Çinar et al. *Fixed Point Theory and Applications* 2013, **2013**:33
http://www.fixedpointtheoryandapplications.com/content/2013/1/33

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

On pointwise and uniform statistical convergence of order α for sequences of functions

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Abstract
In this paper, we introduce the concepts of pointwise and uniform statistical convergence of order α for sequences of real-valued functions. Furthermore, we give the concept of an α-statistically Cauchy sequence for sequences of real-valued functions and prove that it is equivalent to pointwise statistical convergence of order α for sequences of real-valued functions. Also, some relations between $S^\alpha(f)$-statistical convergence and strong $w_p^\beta(f)$-summability are given.

MSC: 40A05; 40C05; 46A45

Keywords: statistical convergence; sequences of functions; Cesàro summability

1 Introduction

The idea of statistical convergence was given by Zygmund [1] in the first edition of his monograph published in Warsaw in 1935. The concept of statistical convergence was introduced by Steinhaus [2] and Fast [3] and later reintroduced by Schoenberg [4] independently. Over the years and under different names, statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory, number theory, measure theory, trigonometric series, turnpike theory and Banach spaces. Later on it was further investigated from the sequence space point of view and linked with summability theory by Başar [5], Connor [6], Et *et al.* [7–9], Fridy [10], Güngör *et al.* [11], Işık [12, 13], Kolk [14], Mohiuddine *et al.* [15–19], Miller and Orhan [20], Mursaleen [21], Rath and Tripathy [22], Salat [23], Savaş [24] and many others. In recent years, generalizations of statistical convergence have appeared in the study of strong integral summability and the structure of ideals of bounded continuous functions on locally compact spaces. Statistical convergence and its generalizations are also connected with subsets of the Stone-Čech compactification of the natural numbers. Moreover, statistical convergence is closely related to the concept of convergence in probability.

The definitions of pointwise and uniform statistical convergence of sequences of real-valued functions were given by Gökhan *et al.* [25, 26] and independently by Duman and Orhan [27]. In the present paper, we introduce and examine the concepts of pointwise and uniform statistical convergence of order α for sequences of real-valued functions. In Section 2 we give a brief overview of statistical convergence of order α and strong p-Cesàro summability. In Section 3 we give the concepts of pointwise and uniform statistical convergence of order α, and the concept α-statistically Cauchy sequence for sequences of real-valued functions and prove that it is equivalent to pointwise statistical convergence of order α for sequences of real-valued functions. We also establish some inclusion relations between $w_p^\beta(f)$ and $S^\alpha(f)$ and between $S^\alpha(f)$ and $S(f)$.

2 Definition and preliminaries

The definitions of statistical convergence and strong p-Cesàro convergence of a sequence of real numbers were introduced in the literature independently of one another and have followed different lines of development since their first appearance. It turns out, however, that the two definitions can be simply related to one another in general and are equivalent for bounded sequences. The idea of statistical convergence depends on the density of subsets of the set $\mathbb{N}$ of natural numbers. The density of a subset E of $\mathbb{N}$ is defined by

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(k) \quad \text{provided the limit exists,}$$

where χ_E is the characteristic function of E. It is clear that any finite subset of $\mathbb{N}$ has zero natural density and $\delta(E^c) = 1 - \delta(E)$.

The α*-density* of a subset E of $\mathbb{N}$ was defined by Çolak [28]. Let α be a real number such that $0 < \alpha \leq 1$. The α*-density* of a subset E of $\mathbb{N}$ is defined by

$$\delta_\alpha(E) = \lim_n \frac{1}{n^\alpha} \left|\{k \leq n : k \in E\}\right| \quad \text{provided the limit exists,}$$

where $|\{k \leq n : k \in E\}|$ denotes the number of elements of E not exceeding n.

