trapping state" (28) found by Zaheer and Zubairy: it is the one for which the total atom-field state  $|\Psi_{ZZ}(\alpha, \gamma, \xi)\rangle$  most closely resembles a "perfect trapping state", in the sense that its most important components  $|\psi_n\rangle$  closely resemble dressed states.

It is clear from expression (30) that the degree of trapping experienced from initial state  $|\Psi_{ZZ}(\alpha,\gamma,\xi)\rangle$  depends both on the relative weight  $\gamma$  of each atomic state and on the phase difference  $(\nu_{\alpha} - \xi)$  between the atomic dipole phase and the field's coherent amplitude. In Fig. 3 we illustrate the change in the weighted dressedness distribution  $D_n$  as a function of these parameters: on the left-hand side, we take equal weights for  $|e\rangle$  and  $|g\rangle$ , satisfying condition (32a), and vary  $(\nu_{\alpha} - \xi)$  from  $\frac{\pi}{2}$  to 0. We can see the quenching of the distribution as the dressedness of the component  $|\psi_{n_{\max}}\rangle$  is increased. On the right-hand side, we keep the phase difference null and vary the relative weight from 0 to  $\frac{\pi}{4}$ , obtaining a similar effect (in both cases, we have chosen  $\alpha = 7$ ). Comparing this figure with Fig. 2, it is apparent that there is a striking similarity between the profile of the  $D_n$  distribution and the envelope of the revivals in the corresponding time evolution, including the appearance of a doublet structure in the limit of the optimal trapping conditions. As we shall see in section IV, this is no coincidence.

#### C. 'Even-odd' entangled states

The achievement of population trapping in the previous examples was seen to be ultimately caused by the quenching of the  $D_n$  distribution, and therefore by the joint properties of the atom-field system. However, since the initial states considered were factorised into atomic and field states, it could also be argued that the trapping was due to a suitable matching of independent atomic and field parameters (i.e., the phase of the field's coherent amplitude matching the atomic dipole phase in the previous example). In order to stress the underlying importance of the dressed-state point of view, we now show that essentially identical evolution and trapping patterns for the population inversion can be obtained even when the atom and field cannot be considered as independent quantum systems.

Consider the following class of states of the atom-field system:

$$|\Psi_{EO}(\alpha,\gamma,\xi)\rangle = \cos(\gamma) |e\rangle_A |even\rangle_f + \sin(\gamma)e^{i\xi} |g\rangle_A |odd\rangle_f, \qquad (33)$$

where  $|even\rangle_f = \frac{1}{\sqrt{2}} (|\alpha\rangle + |-\alpha\rangle)$  and  $|odd\rangle_f = \frac{1}{\sqrt{2}} (|\alpha\rangle - |-\alpha\rangle)$  are respectively even and odd coherent states of the cavity field (here  $|\alpha|$  is assumed large enough so that these states

can be considered normalised). It can be seen that, apart from the limiting cases where  $\gamma \to 0, \frac{\pi}{2}$ , these states are *entangled*. Furthermore, since  $|even\rangle_f (|odd\rangle_f)$  have nonzero amplitudes only for even (odd) photon numbers, an expansion of this state in terms of dressed states  $|n\pm\rangle$  features only those with *even* n (in other words,  $w_{2n-1} = 0$ ). Physically, this implies that states of this form have zero average electric and magnetic fields, and zero average atomic polarisation.

$$\left\langle \hat{a} + \hat{a}^{\dagger} \right\rangle = \left\langle \hat{a} - \hat{a}^{\dagger} \right\rangle = \left\langle \hat{\sigma}_x \right\rangle = \left\langle \hat{\sigma}_y \right\rangle = 0$$
 (34)

Thus, if at t = 0 the atom-filed system is in state  $|\Psi_{EO}(\alpha, \gamma, \xi)\rangle$ , the population inversion evolves according to eq. (18), but with only the even terms present:

$$\langle \hat{\sigma}_z(t) \rangle = \sum_{n=0}^{\infty} w_{2n}^2 \sin(\theta_{2n}) \cos(\phi_{2n}(t)).$$
(35)

Now, it turns out that all these even-indexed dressed-state coordinates have values virtually identical to the ones listed in expression (30) above for the state  $|\Psi_{ZZ}(\alpha, \gamma, \xi)\rangle$  with the same parameters  $\alpha, \gamma, \xi$ . The only difference is a multiplication of the weights  $w_{2n}$  in the present case by a factor of 2, to preserve normalisation in spite of the absence of odd-indexed components. The resulting time-evolution of the atomic inversion, plotted in Fig. 4, shows that the quenching of revivals and appearance of a doublet structure as  $(\nu_{\alpha} - \xi) \rightarrow 0$  proceed in essentially the same manner as in the previous case, the main difference being the doubling of the frequency of revivals [20]. We note, however, that in the present case neither of the parameters  $\nu_{\alpha}, \xi$  can be unambiguously assigned exclusively to the atom or field, so that population trapping in this case must necessarily be understood as a result of the joint atom-field properties of the state in question.

