DETERMINANT INEQUALITIES FOR SIEVED ULTRASPHERICAL POLYNOMIALS

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ABSTRACT. Paul Turan first observed that the Legendre polynomials satisfy the inequality $P_n^2(x) - P_{n-1}(x)P_n(x) > 0$, $-1 < x < 1$. Inequalities of this type have since been proved for both classical and nonclassical orthogonal polynomials. In this paper, we prove such an inequality for sieved orthogonal polynomials of the second kind.

2000 Mathematics Subject Classification. Primary 33C47; Secondary 33E30.

1. Introduction. It was observed by Paul Turán [9] that the Legendre polynomials satisfy the determinant inequality

$$P_n^2(x) - P_{n+1}(x)P_{n-1}(x) > 0, \quad -1 < x < 1, \, n = 0, 1, \ldots. \tag{1.1}$$

G. Szegö [8] gave two very beautiful proofs of Turán's inequality. In the years since Szegö's paper appeared, it has been proved by various authors [5, 6, 7] that inequality (1.1) is satisfied by the classical orthogonal polynomials. In general, let $\{P_n(x)\}$ be a sequence of polynomials orthogonal in an interval $[a, b]$. Then the polynomials must satisfy a recursion

$$P_{n+1}(x) = (A_n x + B_n)P_n(x) - C_n P_{n-1}(x), \quad n = 0, 1, \ldots, \tag{1.2}$$

where we define $P_{-1}(x) = 0$.

We begin with a very simple result shows that, inequalities of Turán type are satisfied by any sequence of orthogonal polynomials.

THEOREM 1.1. *If the polynomials $\{P_n(x)\}$ are orthogonal on $a \le x \le b$, then for each n there exists c_n, $a \le c_n \le b$, such that*

$$\frac{P_n^2(x)}{P_n^2(c_n)} - \frac{P_{n+1}(x)}{P_{n+1}(c_n)} \frac{P_{n-1}(x)}{P_{n-1}(c_n)} \ge 0, \quad a \le x \le b. \tag{1.3}$$

PROOF. Consider the quotient

$$f_n(x) = \frac{P_{n+1}(x)P_{n-1}(x)}{P_n^2(x)}. \tag{1.4}$$

Obviously, the roots of $P_n(x)$ are singularities and apart from these points $f_n(x)$ is continuous. Also, for each root $x_{k,n}$ of $P_n(x)$, $k = 1, 2, \ldots, n$, there is an open interval I_n centered at x_{kn} in which $P_{n+1}(x)P_{n-1}(x) < 0$. This follows from the recursion (1.2).

Consequently, $f_n(x)$ is bounded above and must take on a positive maximum value at a point $c_n \in [a, b]$. Thus

$$\frac{P_{n+1}(c_n)P_{n-1}(c_n)}{P_n^2(c_n)} \geq \frac{P_{n+1}(x)P_{n-1}(x)}{P_n^2(x)}, \quad a \leq x \leq b. \tag{1.5}$$

This proves the theorem.   □

Although Theorem 1.1 asserts that Turán's inequality is a simple consequence of orthogonality, it is generally quite difficult to determine the point c_n referred to in Theorem 1.1. In the case of the classical orthogonal polynomials of Jacobi and their special case, the Gegenbauer polynomials, the point $c_n = 1$ is an endpoint of the interval of orthogonality $[-1, 1]$. Turán's inequality for these classical polynomials is established by using differential identities that are characteristic of classical polynomials. This seems to be the only case that lends itself to that technique.

Obviously, if

$$\delta_n > \max_{a \leq x \leq b} \frac{P_{n+1}(x)P_{n-1}(x)}{P_n^2(x)}, \tag{1.6}$$

then

$$\delta_n P_n^2(x) - P_{n+1}(x)P_{n-1}(x) > 0, \quad a \leq x \leq b. \tag{1.7}$$

Inequalities of the form (1.7) will be called weak Turán inequalities to distinguish them from (1.3) which will simply be referred to as Turán inequalities.