If $x = (x_k)$ is a sequence such that x_k satisfies property $P(k)$ for almost all k except a set of α*-density* zero, then we say that x_k satisfies property $P(k)$ for '*almost all k according to* α' and we abbreviate this by '*a.a.k* (α)'.

It is clear that any finite subset of $\mathbb{N}$ has zero α density and $\delta_\alpha(E^c) = 1 - \delta_\alpha(E)$ does not hold for $0 < \alpha < 1$ in general, the equality holds only if $\alpha = 1$. Note that the α*-density* of any set reduces to the natural density of the set in case $\alpha = 1$.

The order of statistical convergence of a sequence of numbers was given by Gadjiev and Orhan in [29], and after then statistical convergence of order α and strong p-Cesàro summability of order α were studied by Çolak [28].

The statistical convergence of order α is defined as follows. Let $0 < \alpha \leq 1$ be given. The sequence (x_k) is said to be statistically convergent of order α if there is a real number ℓ such that

$$\lim_{n\to\infty} \frac{1}{n^\alpha} \left|\left\{k \leq n : |x_k - \ell| \geq \varepsilon\right\}\right| = 0,$$

for every $\varepsilon > 0$, in which case we say that x is statistically convergent of order α to ℓ. In this case, we write $S^\alpha - \lim x_k = \ell$. The set of all statistically convergent sequences of order α will be denoted by S^α. We write S_0^α to denote the set of all statistically null sequences of order α. It is clear that $S_0^\alpha \subset S^\alpha$ for each $0 < \alpha \leq 1$. The statistical convergence of order α is same with the statistical convergence for $\alpha = 1$.

A sequence $x = (x_k)$ is said to be strongly Cesàro summable to a number ℓ if $\lim_n \frac{1}{n} \times \sum_{k=1}^{n} |x_k - \ell| = 0$. The set of strongly Cesàro summable sequences is denoted by $[C,1]$ and defined as

$$[C,1] = \left\{ x = (x_k) : \lim_n \frac{1}{n} \sum_{k=1}^{n} |x_k - \ell| = 0 \text{ for some } \ell \right\}.$$

There is a natural relationship between statistical convergence and strong p-Cesàro summability.

3 Main result

In this section we give the main results of this article. We give relations between the statistical convergence of order α and the statistical convergence of order β for sequences of functions, the relations between the strong p-Cesàro summability of order α and the strong p-Cesàro summability of order β and the relations between the strong p-Cesàro summability of order α and the statistical convergence of order β for sequences of real-valued functions, where $\alpha \leq \beta$.

Definition 3.1 Let $0 < \alpha \leq 1$ be given. A sequence of functions $\{f_k\}$ is said to be pointwise statistically convergent of order α (or pointwise α-statistically convergent sequence) to the function f on a set A if, for every $\varepsilon > 0$,

$$\lim_n \frac{1}{n^\alpha} \left| \left\{ k \leq n : \left| f_k(x) - f(x) \right| \geq \varepsilon \text{ for every } x \in A \right\} \right| = 0$$

i.e., for every $x \in A$,

$$\left| f_k(x) - f(x) \right| < \varepsilon \quad a.a.k\ (\alpha). \tag{1}$$

In this case, we write $S^\alpha - \lim f_k(x) = f(x)$ on A. $S^\alpha - \lim f_k(x) = f(x)$ means that for every $\delta > 0$ and $0 < \alpha \leq 1$, there is an integer N such that

$$\frac{1}{n^\alpha} \left| \left\{ k \leq n : \left| f_k(x) - f(x) \right| \geq \varepsilon \text{ for every } x \in A \right\} \right| < \delta$$

for all $n > N$ $(= N(\varepsilon, \delta, x))$ and for each $\varepsilon > 0$. The set of all pointwise statistically convergent sequences of functions order α will be denoted by $S^\alpha(f)$. For $\alpha = 1$, we will write $S(f)$ instead of $S^\alpha(f)$ and in the special case $f = 0$, we will write $S_0^\alpha(f)$ instead of $S^\alpha(f)$.