It is also worth remarking that, due to the lack of atomic polarisation, the atomic reduced density operators of these states are always diagonal in the  $|e\rangle$ ,  $|g\rangle$  basis. Therefore, the reduced entropy  $S_a = -Tr\rho_a \ln \rho_a$ , which measures the degree of entanglement between the atom and the field, is entirely determined by the population inversion:

$$S_{a}(t) = -\rho_{ee}(t) \ln \left(\rho_{ee}(t)\right) - \left(1 - \rho_{ee}(t)\right) \ln \left(1 - \rho_{ee}(t)\right)$$
$$= -\frac{1}{2} \left(1 - \langle \hat{\sigma}_{z}(t) \rangle\right) \ln \left(\frac{1}{2} \left(1 - \langle \hat{\sigma}_{z}(t) \rangle\right)\right)$$
$$-\frac{1}{2} \left(1 + \langle \hat{\sigma}_{z}(t) \rangle\right) \ln \left(\frac{1}{2} \left(1 + \langle \hat{\sigma}_{z}(t) \rangle\right)\right). \tag{36}$$

Thus, in this case the existence of population trapping is equivalent to the atom and field remaining (nearly) maximally entangled during their entire time-evolution. For any given state  $|\Psi_{EO}(\alpha, \gamma, \xi)\rangle$ , a lower bound to the value of  $S_a(t)$  at any given instant of the evolution is then given by:

$$S_{\min} = -\frac{1}{2} \left( 1 - M \right) \ln \left( \frac{1}{2} \left( 1 - M \right) \right) - \frac{1}{2} \left( 1 + M \right) \ln \left( \frac{1}{2} \left( 1 + M \right) \right)$$
(37)

where M is the bound defined in eq. (20). For instance, in the case of the 'trapping state' depicted in Fig. 4, where,  $\alpha = 7$ ,  $\gamma = \frac{\pi}{4}$ ,  $\xi = 0$ , we have  $S_{\min} = 0.69005$ , very close to the maximum possible value  $\ln 2 \simeq 0.69315$ . Finally, we note that despite the existence of this lower bound, it is still possible to devise schemes by which such entangled atom-field states can be constructed [16,17].

## IV. POISSON SUMMATION FORMULA FOR REVIVALS IN THE CASE OF POPULATION TRAPPING

In [8], Fleischhauer and Schleich obtained approximate analytical expressions for the evolution of the atomic inversion, using a stationary-phase method based on the *Poisson* Summation Formula [22]. They showed that in many cases the inversion can be written as a sum in which each term  $\omega_k(t)$  is non-negligible only during a certain extension of time. This is in contrast for instance with the expressions used in section II A above, where each term in the summation is periodic.

Their work was restricted to initial states of the form  $|g\rangle_A \otimes |\Phi\rangle_F$ , in which the inversion is given by eq. (9). They were able to show that, if the photon distribution  $P_n$  of  $|\Phi\rangle_F$  is sufficiently smooth, then the  $k^{th}$  term in the Poisson sum for the inversion is given by

$$\omega_k(t) = -\frac{\lambda t}{\pi\sqrt{2k^3}} P\left(n = \frac{\lambda^2 t^2}{4\pi^2 k^2}\right) \cos\left(\frac{\lambda^2 t^2}{2\pi k} - \frac{\pi}{4}\right)$$
(38)

where P(n) is a continuous interpolation of  $P_n$ .

The main interest of this formulation is the fact that, if  $P_n$  is also sufficiently narrow, then the term  $\omega_k(t)$  describes to a high accuracy the evolution of the inversion during the  $k^{th}$  revival. This can be readily seen from the formula above: this term describes a rapid oscillation in time, modulated by an envelope centred around  $t \simeq \frac{2\pi k \sqrt{\langle n \rangle}}{\lambda}$  and whose format is essentially that of  $P_n$ . Therefore, for narrow enough  $P_n$ , some of the terms  $\omega_k(t)$ become completely disjoint from the rest, describing an independent revival. Fleischhauer and Schleich were able to use this formulation to derive many interesting results, such as the decrease in amplitude and increase in width of each successive revival, and also the number of revivals that can be resolved for a given state, before they become scrambled due to their increasing width and consequent interference with each other. Finally, they were also able to extend the technique to some cases where  $P_n$  is not smooth, such as in squeezed states.

Using our dressed-state formulation for the population inversion, eq. (18), we are able to extend the Poisson Summation Formula method to even more general initial states, including ones with atomic or atom-field coherence. In the Appendix we show that, under suitable conditions, the behaviour of the atomic inversion during its  $k^{th}$  revival is described by:

$$\langle \sigma_z \left( t \right) \rangle \simeq \left( \frac{\lambda t}{\pi \sqrt{2k^3}} \right) \left[ D\left( n \right) \cos \left( \phi_n \left( 0 \right) + \frac{\lambda^2 t^2}{2\pi k} - \frac{\pi}{4} \right) \right] \bigg|_{n+1 = \frac{\lambda^2 t^2}{4\pi^2 k^2}}$$
(39)

where D(n) is a continuous interpolation of the 'weighted dressedness' distribution  $D_n$  of the initial state. Thus, in general it is this distribution, not  $P_n$ , that is reflected in the shape of each revival. In particular, in the case of population trapping, the quenching of the revivals mirrors that of  $D_n$ . This explains the similarity between the profiles in Figs. 2 and 3, including the doublet structure of the revivals in the limit of maximum trapping.