2. A weak Turán inequality for sieved ultraspherical polynomials of the second kind.
The sieved ultraspherical polynomials were discovered by Al-Salam et al. [1]. Ismail [2, 4] investigated them at great length. If $k \geq 2$ is an integer, then the sieved ultraspherical polynomials of the second kind, $B_n^\lambda(x; k)$, satisfy

$$B_{n+1}^\lambda(x; k) = 2xB_n^\lambda(x; k) - B_{n-1}^\lambda(x; k), \quad n + 1 \neq mk,$$

$$mB_{mk}^\lambda(x; k) = 2x(m + \lambda)B_{mk-1}^\lambda(x; k) - (m + 2\lambda)B_{mk-2}^\lambda(x; k), \quad m > 0, \tag{2.1}$$

where $B_0^\lambda(x; k) = 1$, $B_1^\lambda(x; k) = 2x$, $k \geq 2$.

Charris and Ismail [4] proved the following remarkable formula that is critical in deriving a weak Turán inequality for these polynomials,

$$B_{mk+j}^\lambda(x; k) = U_j(x)C_m^{\lambda+1}(T_k(x)) + U_{k-j-2}(x)C_{m-1}^{\lambda+1}(T_k(x)),$$

$$j = 0, 1, \ldots, k - 1; \; m = 0, 1, 2, \ldots. \tag{2.2}$$

In (2.2), $U_j(x)$ and $T_k(x)$ are the Chebychev polynomials of the second kind and the first kind, respectively, and $C_m^\lambda(x)$ are the ultraspherical polynomials. We recall that if $x = \cos\theta$, then

$$T_k(x) = \cos k\theta, \qquad U_j(x) = \frac{\sin(j+1)\theta}{\sin\theta}. \tag{2.3}$$

We note that $U_{-1}(x) = 0$ and $U_{-j}(x) = -(\sin(j-1)\theta/\sin\theta) = -U_{j-2}(x)$, $j \neq 1$. First, we remark that the sieved ultraspherical polynomials do not in general satisfy $\Delta_n(x) = (B_n(x; k))^2 - B_{n+1}(x; k)B_{n-1}(x; k) > 0$. The inequality fails when $n = mk + k - 1$

because a lengthy calculation, which we omit, shows that

$$\Delta_{mk+k-1}(1) = \frac{(2\lambda+2)_{m-1}(2\lambda+1)_{m+1}(2\lambda-2k+1)}{(m+1)!m!}. \tag{2.4}$$

This last quantity may become negative depending on k and λ. Note that $(a)_n$ is defined by $(a)_n = a(a+1)\cdots(a+n-1)$.

THEOREM 2.1. *Let* $\Delta_n = \delta_n[B_n^\lambda(x;k)]^2 - B_{n+1}^\lambda(x;k)B_{n-1}^\lambda(x;k)$, *where* $k \geq 2$ *and* $\delta_n = 1$ *if* $n = mk+j$, $j = 0,1,\dots,k-2$, $m = 0,1,\dots$, *and* $\delta_{mk+k-1} = (m+\lambda+1)/(m+1)$. *Then* $\Delta_n > 0$ *for* $-1 \leq x \leq 1$, $n = 0,1,\dots$.

PROOF. First, we prove that $\Delta_n > 0$ for $n = mk+j$, $j = 0,1,\dots,k-2$. Using elementary trigonometric identities, we can show that

$$U_j^2 - U_{j-1}U_{j+1} = 1, \qquad 2U_jU_{k-j-2} - U_{j+1}U_{k-j-1} - U_{j-1}U_{k-j-3} = -2T_k. \tag{2.5}$$