The statistical convergence of order α for a sequence of functions is well defined for $0 < \alpha \leq 1$. But it is not well defined for $\alpha > 1$. For this, let $\{f_k\}$ be defined as follows:

$$f_k(x) = \begin{cases} 1, & k = 2n, \\ x^k, & k \neq 2n, \end{cases} \qquad n = 1, 2, 3, \ldots, x \in \left[0, \frac{1}{2} \right].$$

Then both

$$\lim_{n \to \infty} \frac{1}{n^\alpha} \left| \left\{ k \leq n : \left| f_k(x) - 1 \right| \geq \varepsilon \text{ for every } x \in A \right\} \right| = \lim_{n \to \infty} \frac{n}{2n^\alpha} = 0$$

and

$$\lim_{n \to \infty} \frac{1}{n^\alpha} \left| \left\{ k \leq n : \left| f_k(x) - 0 \right| \geq \varepsilon \text{ for every } x \in A \right\} \right| = \lim_{n \to \infty} \frac{n}{2n^\alpha} = 0$$

for $\alpha > 1$, so that $\{f_k\}$ statistically converges of order α both to 1 and 0, *i.e.*, $S^\alpha - \lim f_k(x) = 1$ and $S^\alpha - \lim f_k(x) = 0$, which is impossible.

Theorem 3.2 *Let* $0 < \alpha \leq 1$ *and* $\{f_k\}$, $\{g_k\}$ *be sequences of real-valued functions defined in a set* A.

(i) *If* $S^\alpha - \lim f_k(x) = f(x)$ *and* $c \in R$, *then* $S^\alpha - \lim cf_k(x) = cf(x)$.

(ii) *If* $S^\alpha - \lim f_k(x) = f(x)$ *and* $S^\alpha - \lim g_k(x) = g(x)$, *then* $S^\alpha - \lim(f_k(x) + g_k(x)) = f(x) + g(x)$.

Proof (i) The proof is clear in case $c = 0$. Suppose that $c \neq 0$ and $S^\alpha - \lim f_k(x) = f(x)$, then there exists $\varepsilon > 0$ such that

$$\left|f_k(x) - f(x)\right| < \frac{\varepsilon}{|c|} \quad a.a.k\ (\alpha),$$

and hence

$$\left|cf_k(x) - cf(x)\right| < \varepsilon \quad a.a.k\ (\alpha).$$

This implies that $S^\alpha - \lim cf_k(x) = cf(x)$.

The proof of (ii) follows from the following inequalities:

$$\begin{aligned} &\frac{1}{n^\alpha}\left|\left\{k \leq n : \left|f_k(x) + g_k(x) - \big(f(x) + g(x)\big)\right| \geq \varepsilon \text{ for every } x \in A\right\}\right| \\ &\quad \leq \frac{1}{n^\alpha}\left|\left\{k \leq n : \left|f_k(x) - f(x)\right| \geq \frac{\varepsilon}{2} \text{ for every } x \in A\right\}\right| \\ &\qquad + \frac{1}{n^\alpha}\left|\left\{k \leq n : \left|g_k(x) - g(x)\right| \geq \frac{\varepsilon}{2} \text{ for every } x \in A\right\}\right|. \end{aligned}$$

It is easy to see that every convergent sequence of functions is statistically convergent of order α, that is, $c(f) \subset S^\alpha(f)$ for each $0 < \alpha \leq 1$. But the converse of this does not hold. For example, the sequence $\{f_k\}$ defined by

$$f_k(x) = \begin{cases} 1, & k = n^3, \\ \frac{2kx}{1+k^2x^2}, & k \neq n^3 \end{cases}$$

is statistically convergent of order α with $S^\alpha - \lim f_k(x) = 0$ for $\alpha > \frac{1}{3}$, but it is not convergent. □