In order to display the accurateness of this expression, we use it to calculate the revivals in the case of the Zaheer-Zubairy states  $|\Psi_{ZZ}(\alpha, \gamma, \xi)\rangle$  introduced above. From expressions (30) for the dressed coordinates, we see that in this case the weighted dressedness distribution is given by

$$D_n(\alpha, \gamma, \xi) = \sqrt{Q_1^2(n) + Q_2^2(n) - 2Q_1(n) \cdot Q_2(n) \cdot \cos 2(\nu_\alpha - \xi)}$$
(40)

where

$$Q_1(n) = \exp\left(-\left|\alpha\right|^2\right) \frac{\left|\alpha\right|^{2n}}{(n)!} \cos^2\left(\gamma\right)$$
(41a)

$$Q_2(n) = \exp\left(-|\alpha|^2\right) \frac{|\alpha|^{2(n+1)}}{(n+1)!} \sin^2(\gamma)$$
(41b)

For sufficiently large  $\alpha$ , these Poissonian distributions are well approximated by Gaussians:

$$Q_1(n) \simeq \frac{\cos^2(\gamma)}{\sqrt{2\pi} |\alpha|} \exp\left(\frac{-\left(n - |\alpha|^2\right)^2}{2 |\alpha|^2}\right)$$
(42a)

$$Q_2(n) \simeq \frac{\sin^2(\gamma)}{\sqrt{2\pi} |\alpha|} \exp\left(\frac{-\left(n+1-|\alpha|^2\right)^2}{2 |\alpha|^2}\right).$$
(42b)

Extending these to continuous values of n and substituting in (40) (39) we obtain an analytical expression for the  $k^{th}$  revival in the inversion. In Fig. 5 we plot the first two of these (k = 1, 2) in the cases of maximum and minimum population trapping, alongside the corresponding exact evolutions obtained numerically. Despite a little distortion, agreement is seen to be very good.

A similar calculation is also possible in the case of the entangled 'even-odd' states  $|\Psi_{EO}(\alpha, \gamma, \xi)\rangle$  presented in the previous section. In this case, formula (39) given above for the inversion is not valid, due to the strong oscillations in  $D_n$  [ $D_n = 0$  for odd n]. However, it is possible to adapt our calculations to this situation (see section A 1 in the Appendix), obtaining the expression:

$$\langle \sigma_z \left( t \right) \rangle \simeq \left( \frac{\lambda t}{\pi \sqrt{k^3}} \right) \left[ D\left( m \right) \cos \left( \phi_{2m}(0) + \frac{\lambda^2 t^2}{2\pi k} + \pi k - \frac{\pi}{4} \right) \right] \Big|_{m+1 = \frac{\lambda^2 t^2}{\pi^2 k^2}}$$
(43)

for the inversion during the  $k^{th}$  revival. Here D(m) is a continuous interpolation of the evenindexed terms of  $D_n$ , renumbered with the new index  $m(D(m=3) = D_6$ , for instance). Comparing with expression (39) above for the Zaheer-Zubairy states, we can see that the main difference in the present case is that the frequency of revivals is *doubled*: the  $k^{th}$ revival occurs around  $t_k^{EO} \simeq \frac{\pi k \sqrt{\langle m \rangle + 1}}{\lambda}$ , compared to  $t_k^{ZZ} = \frac{2\pi k \sqrt{\langle n \rangle + 1}}{\lambda}$  in the previous case. This difference is due essentially to the doubling of the separation between adjacent Rabi frequencies in the present case [21], and is illustrated by comparing Figs. 2 and 4.

## V. CONCLUSION

Intuitive pictures of the interaction between a two-level atom and an electric field commonly involve the expectation that the atomic level populations must change as both systems exchange excitations over the course of time [24]. This is due to the absence of further

atomic levels, which precludes the existence of destructive interference between different atomic transitions. However, in a fully-quantised interaction model such as the Jaynes-Cummings model, it is indeed possible to have states in which the atomic populations are completely or nearly completely trapped. This can be ultimately traced to the fact that the eigenstates of this model are entangled. We have shown that, by giving up the traditional point of view based on the individuality of each subsystem and assuming instead one based on these entangled dressed states, it is possible to obtain a quantitative understanding of population trapping in this model. This is achieved via the introduction of a set of joint atom-field state variables, the 'weighted dressedness' distribution  $D_n$  (eq. 19). We have shown that, for general initial conditions, this distribution governs the evolution of the atomic inversion, assuming a role commonly attributed to the photon number distribution  $P_n$  (to which it reduces in particular cases). Using  $D_n$ , we have obtained an upper bound to the amplitude of population oscillations that can be expected from a given state at any instant of its evolution. We have also been able to obtain an approximate analytical description of the behaviour of the atomic inversion during partially suppressed revivals. We have found that in general the shape of revival envelopes is a direct reflection of the form of (a continuous interpolation of )  $D_n$ . In the particular case of a field initially in a coherent state, this explains the appearance of a doublet structure in the revivals in the limit of greatest population trapping.

## VI. ACKNOWLEDGEMENTS

We would like to thank Dr. S.M. Dutra and Prof. P.L. Knight for helpful comments and discussions. This work was supported by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) and Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP), Brazil.