Using (2.2), we have

$$
\begin{aligned}
\Delta_{mk+j}(x) &= \left(U_jC_m^{\lambda+1}(T_k) + U_{k-j-2}C_{m-1}^{\lambda+1}(T_k)\right)^2 - \left(U_{j+1}C_m^{\lambda+1}(T_k) + U_{k-j-3}C_{m-1}^{\lambda+1}(T_k)\right) \\
&\quad \cdot \left(U_{j-1}C_m^{\lambda+1}(T_k) + U_{k-j-1}C_{m-1}^{\lambda+1}(T_k)\right) \\
&= \left(U_j^2 - U_{j-1}U_{j+1}\right)\left(C_m^{\lambda+1}(T_k)\right)^2 + \left(U_{k-j-2}^2 - U_{k-j-3}U_{k-j-1}\right)\left(C_{m-1}^{\lambda+1}(T_k)\right)^2 \\
&\quad + \left(2U_jU_{k-j-2} - U_{j+1}U_{k-j-1} - U_{j-1}U_{k-j-3}\right)C_m^{\lambda+1}(T_k)C_{m-1}^{\lambda+1}(T_k).
\end{aligned} \tag{2.6}
$$

From (2.5), the above equation yields

$$
\begin{aligned}
\Delta_{mk+j}(x) &= \left(C_m^{\lambda+1}(T_k)\right)^2 - 2T_kC_m^{\lambda+1}(T_k)C_{m-1}^{\lambda+1}(T_k) + \left(C_{m-1}^{\lambda+1}(T_k)\right)^2 \\
&= \left(C_m^{\lambda+1}(T_k) - T_kC_{m-1}^{\lambda+1}(T_k)\right)^2 + \left(1 - T_k^2\right)\left(C_{m-1}^{\lambda+1}(T_k)\right)^2 \\
&= \left(C_m^{\lambda+1}(T_k) - T_kC_{m-1}^{\lambda+1}(T_k)\right)^2 + \sin^2 k\theta\left(C_{m-1}^{\lambda+1}(T_k)\right)^2 > 0.
\end{aligned} \tag{2.7}
$$

For the case $n = mk+k-1$, we need the recursion for ultraspherical polynomials

$$(n+1)C_{n+1}^\lambda(x) = 2(n+\lambda)xC_n^\lambda(x) - (n+2\lambda-1)C_{n-1}^\lambda(x), \quad n \geq 1, \tag{2.8}$$

where $C_0^\lambda = 1$, $C_1^\lambda = 2\lambda x$.

Using elementary trigonometric identities, we obtain

$$U_n = U_{n-2} + 2T_n, \tag{2.9}$$

$$\left(U_{n-1}\right)^2 - \left(U_{n-2}\right)^2 - 2T_nU_{n-2} = 1. \tag{2.10}$$

Using (2.2), we get the following formula when $j = k-1$, writing $\theta = (m+\lambda+1)/(m+1)$ for simplicity,

$$
\begin{aligned}
\Delta_{mk+k-1}(x) &= \theta\left(B_{mk+k-1}^\lambda\right)^2 - B_{mk+k-2}^\lambda B_{mk+k}^\lambda \\
&= \theta\left(U_{k-1}C_m^{\lambda+1}(T_k)\right)^2 - \left(U_{k-2}C_m^{\lambda+1}(T_k) + C_{m-1}^{\lambda+1}(T_k)\right)\left(C_{m+1}^{\lambda+1}(T_k) + U_{k-2}C_m^{\lambda+1}(T_k)\right) \\
&= \theta\left(U_{k-1}\right)^2\left(C_m^{\lambda+1}(T_k)\right)^2 - U_{k-2}C_m^{\lambda+1}(T_k)C_{m+1}^{\lambda+1}(T_k) - \left(U_{k-2}\right)^2C_m^{\lambda+1}(T_k) \\
&\quad - C_{m-1}^{\lambda+1}(T_k)C_{m+1}^{\lambda+1}(T_k) - U_{k-2}C_{m-1}^{\lambda+1}(T_k)C_m^{\lambda+1}(T_k).
\end{aligned} \tag{2.11}
$$