Definition 3.3 Let α be any real number such that $0 < \alpha \leq 1$ and let $\{f_k\}$ be a sequence of functions on a set A. The sequence $\{f_k\}$ is a statistically Cauchy sequence of order α (or α-statistically Cauchy sequence) provided that for every $\varepsilon > 0$, there exists a number N $(= N(\varepsilon, x))$ such that

$$\left|f_k(x) - f_N(x)\right| < \varepsilon \quad a.a.k\ (\alpha),$$

i.e.,

$$\lim_n \frac{1}{n^\alpha}\left|\left\{k \leq n : \left|f_k(x) - f_N(x)\right| \geq \varepsilon \text{ for every } x \in A\right\}\right| = 0.$$

Theorem 3.4 *Let $\{f_k\}$ be a sequence of functions defined on a set A. The following statements are equivalent*:

(i) $\{f_k\}$ *is a pointwise α-statistically convergent sequence on A*;

(ii) $\{f_k\}$ *is a α-statistically Cauchy sequence on A*;

(iii) $\{f_k\}$ *is a sequence of functions for which there is a pointwise convergent sequence of order α, a sequence of functions $\{g_k\}$ such that $f_k(x) = g_k(x)$ a.a.k (α) for every $x \in A$.*

Proof (i) $\Rightarrow$ (ii) Suppose that $S^\alpha - \lim f_k(x) = f(x)$ on A and let $\varepsilon > 0$. Then $|f_k(x) - f(x)| < \frac{\varepsilon}{2}$ *a.a.k* (α) and if N is chosen so that $|f_N(x) - f(x)| < \frac{\varepsilon}{2}$, then we have

$$\left|f_k(x) - f_N(x)\right| \leq \left|f_k(x) - f(x)\right| + \left|f_N(x) - f(x)\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad a.a.k\ (\alpha)$$

for every $x \in A$. Hence $\{f_k\}$ is an α-statistically Cauchy sequence.

Next, assume (ii) is true and choose N so that the band $I = [f_N(x) - 1, f_N(x) + 1]$ contains $f_k(x)$ *a.a.k* (α) for every $x \in A$. Also, apply (ii) to choose M so that $I' = [f_M(x) - \frac{1}{2}, f_M(x) + \frac{1}{2}]$ contains $f_k(x)$ *a.a.k* (α) for every $x \in A$. We assert that

$$I_1 = I \cap I' \text{ contains } f_k(x)\ a.a.k\ (\alpha) \quad \text{for every } x \in A;$$

for

$$\begin{aligned}&\left\{k \leq n : f_k(x) \notin I \cap I' \text{ for every } x \in A\right\}\\ &\quad = \left\{k \leq n : f_k(x) \notin I \text{ for every } x \in A\right\} \cup \left\{k \leq n : f_k(x) \notin I' \text{ for every } x \in A\right\}\end{aligned}$$

so

$$\begin{aligned}&\lim_{n\to\infty} \frac{1}{n^\alpha}\left|\left\{k \leq n : f_k(x) \notin I \cap I' \text{ for every } x \in A\right\}\right|\\ &\quad \leq \lim_{n\to\infty} \frac{1}{n^\alpha}\left|\left\{k \leq n : f_k(x) \notin I \text{ for every } x \in A\right\}\right|\\ &\qquad + \lim_{n\to\infty} \frac{1}{n^\alpha}\left|\left\{k \leq n : f_k(x) \notin I' \text{ for every } x \in A\right\}\right| = 0.\end{aligned}$$