## APPENDIX A: APPROXIMATE EXPRESSIONS FOR REVIVALS IN THE ATOMIC POPULATION INVERSION FOR GENERALISED INITIAL CONDITIONS

In this Appendix, we calculate approximate expressions for the revivals in the atomic population inversion, which are valid for a wide variety of *pure* initial conditions of the atom-field system. These calculations generalise the method presented by Fleischhauer and Schleich [8], who assumed an initially factorised state of the form  $|g\rangle_A \otimes |\psi\rangle_F$ . The result they obtained for the  $k^{th}$  revival is:

$$\langle \hat{\sigma}_z(t) \rangle \simeq -P\left(n = \frac{\lambda^2 t^2}{4\pi^2 k^2}\right) \frac{\lambda t}{\pi\sqrt{2k^3}} \cos\left(\frac{\lambda^2 t^2}{2\pi k} - \frac{\pi}{4}\right)$$
 (A1)

where  $P_n$  is the photon distribution of state  $|\psi\rangle_C$ . In the case of more general initial conditions, including ones with atomic and atom-field coherence, we shall see that the form of the revival envelope depends not on the state's photon-number distribution, but on the 'weighted dressedness' distribution  $D_n$  given in equation (19).

As was shown in section IIB1, the inversion, written in terms of dressed-state coordinates, has the form:

$$\left\langle \hat{\sigma}_{z}\left(t\right)\right\rangle = -w_{-1} + \sum_{n=0}^{\infty} D_{n} \cos\left(\phi_{n}\left(t\right)\right) \tag{A2}$$

where

$$\phi_n(t) = \phi_n(0) - \Omega_n t \tag{A3}$$

This expression can be rewritten according to the Poisson Summation Formula (see Courant and Hilbert [22], p. 76)

$$\left\langle \hat{\sigma}_{z}\left(t\right)\right\rangle = \sum_{k=-\infty}^{\infty} \omega_{k}\left(t\right) + \tau_{0}\left(t\right) \tag{A4}$$

where

$$\omega_k(t) = \int_0^\infty dn D(n) \cos(\phi_n(0) - \Omega_n t) e^{2i\pi kn}$$
(A5a)

$$\tau_0(t) = \frac{1}{2} D(0) \cos(\phi_0(0) - 2\lambda t) - w_{-1}$$
(A5b)

and where D(n) is any 'reasonable' function of a continuous variable n (continuous, differentiable, etc.), that interpolates between the values of  $D_n$  at the points where n integer.

Noting that the sum in k in (A4) extends to  $\pm \infty$ , so that the expression is invariant under  $k \leftrightarrow -k$ , it is possible to substitute eq. (A5a) by:

$$\omega_{k}(t) = \int_{0}^{\infty} dn D(n) \cos(\phi_{n}(0) - 2S_{k}(n, t)) =$$

$$= \int_{0}^{\infty} dn D(n) \cos(\phi_{n}(0)) \cos(2S_{k}(n, t)) +$$
(A6)

$$+\int_{0}^{\infty} dn \sin\left(\phi_{n}\left(0\right)\right) \sin\left(2S_{k}\left(n,t\right)\right) \tag{A7}$$

$$= Re \int_{0}^{\infty} dn D_{1}(n) \exp(2iS_{k}(n,t)) + Im \int_{0}^{\infty} dn D_{2}(n) \exp(2iS_{k}(n,t))$$
(A8)

where we have defined

$$S_k(n,t) = \pi kn - \lambda t \sqrt{n+1}$$
(A9a)

$$D_1(n) = D(n)\cos(\phi_n(0)) \tag{A9b}$$

$$D_2(n) = D(n)\sin(\phi_n(0)).$$
(A9c)

In this way, the inversion may be rewritten (without any approximation) as

$$\left\langle \hat{\sigma}_{z}\left(t\right)\right\rangle = \sum_{k=-\infty}^{\infty} \omega_{k}^{1}\left(t\right) + \omega_{k}^{2}\left(t\right) + \tau_{0}\left(t\right)$$
(A10)

where

$$\omega_k^1(t) \equiv Re \int_0^\infty dn D_1(n) \exp\left(2iS_k(n,t)\right)$$
(A11a)

$$\omega_k^2(t) \equiv Im \int_0^\infty dn D_2(n) \exp\left(2iS_k(n,t)\right)$$
(A11b)

Now, assuming the envelopes  $D_1(n)$  and  $D_2(n)$  are sufficiently smooth if compared to the oscillating functions  $\cos(2S_k(n,t))$ ,  $\sin(2S_k(n,t))$ , we may apply the method of stationary phases, approximating these expressions by:

$$\omega_k^1(t) \simeq D_1 \left(n = n_k\right) Re \left\{ \begin{array}{c} \exp\left(2iS_k \left(n = n_k\right)\right) \times \\ \times \int_0^\infty dn \exp\left[i \frac{\partial^2 S_k}{\partial n^2}\Big|_{n = n_k} \left(n - n_k\right)^2\right] \end{array} \right\}$$
(A12a)

$$\omega_k^2(t) \simeq D_2(n = n_k) Im \left\{ \begin{array}{c} \exp\left(2iS_k(n = n_k)\right) \times \\ \times \int_0^\infty dn \exp\left[i \left.\frac{\partial^2 S_k}{\partial n^2}\right|_{n = n_k} (n - n_k)^2\right] \end{array} \right\}$$
(A12b)

where  $n_k$  is the point at which  $\frac{\partial S_k}{\partial n} = 0$ :

$$n_k + 1 = \frac{\lambda^2 t^2}{4\pi^2 k^2} \tag{A13}$$

We note that these expressions are invalid for k = 0, since in this case the phase is always stationary (=0). Substituting in (A9a), we obtain:

$$S_k \left( n = n_k \right) = -\left( \pi k + \frac{\lambda^2 t^2}{4\pi k} \right) \tag{A14}$$

$$\left. \frac{\partial^2 S_k}{\partial n^2} \right|_{n=n_k} = 2 \frac{\pi^3 k^3}{\lambda^2 t^2} \equiv F \tag{A15}$$