From (2.8), the above equation yields

$$
\begin{aligned}
\Delta_{mk+k-1}(x) &= \left\{\theta(U_{k-1})^2 - (U_{k-2})^2\right\}(C_m^{\lambda+1}(T_k))^2 - U_{k-2}C_m^{\lambda+1}(T_k) \\
&\quad \cdot \left\{\frac{2(m+\lambda+1)}{m+1}T_k C_m^{\lambda+1}(T_k) - \frac{m+2\lambda+1}{m+1}C_{m-1}^{\lambda+1}(T_k)\right\} \\
&\quad - C_{m-1}^{\lambda+1}(T_k) \cdot \left\{\frac{2(m+\lambda+1)}{m+1}T_k C_m^{\lambda+1}(T_k) - \frac{m+2\lambda+1}{m+1}C_{m-1}^{\lambda+1}(T_k)\right\} \\
&\quad - U_{k-2}C_{m-1}^{\lambda+1}(T_k)C_m^{\lambda+1}(T_k) \\
&= \left\{\theta(U_{k-1})^2 - (U_{k-2})^2 - \frac{2(m+\lambda+1)}{m+1}T_k U_{k-2}\right\}(C_m^{\lambda+1}(T_k))^2 \\
&\quad + \left(\frac{2\lambda}{m+1}U_{k-2} - \frac{2(m+\lambda+1)}{m+1}T_k\right)C_m^{\lambda+1}(T_k)C_{m-1}^{\lambda+1}(T_k) \\
&\quad + \frac{m+2\lambda+1}{m+1}(C_{m-1}^{\lambda+1}(T_k))^2.
\end{aligned}
\tag{2.12}
$$

Multiplying by $(m+1)$ on both sides and using (2.10), we now have

$$
\begin{aligned}
&(m+1)\Delta_{mk+k-1}(x) \\
&= \{(m+\lambda+1) + \lambda U_{k-2}^2\}(C_m^{\lambda+1}(T_k))^2 + 2\{\lambda U_{k-2} - (m+\lambda+1)T_k\} \\
&\quad \cdot C_m^{\lambda+1}(T_k)C_{m-1}^{\lambda+1}(T_k) + (m+2\lambda+1)(C_{m-1}^{\lambda+1}(T_k))^2 \\
&= \left\{(m+\lambda+1) + \lambda U_{k-2}^2\right\}\left[C_m^{\lambda+1}(T_k) + \frac{\lambda U_{k-2} - (m+\lambda+1)T_k}{(m+\lambda+1) + \lambda U_{k-2}^2}C_{m-1}^{\lambda+1}(T_k)\right]^2 \\
&\quad + \left[\left\{(m+\lambda+1) + \lambda U_{k-2}^2\right\}(m+2\lambda+1) - \{\lambda U_{k-2} - (m+\lambda+1)T_k\}^2\right](C_{m-1}^{\lambda+1}(T_k))^2.
\end{aligned}
\tag{2.13}
$$

To complete the proof, we show that the last coefficient of $\left(C_{m-1}^{\lambda+1}(T_k)\right)^2$ must be positive.

We have

$$
\begin{aligned}
&\left\{(m+\lambda+1) + \lambda(U_{k-2})^2\right\}(m+2\lambda+1) - \{\lambda U_{k-2} - (m+\lambda+1)T_k\}^2 \\
&= (m+\lambda+1)(m+2\lambda+1) + \lambda(m+2\lambda+1)(U_{k-2})^2 \\
&\quad - \lambda^2(U_{k-2})^2 + 2\lambda(m+\lambda+1)U_{k-2}T_k - (m+\lambda+1)^2(T_k)^2 \\
&= (m+\lambda+1)^2(1 - (T_k)^2) + \lambda(m+\lambda+1) \\
&\quad + \lambda(m+\lambda+1)(U_{k-2})^2 + 2\lambda(m+\lambda+1)U_{k-2}T_k \\
&= (m+\lambda+1)^2(1 - (T_k)^2) + \lambda(m+\lambda+1)\left\{1 + (U_{k-2})^2 + 2U_{k-2}T_k\right\}.
\end{aligned}
\tag{2.14}
$$

By using (2.10), the above equation yields

$$
\begin{aligned}
(m+\lambda+1)^2(1-x^2)(U_{k-1})^2 + \lambda(m+\lambda+1)(U_{k-1})^2 \\
= (m+\lambda+1)(U_{k-1})^2\{(m+\lambda+1)(1-x^2) + \lambda\} > 0 \quad (2.15)
\end{aligned}
$$

(in view of $1 - (T_k)^2 = (1 - x^2)(U_{k-1})^2$). This completes the proof of Theorem 2.1.