Therefore, I_1 is a closed band of height less than or equal to 1 that contains $f_k(x)$ *a.a.k* (α) for every $x \in A$. Now we proceed by choosing $N(2)$ so that $I'' = [f_{N(2)}(x) - \frac{1}{4}, f_{N(2)}(x) + \frac{1}{4}]$ contains $f_k(x)$ *a.a.k* (α), and by the preceding argument, $I_2 = I_1 \cap I''$ contains $f_k(x)$ *a.a.k* (α) for every $x \in A$ and I_2 has height less than or equal to $\frac{1}{2}$. Continuing inductively, we construct a sequence $\{I_m\}_{m=1}^\infty$ of closed band such that for each m, $I_m \supseteq I_{m+1}$, the height of I_m is not greater than $2^{1-m}$and $f_k(x) \in I_m$ *a.a.k* (α) for every $x \in A$. Thus there exists a function $f(x)$, defined on A, such that $\{f(x)\}$ is equal to $\bigcap_{m=1}^\infty I_m$. Using the fact that $f_k(x) \in I_m$ *a.a.k* (α) for every $x \in A$, we choose an increasing positive integer sequence $\{T_m\}_{m=1}^\infty$ such that

$$\frac{1}{n^\alpha}\left|\left\{k \leq n : f_k(x) \notin I_m \text{ for every } x \in A\right\}\right| < \frac{1}{m} \quad \text{if } n > T_m. \tag{2}$$

Now define a subsequence $(z_k(x))$ of $(f_k(x))$ consisting of all terms $f_k(x)$ such that $k > T_1$ and if $T_m < k \leq T_{m+1}$ then $f_k(x) \notin I_m$ for every $x \in A$. Next, define the sequence of functions

$(g_k(x))$ by

$$g_k(x) = \begin{cases} f(x) & \text{if } f_k(x) \text{ is a term of } z_k(x), \\ f_k(x) & \text{otherwise} \end{cases}$$

for every $x \in A$. Then $\lim_{k\to\infty} g_k(x) = f(x)$ on A; for if $\varepsilon > \frac{1}{m} > 0$ and $k > T_m$, then either $f_k(x)$ is a term of $(z_k(x))$ or $g_k(x) = f_k(x) \in I_m$ on A and $|g_k(x) - f_k(x)| \leq$ height of $I_m \leq 2^{1-m}$ for every $x \in A$. We also assert that $g_k(x) = f_k(x)$ *a.a.k* (α) for every $x \in A$. To verify this, we observe that if $T_m < n \leq T_{m+1}$, then

$$\begin{aligned} &\{k \leq n : f_k(x) \neq g_k(x) \text{ for every } x \in A\} \\ &\quad \subseteq \{k \leq n : f_k(x) \notin I_m \text{ for every } x \in A\}. \end{aligned}$$

So, by (2)

$$\begin{aligned} &\frac{1}{n^\alpha}\left|\{k \leq n : f_k(x) \neq g_k(x) \text{ for every } x \in A\}\right| \\ &\quad \leq \frac{1}{n^\alpha}\left|\{k \leq n : f_k(x) \notin I_m \text{ for every } x \in A\}\right| < \frac{1}{m}. \end{aligned}$$

Hence, the limit is 0 as $n \to \infty$ and $f_k(x) = g_k(x)$ *a.a.k* (α) for every $x \in A$. Therefore, (ii) implies (iii).

Finally, assume that (iii) holds, say $f_k(x) = g_k(x)$ *a.a.k* (α) for every $x \in A$ and $\lim_{k\to\infty} g_k(x) = f(x)$ on A. Let $\varepsilon > 0$. Then for each n,

$$\begin{aligned} &\{k \leq n : |f_k(x) - f(x)| \geq \varepsilon \text{ for every } x \in A\} \\ &\quad \subseteq \{k \leq n : f_k(x) \neq g_k(x) \text{ for every } x \in A\} \\ &\quad \cup \{k \leq n : |g_k(x) - f(x)| \geq \varepsilon \text{ for every } x \in A\} \end{aligned}$$

since $\lim_{k\to\infty} g_k(x) = f(x)$ on A, the latter set contains a fixed number of integers, say $l = l(\varepsilon, x)$. Therefore,