Now, the integral in (A12a, b) may be written in terms of the Fresnel integrals [23]:

$$C(x) = \sqrt{\frac{2}{\pi}} \int_0^x dy \cos\left(y^2\right) dy$$
 (A16a)

$$S(x) = \sqrt{\frac{2}{\pi}} \int_0^x dy \sin\left(y^2\right) dy$$
 (A16b)

For instance, for F > 0:

$$\int_{0}^{\infty} dn \exp\left(iF\left(n-n_{k}\right)^{2}\right) = \sqrt{\frac{\pi}{2F}} \left[C\left(x\to\infty\right) + C\left(\sqrt{F}n_{k}\right) + i\left(S\left(x\to\infty\right) + S\left(\sqrt{F}n_{k}\right)\right)\right]$$
(A17)

The asymptotic form of the Fresnel integrals for  $x \to \infty$  [23] is:

$$C(x) \simeq \frac{1}{2} + \sqrt{\frac{1}{2\pi}} \frac{\sin(x^2)}{x} + O\left(\frac{1}{x^2}\right)$$
 (A18a)

$$S(x) \simeq \frac{1}{2} + \sqrt{\frac{1}{2\pi} \frac{\cos(x^2)}{x}} + O\left(\frac{1}{x^2}\right)$$
 (A18b)

Assuming  $\sqrt{|F|}n_k \gg 1$  and taking these approximations to zeroth order, we have:

$$\int_0^\infty dn \exp\left(iF\left(n-n_k\right)^2\right) \simeq \sqrt{\frac{\pi}{2F}} \left(1+i\right) \text{ (valid for } F>0\text{)}$$
(A19)

Similarly, for F < 0

$$\int_0^\infty dn \exp\left(iF\left(n-n_k\right)^2\right) \simeq \sqrt{\frac{\pi}{2|F|}} \left(1-i\right) \tag{A20}$$

Using (A15) and (A13), condition  $\sqrt{|F|}n_k \gg 1$  becomes:

$$\lambda t \gg 2\left(\sqrt{\pi |k|} + \sqrt{2\pi |k| + 4\pi^2 k^2}\right)$$
 (A21a)

ou 
$$\ll 2\left(\sqrt{\pi |k|} - \sqrt{2\pi |k| + 4\pi^2 k^2}\right)$$
 (A21b)

for k = 1, for example, this requires  $\lambda t \gg 17$ 

Substituting the asymptotic expressions (A19), (A20) in (A12a, b), we obtain for k > 0 (or k < 0):

$$\omega_k^1(t) \simeq D_1(n=n_k) \sqrt{\frac{\pi}{2F}} \left[ \cos\left(2S_k|_{n=n_k}\right) \mp \sin\left(2S_k|_{n=n_k}\right) \right]$$
(A22a)

$$\omega_k^2(t) \simeq D_2(n = n_k) \sqrt{\frac{\pi}{2F}} \left[ \cos\left(2S_k|_{n=n_k}\right) \pm \sin\left(2S_k|_{n=n_k}\right) \right]$$
(A22b)

Thus, using (A15) and (A10), the inversion may be written:

$$\left\langle \hat{\sigma}_{z}\left(t\right) \right\rangle = \sum_{k=-\infty,k\neq0}^{\infty} \left(\frac{\lambda t}{2\pi k^{\frac{3}{2}}}\right) \begin{bmatrix} D_{1}\left(n=n_{k}\right) \left[\cos\left(2S_{k}|_{n=n_{k}}\right) \mp \sin\left(2S_{k}|_{n=n_{k}}\right)\right] + \\ D_{2}\left(n=n_{k}\right) \left[\cos\left(2S_{k}|_{n=n_{k}}\right) \pm \sin\left(2S_{k}|_{n=n_{k}}\right)\right] \end{bmatrix} + \omega_{0}^{1} + \omega_{0}^{2} + \tau_{0}$$
(A23)