$\square$

REMARK 2.2. The factor $(m + \lambda + 1)/(m + 1)$ that appears in the statement of Theorem 2.1 cannot be improved with a smaller number. This is because of the asymptotic relation

$$\lim_{|x| \to \infty} \frac{B_{mk+k-2}^{\lambda}(x) B_{mk+k}^{\lambda}(x)}{\left(B_{mk+k-1}^{\lambda}(x)\right)^2} = \frac{m + \lambda + 1}{m + 1}. \tag{2.16}$$

3. A further determinant of sieved ultraspherical polynomials. In [3], Bustoz and Savage proved an inequality for ultraspherical polynomials of the form

$$C_n^{\alpha}(x) C_{n+1}^{\beta}(x) - C_{n+1}^{\alpha}(x) C_n^{\beta}(x) > 0, \quad 0 < x < 1, \ 0 < \beta - \alpha \le 1. \tag{3.1}$$

This inequality was used to answer a conjecture of Askey and Gasper regarding a trigonometric kernel. The question arises if a similar inequality might hold for sieved ultraspherical polynomials. Here, we prove that the analogous inequality does not hold and we determine the location of sign changes for the corresponding sieved expression. We begin with some lemmas.

LEMMA 3.1. *Define* $\Delta_n(x) = C_n^{\lambda}(x) C_{n+1}^{\lambda+1}(x) - C_{n+1}^{\lambda}(x) C_n^{\lambda+1}(x)$, *where* $\{C_n^{\lambda}(x)\}$ *are the ultraspherical polynomials. Then*

$$\Delta_n(x) = a_n x \left[C_n^{\lambda}(x) \right]^2 + b_n \Delta_{n-2}(x), \quad n \ge 2, \tag{3.2}$$

where $a_n = (2(n + \lambda)/\lambda(n + 1))$, $b_n = ((n + 2\lambda - 1)(n + 2\lambda - 2)/n(n + 1))$.

PROOF. Each side of (3.2) is a polynomial of degree $2n + 1$. Thus, identity holds in (3.2) if both sides are equal at $2n + 1$ points. We prove equality at the roots $C_n^{\lambda}(x) = 0$, $C_{n+1}^{\lambda}(x) = 0$. First, note that (3.2) holds when $x = 0$. Thus, we may focus on the nonzero roots of $C_n^{\lambda}(x)$ and $C_{n+1}^{\lambda}(x)$. Let $C_n^{\lambda}(a) = 0$. The left-hand side of (3.2) reduces to $\Delta_n(a) = -P_{n+1}^{\lambda}(a) P_n^{\lambda+1}(a)$. We show that the right-hand side of (3.2) has the same value. From the recursion (2.8), it follows that

$$C_{n-1}^{\lambda}(a) = -\frac{(n+1)C_{n+1}^{\lambda}(a)}{n+2\lambda-1}, \qquad C_{n-2}^{\lambda}(a) = -\frac{2(n+\lambda-1)(n+1)aC_{n+1}^{\lambda}(a)}{(n+2\lambda-2)(n+2\lambda-1)}. \tag{3.3}$$

From the relation

$$(n+\lambda)C_n^{\lambda}(x) = \lambda\left[C_n^{\lambda+1}(x) - C_{n-2}^{\lambda+1}(x)\right], \tag{3.4}$$

it follows that

$$C_{n-2}^{\lambda+1}(a) = C_n^{\lambda+1}(a), \tag{3.5}$$

and from the recursion (2.8) we get

$$C_{n-1}^{\lambda+1}(a) = \frac{C_n^{\lambda+1}(a)}{a}. \tag{3.6}$$

Then it follows that the right-hand side of (3.2) reduces to $-C_{n+1}^{\lambda}(a) C_n^{\lambda+1}(a)$. Thus, (3.2) holds at the n roots of $C_n^{\lambda}(x)$. In a very similar fashion, it can be shown that (3.2) holds at the $n + 1$ roots of $C_{n+1}^{\lambda}(x)$. This proves the lemma. $\qquad \square$

By iterating (3.2) we get the following corollary.