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{n^\alpha}\left|\{k \leq n : |f_k(x) - f(x)| \geq \varepsilon \text{ for every } x \in A\}\right| \\ &\quad \leq \lim_{n\to\infty} \frac{1}{n^\alpha}\left|\{k \leq n : f_k(x) \neq g_k(x) \text{ for every } x \in A\}\right| + \lim_{n\to\infty} \frac{l}{n^\alpha} = 0 \end{aligned}$$

because $f_k(x) = g_k(x)$ *a.a.k* (α) for every $x \in A$. Hence $|f_k(x) - f(x)| < \varepsilon$ *a.a.k* (α) for every $x \in A$, so (i) holds and the proof is complete. □

Corollary 3.5 *If* $\{f_k\}$ *is a sequence of functions such that* $S^\alpha - \lim f_k(x) = f(x)$ *on* A, *then* $\{f_k\}$ *has a subsequence* $\{f_{k(n)}(x)\}$ *such that* $\lim_{n\to\infty} f_{k(n)}(x) = f(x)$ *on* A.

Theorem 3.6 *Let* $0 < \alpha \leq \beta \leq 1$. *Then* $S^\alpha(f) \subseteq S^\beta(f)$ *and the inclusion is strict for some* α *and* β *such that* $\alpha < \beta$.

Proof If $0 < \alpha \leq \beta \leq 1$, then

$$\begin{aligned}&\frac{1}{n^\beta}\left|\left\{k \leq n : \left|f_k(x) - f(x)\right| \geq \varepsilon \text{ for every } x \in A\right\}\right| \\ &\quad \leq \frac{1}{n^\alpha}\left|\left\{k \leq n : \left|f_k(x) - f(x)\right| \geq \varepsilon \text{ for every } x \in A\right\}\right|\end{aligned}$$

for every $\varepsilon > 0$ and this gives that $S^\alpha(f) \subseteq S^\beta(f)$. To show that the inclusion is strict, consider the sequence $\{f_k\}$ defined by

$$f_k(x) = \begin{cases} 1, & k = n^2, \\ \frac{k^2 x}{1+k^3x^2}, & k \neq n^2, \end{cases} \quad n = 1, 2, 3, \ldots, x \in [0,1].$$

Hence we can write for $\frac{1}{2} < \alpha \leq 1$

$$\begin{aligned}&\frac{1}{n^\alpha}\left|\left\{k \leq n : \left|f_k(x) - 0\right| \geq \varepsilon \text{ for every } x \in [0,1]\right\}\right| \\ &\quad = \frac{1}{n^\alpha}\left|\left\{k \leq n : \left|f_k(x)\right| \geq \varepsilon \text{ for every } x \in [0,1]\right\}\right| \leq \frac{\sqrt{n}}{n^\alpha} \to 0.\end{aligned}$$

Then $S^\beta - \lim f_k(x) = 0$, *i.e.*, $x \in S^\beta(f)$ for $\frac{1}{2} < \beta \leq 1$, but $x \notin S^\alpha(f)$ for $0 < \alpha \leq \frac{1}{2}$. □

If we take $\beta = 1$ in Theorem 3.6, then we obtain the following result.

Corollary 3.7 *If a sequence of functions $\{f_k\}$ is statistically convergent of order α, to the function f for some $0 < \alpha \leq 1$, then it is statistically convergent to the function f.*

Definition 3.8 Let α be any real number such that $0 < \alpha \leq 1$ and let p be a positive real number. A sequence of functions $\{f_k\}$ is said to be strongly p-Cesàro summable of order α if there is a function f such that

$$\lim_{n\to\infty} \frac{1}{n^\alpha} \sum_{k=1}^{n} \left|f_k(x) - f(x)\right|^p = 0.$$

In this case, we write $w_p^\alpha - \lim f_k(x) = f(x)$ on A. The strong p-Cesàro summability of order α reduces to the strong p-Cesàro summability for $\alpha = 1$. The set of all strongly p-Cesàro summable sequences of functions of order α will be denoted by $w_p^\alpha(f)$. We write $w_{o,p}^\alpha(f)$ in case $f(x) = 0$.