(where the upper (lower) sign is valid for the terms with k > 0 (< 0). Finally, substituting the value (A14) for  $S_k|_{n=n_k}$ :

$$\langle \hat{\sigma}_{z} (t) \rangle = \sum_{k=-\infty, k\neq 0}^{\infty} \left( \frac{\lambda t}{2\pi k^{\frac{3}{2}}} \right) \begin{bmatrix} \left( D_{1} + D_{2} |_{n=n_{k}} \right) \cos \left( 2\pi k + \frac{\lambda^{2} t^{2}}{2\pi k} \right) \pm \\ \pm \left( D_{1} - D_{2} |_{n=n_{k}} \right) \sin \left( 2\pi k + \frac{\lambda^{2} t^{2}}{2\pi k} \right) \end{bmatrix} + \omega_{0}^{1} + \omega_{0}^{2} + \tau_{0}$$

$$= \sum_{k=-\infty, k\neq 0}^{\infty} \left( \frac{\lambda t}{2\pi k^{\frac{3}{2}}} \right) D(n) |_{n=n_{k}} \begin{bmatrix} \left( \cos \left( \phi_{n} \left( 0 \right) \right) + \sin \left( \phi_{n} \left( 0 \right) \right) |_{n=n_{k}} \right) \cos \left( \frac{\lambda^{2} t^{2}}{2\pi k} \right) \pm \\ \pm \left( \cos \left( \phi_{n} \left( 0 \right) \right) - \sin \left( \phi_{n} \left( 0 \right) \right) |_{n=n_{k}} \right) \sin \left( \frac{\lambda^{2} t^{2}}{2\pi k} \right) \end{bmatrix} + \\ + \omega_{0}^{1} + \omega_{0}^{2} + \tau_{0}$$

$$(A24)$$

Finally, since:

$$(\cos(x) + \sin(x))\cos(y) \pm (\cos(x) - \sin(x))\cos(y) = \sqrt{2}\cos\left(x \pm y - \frac{\pi}{4}\right)$$
 (A25)

then the approximate expression for the inversion is:

$$\left\langle \hat{\sigma}_{z}\left(t\right) \right\rangle = \sum_{k=-\infty,k\neq0}^{\infty} \left(\frac{\lambda t}{\sqrt{2\pi k^{\frac{3}{2}}}}\right) \left[ D\left(n\right) \cos\left(\phi_{n}\left(0\right) \pm \frac{\lambda^{2}t^{2}}{2\pi k} - \frac{\pi}{4}\right) \right] \bigg|_{n+1=\frac{\lambda^{2}t^{2}}{4\pi^{2}k^{2}}} + \omega_{0}^{1} + \omega_{0}^{2} + \tau_{0}$$
(A26)

where the upper (lower) sign is valid for the terms with k > 0 (< 0), and where  $\tau_0$  and  $\omega_0^1 + \omega_0^2$  are given by

$$\tau_0(t) = \frac{1}{2} D(0) \cos(\phi_0(0) - 2\lambda t) - w_{-1}$$
(A27a)

$$\omega_0^1 + \omega_0^2(t) = \omega_0(t) = \int_0^\infty dn D(n) \cos\left(\phi_n(0) - 2\lambda t \sqrt{n+1}\right)$$
(A27b)

The first of these two last terms represents the contribution to the inversion of the states in which the field is in a vacuum state. For the initial conditions which satisfy the approximations that have been made above, this term will normally be negligible (see below). The second term assumes non-negligible values only in the vicinity of t = 0. This is due to the fact that it is an integral over oscillating functions with different frequencies, which rapidly cancel each other out; also, since these frequencies form a continuum, they cannot re-phase substantially at subsequent times. Fleischhauer and Schleich have thus conjectured that zero-order terms of the Poisson Formula such as this always describe the first collapse of the inversion shortly after t = 0.

The remaining terms (  $k \neq 0$  ) each describe a modulated oscillation in the inversion, with an envelope given essentially by the shape of D(n) and assuming non-negligible values only in an interval around  $t \simeq \frac{2\pi k \sqrt{\langle n \rangle + 1}}{\lambda}$  (here  $\langle n \rangle$  represents a value of n around the peak of D(n)). If D(n) is sufficiently narrow, the first few of these modulated oscillations will be well-separated in time, thus constituting an independent *revival* during which the inversion is described by

$$\left\langle \hat{\sigma}_{z}\left(t\right) \right\rangle \simeq \left(\frac{\lambda t}{\pi\sqrt{2k^{3}}}\right) \left[ D\left(n\right)\cos\left(\phi_{n}\left(0\right) \pm \frac{\lambda^{2}t^{2}}{2\pi k} - \frac{\pi}{4}\right) \right] \bigg|_{n+1=\frac{\lambda^{2}t^{2}}{4\pi^{2}k^{2}}}$$
(A28)

When the initial state is of the form  $|g\rangle_A \otimes |\psi\rangle_C$ , we have

$$\phi_n(0) \to \pi , \ D(n) \to P_{n+1}$$
 (A29)

( $P_n$  being the photon distribution of  $|\psi\rangle_C$ ), so that Fleischhauer and Schleich's result (A1) is recovered (eq. (2.8b) in [8]).

This expression is valid as long as:

- a) The distributions  $D_n \cos(\phi_n(0))$  and  $D_n \sin(\phi_n(0))$  vary slowly with n if compared with  $\cos S_k(n,t) = \cos(\pi kn \lambda t \sqrt{n+1})$ .
- b) The value of t obeys conditions (A21a, b). This implies that, for the  $k^{th}$  term of the sum above to describe well the  $k^{th}$  revival, D(n) must assume its largest values in the region of n where:

$$n+1 \gg \frac{4\left(3\pi k + 4\pi^2 k^2 + 2\pi k\sqrt{2+4\pi k}\right)}{4\pi^2 k^2} = 4 + \frac{1}{2\pi k} \left(\frac{3}{2} + \sqrt{2+4\pi k}\right)$$
(A30)

Thus, the approximation can be expected to be good for all revivals if the initial state has at least 10 or so photons on average in the field. This will usually also ensure that the component  $\tau_0$ , which depends on D(0), (eq. (A27a)), can be ignored.