COROLLARY 3.2.

$$\Delta_n(x) = a_n x \left[C_n^\lambda(x) \right]^2 + x \sum_{k=1}^{[n/2]} d_k \left[C_{n-2k}^\lambda(x) \right]^2, \quad \lambda > 0, \ n \geq 2, \qquad (3.7)$$

where $a_n = (2(n+\lambda)/\lambda(n+1))$,

$$d_k = a_{n-2k} \prod_{j=0}^{k-1} b_{n-2j}, \qquad b_n = \frac{(n+2\lambda-1)(n+2\lambda-2)}{n(n+1)}. \qquad (3.8)$$

LEMMA 3.3. *Let $\{U_n(x)\}$ denote the sequence of Chebychev polynomials of the second kind. Then*

$$U_j(x) U_{k-j-1}(x) - U_{j-1}(x) U_{k-j-2}(x) = U_{k-1}(x). \qquad (3.9)$$

PROOF. We write $x = \cos\theta$ so that $U_n(x) = (\sin(n+1)\theta/\sin\theta)$ and apply trigonometric identities. □

THEOREM 3.4. *Let $D_{mk+j}(x) = B_{mk+j-1}^\lambda(x) B_{mk+j}^{\lambda+1}(x) - B_{mk+j}^\lambda(x) B_{mk+j-1}^{\lambda+1}(x)$, $k \geq 2$, $m \geq 3$, and $j = 0, 1, \ldots, k-1$. If $\lambda > 0$, then*

$$D_{mk+j}(x) = U_{k-1}(x) T_k(x) \left\{ a_{m-1} \left[c_{m-1}^{\lambda+1}(T_k(x)) \right]^2 + \sum_{k=1}^{[(m-1)/2]} \gamma_k \left[C_{m-1-2k}^{\lambda+1}(T_k(x)) \right]^2 \right\},$$
$$(3.10)$$

where $T_k(x)$, $U_{k-1}(x)$ are Chebychev polynomials of the first and second kind,

$$a_{m-1} = \frac{2(m+\lambda)}{m(\lambda+1)}, \qquad \gamma_k = a_{m-2k-1} \prod_{j=0}^{k-1} b_{m-2j-1}, \qquad b_{m-1} = \frac{(m+2\lambda)(m+2\lambda-1)}{m(m-1)}. \qquad (3.11)$$

PROOF. From Lemma 3.3 after reducing, we get

$$D_{mk+j}(x) = U_{k-1}(x) \left[C_{m-1}^{\lambda+1}(T_k(x)) C_m^{\lambda+2}(T_k(x)) - C_m^{\lambda+1}(T_k(x)) C_{m-1}^{\lambda+2}(T_k(x)) \right]. \quad (3.12)$$

Now, apply Corollary 3.2. This gives the statement of the theorem.

It is clear from Theorem 3.4 that the determinants $D_{mk+j}(x)$ change sign at the roots of the Chebychev polynomials $T_k(x)$, $U_{k-1}(x)$. □
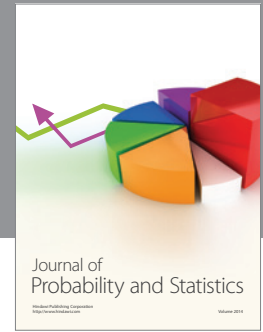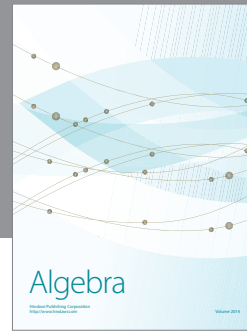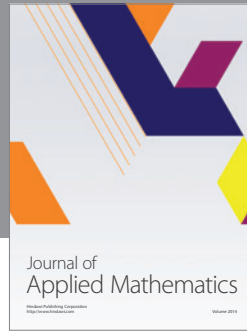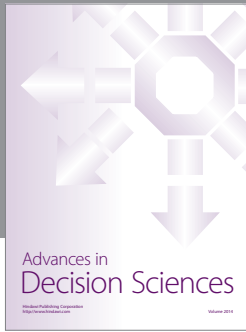
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