Theorem 3.9 *Let $0 < \alpha \leq \beta \leq 1$ and p be a positive real number. Then $w_p^\alpha(f) \subseteq w_p^\beta(f)$ and the inclusion is strict for some α and β such that $\alpha < \beta$.*

Proof Let the sequence $\{f_k\}$ be strongly p-Cesàro summable of order α. Then, given α and β such that $0 < \alpha \leq \beta \leq 1$ and a positive real number p, we may write

$$\frac{1}{n^\beta} \sum_{k=1}^{n} \left|f_k(x) - f(x)\right|^p \leq \frac{1}{n^\alpha} \sum_{k=1}^{n} \left|f_k(x) - f(x)\right|^p,$$

and this gives that $w_p^\alpha(f) \subseteq w_p^\beta(f)$.

To show that the inclusion is strict, consider the sequence $\{f_k\}$ defined by

$$f_k(x) = \begin{cases} \frac{1}{1+kx}, & k = n^2, \\ 0, & k \neq n^2, \end{cases} \quad x \in \left[0, \frac{1}{k}\right].$$

Then

$$\frac{1}{n^\beta} \sum_{k=1}^{n} \left|f_k(x) - 0\right|^p \leq \frac{\sqrt{n}}{n^\beta} = \frac{1}{n^{\beta-\frac{1}{2}}}$$

since $1/(n^{\beta-\frac{1}{2}}) \to 0$ as $n \to \infty$, then $w_p^\beta - \lim f_k(x) = 0$, *i.e.*, the sequence $\{f_k\}$ is strongly p-Cesàro summable of order α for $\frac{1}{2} < \beta \leq 1$, but since

$$\frac{\sqrt{n}}{2n^\alpha} \leq \frac{1}{n^\alpha} \sum_{k=1}^{n} \left|f_k(x) - 0\right|^p$$

and $\sqrt{n}/2n^\alpha \to \infty$, $n \to \infty$, the sequence $\{f_k\}$ is not strongly p-Cesàro summable of order α for $0 < \alpha < \frac{1}{2}$. □

Corollary 3.10 *Let* $0 < \alpha \leq \beta \leq 1$ *and* p *be a positive real number. Then*
(i) *if* $\alpha = \beta$, *then* $w_p^\alpha(f) = w_p^\beta(f)$;
(ii) $w_p^\alpha(f) \subseteq w_p(f)$ *for each* $\alpha \in (0,1]$ *and* $0 < p < \infty$.

Theorem 3.11 *Let* $0 < \alpha \leq 1$ *and* $0 < p < q < \infty$. *Then* $w_q^\alpha(f) \subseteq w_p^\alpha(f)$.

Proof Omitted. □

Theorem 3.12 *Let* α *and* β *be fixed real numbers such that* $0 < \alpha \leq \beta \leq 1$ *and* $0 < p < \infty$. *If a sequence of functions* $\{f_k\}$ *is strongly p-Cesàro summable of order* α *to the function* f, *then it is statistically convergent of order* β *to the function* f.

Proof For any sequence of functions $\{f_k\}$ defined on A, we can write

$$\sum_{k=1}^{n} \left|f_k(x) - f(x)\right|^p \geq \left|\left\{k \leq n : \left|f_k(x) - f(x)\right| \geq \varepsilon \text{ for every } x \in A\right\}\right| \cdot \varepsilon^p$$

and so that

$$\begin{aligned} \frac{1}{n^\alpha} \sum_{k=1}^{n} \left|f_k(x) - f(x)\right|^p &\geq \frac{1}{n^\alpha}\left|\left\{k \leq n : \left|f_k(x) - f(x)\right| \geq \varepsilon \text{ for every } x \in A\right\}\right| \cdot \varepsilon^p \\ &\geq \frac{1}{n^\beta}\left|\left\{k \leq n : \left|f_k(x) - f(x)\right| \geq \varepsilon \text{ for every } x \in A\right\}\right| \cdot \varepsilon^p. \end{aligned}$$ □