It is straightforward to show that in the examples of section IV, where the dressed-state coordinates are given by expressions (30) and where  $|\alpha| = 7$ , both of these conditions are satisfied.

## 1. Variation for 'even-odd' states [where $w_{2n-1} = 0$ ]

In the case of 'even-odd'-type states such as  $|\Psi_{EO}(\alpha, \gamma, \xi)\rangle$ , where only the dressed-state coordinates  $w_n$  with *even* index are non-null  $(w_{2n-1} = 0)$ , condition (a) above is violated and expression (A26) is thus invalid. Nevertheless, a similar analytical expression for the inversion can still be derived if eq. (A2) is rewritten considering only the terms with even n. In this case, it is straightforward to show that, as long as the 'continuous versions' of distributions  $D_{2n} \cos(\phi_{2n}(0))$  and  $D_{2n} \sin(\phi_{2n}(0))$  are sufficiently smooth as a function of n, then the same stationary-phase method as was used above can be applied, resulting in:

$$\left\langle \hat{\sigma}_{z}\left(t\right) \right\rangle \simeq \sum_{k=-\infty,k\neq0}^{\infty} \left(\frac{\lambda t}{\pi k^{\frac{3}{2}}}\right) \left[ D\left(m\right) \cos\left(\phi_{2m}\left(0\right) \pm \frac{\lambda^{2} t^{2}}{2\pi k} + \pi k - \frac{\pi}{4}\right) \right] \Big|_{m+1=\frac{\lambda^{2} t^{2}}{\pi^{2} k^{2}}} + \omega_{0}^{1} + \omega_{0}^{2} + \tau_{0}$$
(A31)

Here D(m) is a continuous interpolation of the even-indexed terms of  $D_n$ , renumbered with the new index m ( $D(m = 3) = D_6$ , for instance).  $\tau_0$  and  $\omega_0^1 + \omega_0^2$  are still given by (A27a, b), except one must substitute  $n \to m$ .

Once again, it is possible to show that the approximations realised in the course of obtaining this formula remain valid as long as D(m) assumes significant values only for  $2m \gtrsim 10$ . (Thus, whenever the formula is applicable the term  $\tau_0$  can be ignored).

## REFERENCES

- [1] E.T. Jaynes and F.W. Cummings, Proc. IEEE **51**, 89 (1963).
- [2] B.W. Shore and P.L. Knight, J. Mod. Opt.40, 1195 (1993).
- [3] J.H. Eberly, N.B. Narozhny, and J.J.Sanchez-Mondragon, Phys. Rev. Lett. 44, 1323 (1980); N.B. Narozhny, J.J. Sanchez-Mondragon, and J.H. Eberly, Phys. Rev. A 23, 236 (1981); H.I. Yoo, J.J. Sanchez-Mondragon and J.H. Eberly, J. Phys A 14 ,1383 (1981).
- [4] S.J.D. Phoenix and P.L. Knight, Ann. Phys. (N.Y.), 186, 381 (1988); Phys. Rev. A 44, 6023 (1991); A. Ekert and P.L. Knight, Am. J. Phys. 63, 415 (1995).
- [5] See for instance: S. Haroche and J.M. Raimond, in *Adv. At. Mol. Opt. Phys., Suppl. 2* (Academic Press, New York, 1994) and references therein.
- [6] C.A. Blockley, D.F. Walls and H. Risken, Europhys. Lett. 17, 509 (1992); D.M. Meekhof,
   C.Monroe, B.E. King, W.M. Itano and D.J. Wineland, Phys. Rev. Lett. 76, 1796; 77, 2346 (1996)
- [7] M. Brune, F. Schmidt-Kaler, A. Maali, J. Dreyer, J.M. Raimond, and S. Haroche, Phys. Rev. Lett. 76, 1800 (1996).
- [8] M. Fleischhauer and W.P. Schleich, Phys. Rev. A 47, 4258 (1993).
- [9] K. Zaheer and M.S. Zubairy, Phys. Rev. A **39**, 2000 (1989).
- [10] Julio Gea-Banacloche, Phys. Rev. A 44, 5913 (1991).
- [11] J.I. Cirac and L.L. Sánchez-Soto, Phys. Rev. A 42, 2851 (1990).
- [12] J.H. Shapiro and S.R. Shepard, Phys. Rev. A 43, 3795 (1991).
- [13] L. Susskind and J. Glogower, Physics, 1, 49 (1964).
- [14] J.I. Cirac and L.L. Sánchez-Soto, Phys. Rev. A 44, 3317 (1991).
- [15] P. Meystre and M. Sargent III, Elements of Quantum Optics, 2nd ed. (Springer-Verlag, Berlin, 1991).
- [16] D. Jonathan, M.Sc. thesis, Universidade Estadual de Campinas, 1997.
- [17] D. Jonathan, K. Furuya and A.Vidiella-Barranco, in preparation.
- [18] A similar bound on the degree of deviation of the field photon statistics from an initial Poissonian distribution has been given in M.Hillery, Phys. Rev. A **35**, 4186 (1987).
- [19] We remark that expression (16a) given in [11] for the atomic state in this case is actually mistaken, and should be replaced with the one given in expression (21) above.
- [20] This phenomenon, which we address quantitatively in the next section, is essentially due to the increased spacing between adjacent Rabi functions. See [21].
- [21] A. Vidiella-Barranco, H. Moya-Cessa and V. Buzek, J. Mod. Opt. **39**, 1441 (1992).
- [22] R. Courant and D. Hilbert, Methods of Mathematical Physics, vol I (Interscience, 1953).
- [23] I.S. Gradsteyn and L.M. Ryzhik, Tables of Integrals, Series and Products (Academic Press, New York, 1965).
- [24] H.-I. Yoo and J.H. Eberly, Phys. Rep. **118**, 239 (1985).

## FIGURE CAPTIONS

**FIG. 1.** 'Dressed-state coordinates' for the Jaynes-Cummings model. Any state of the atom-field system can be represented in terms of the set of parameters  $\theta_n$ ,  $\phi_n$ ,  $w_n$ ,  $\xi_n$  as defined in eqs.(10) and (11). For each n, the first two of these can be pictured as forming a Bloch-type unit sphere, while the second two parametrise a circle of radius  $w_n$ . Under the effect of the Jaynes-Cummings Hamiltonian,  $w_n$  and  $\theta_n$  are constants of the motion, while  $\phi_n$  and  $\xi_n$  undergo periodic motion at a frequency determined by the Rabi frequency  $\Omega_n$ .

**FIG. 2.**Evolution of the atomic population inversion for an initial state where the field is in a coherent state and the atom is in an equally-weighed coherent superposition of  $|e\rangle$  and  $|g\rangle$ , with relative phase  $\xi$  (see eq. (29)). The amplitude of the coherent state is  $\alpha = |\alpha| e^{i\nu_{\alpha}}$ , where  $|\alpha| = 7$  (49 photons on average in the field). In (a), the relative phase  $(\nu_{\alpha} - \xi)$  is equal to  $\frac{\pi}{2}$ , and the evolution follows the familiar collapse-revival pattern. As this phase difference is lowered to  $\frac{\pi}{10}$  (b) and finally zero (c), the revivals are quenched, and the atomic populations become effectively trapped (notice the scale change in (c)). Simultaneously to this flattening, the revivals also assume a doublet structure.

**FIG. 3.** 'Weighted dressedness' distributions  $D_n = w_n^2 \sin \theta_n$  for states where the field is in a coherent state  $|\alpha\rangle$  with amplitude  $\alpha = 7e^{i\nu\alpha}$  and the atom is in a coherent superposition  $\cos \gamma |e\rangle + e^{-i\xi} \sin \gamma |g\rangle$ . In (a),  $\gamma$  is fixed at  $\frac{\pi}{4}$  (equal weights), while the relative phase  $(\nu_\alpha - \xi)$ varies from  $\frac{\pi}{2}$  (I) to  $\frac{\pi}{4}$  (II) to zero (III). The resulting flattening of  $D_n$  is due to the increasing dressedness of the components with the greatest weights  $w_n$ . In (b),  $(\nu_\alpha - \xi)$  is fixed at zero, while  $\gamma$  varies from  $\frac{\pi}{2}$  (1) to  $\frac{\pi}{3}$  (2) to  $\frac{\pi}{4}$  (III). In this case, the flattening corresponds to the matching of the points  $n_{\text{max}}$  of maximum weight and  $n_{\text{min}}$  of maximum dressedness (see eqs. 30). The limit of maximum flattening corresponds to the greatest degree of population trapping in the time evolution. The appearance of a doublet structure in this limit is due to the most important component of the state becoming completely dressed ( $\sin \theta_{n_{\text{max}}} \to 0$ ). This is reflected in the shape of the quenched revivals, as can be seen in Fig. 2.

**FIG. 4.** Evolution of the atomic population inversion when the atom-field system is initially in the entangled 'even-odd' state  $|\Psi_{EO}(\alpha, \gamma, \xi)\rangle$  given in expression (33). The parameters in (a)-(c) are the same as those in Fig. 2. As in that case, a quenching of revivals and appearance of a doublet structure is observed as  $(\nu_{\alpha} - \xi) \rightarrow 0$ . The main difference in the present case is the halving of the interval between adjacent revivals, due to the doubled spacing between the Rabi frequencies present.

**FIG. 5.** First and second revivals in the inversion for an initial state where the atom is in a coherent superposition of  $|e\rangle$  and  $|g\rangle$  and the field is in a coherent state  $|\alpha\rangle$  with amplitude  $\alpha = 7$ . The top two graphs are close-ups of Figs. 2(a) and 2(c) and depict the numerical calculation of the Jaynes-Cummings sum (18). The bottom two plot the approximate analytical expression (39), obtained from the Poisson Summation Formula. Despite a little distortion, agreement is seen to be good. Note that, at the tail end of (a1) and (a2), the beginning of the third revival can already be seen interfering with the second one, while in (b1) and (b2) only the analytical expressions for first two revivals are plotted.



Fig. 1

Fig. 2