Corollary 3.13 *Let* α *be a fixed real number such that* $0 < \alpha \leq 1$ *and* $0 < p < \infty$. *If a sequence of functions* $\{f_k\}$ *is strongly p-Cesàro summable of order* α *to the function* f, *then it is statistically convergent of order* α *to the function* f.

Definition 3.14 Let α be any real number such that $0 < \alpha \leq 1$. A sequence of functions $\{f_k\}$ is said to be uniformly statistically convergent of order α or uniformly (α-statistically convergent sequence) to the function f on a set A if, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n^\alpha} \left| \left\{ k \leq n : \left| f_k(x) - f(x) \right| \geq \varepsilon \text{ for all } x \in A \right\} \right| = 0,$$

i.e., for all $x \in A$,

$$\left| f_k(x) - f(x) \right| < \varepsilon \quad a.a.k\ (\alpha). \tag{3}$$

In this case, we write

$$S^\alpha - \lim f_k(x) = f(x) \quad \text{uniformly on } A \text{ or } S_u^\alpha - \lim f_k(x) = f(x) \text{ on } A.$$

The set of all uniformly α-statistically convergent sequences will be denoted by $S_u^\alpha(f)$.

Theorem 3.15 *Let f and f_k, for all $k \in \mathbb{N}$, be continuous functions on $A = [a,b] \subset R$ and $0 < \alpha \leq 1$. Then $S^\alpha - \lim f_k(x) = f(x)$ uniformly on A if and only if $S^\alpha - \lim c_k = 0$, where $c_k = \max_{x\in A} |f_k(x) - f(x)|$.*

Proof Suppose that $S^\alpha - \lim f_k(x) = f(x)$ uniformly on A. Since $|f_k(x) - f(x)|$ is continuous on A for each $k \in \mathbb{N}$, it has absolute maximum value at some point $x_k \in A$, *i.e.*, there exist $x_1, x_2, \ldots \in A$ such that $c_1 = |f_1(x_1) - f(x_1)|$, $c_2 = |f_2(x_2) - f(x_2)|, \ldots$, *etc.* Thus we may write $c_k = |f_k(x_k) - f(x_k)|$, $k = 1, 2, \ldots$. From the definition of uniform α-statistical convergence, we may write, for every $\varepsilon > 0$,

$$\left| f_k(x_k) - f(x_k) \right| < \varepsilon \quad a.a.k\ (\alpha).$$

Hence, $S^\alpha - \lim c_k = 0$.

The necessity is trivial. □

It follows from (3) that if $\lim f_k(x) = f(x)$ uniformly on A, then $S^\alpha - \lim f_k(x) = f(x)$ uniformly on A. But the converse is not true, for this consider the sequence defined by

$$f_k(x) = \begin{cases} 1, & k = n^2, \\ \frac{k}{k^2 + k^2 x^2} & \text{otherwise,} \end{cases} \qquad k = 1, 2, 3, \ldots, x \in [0,1].$$

Then if $x \in [0,1]$ and $\alpha \in [\frac{1}{2}, 1]$, then $\{f_k\}$ is uniformly α-statistically convergent to $f(x) = 0$ on $[0,1]$ since $S^\alpha - \lim c_k = 0$, where

$$c_k = \max_{x\in[0,1]} \left| f_k(x) - 0 \right| = \begin{cases} 2, & k = n^2, \\ \frac{1}{k} & \text{otherwise,} \end{cases}$$

but $(f_k(x))$ is not uniformly convergent on $[0,1]$ since $\lim_{k\to\infty} c_k$ does not exist.

Corollary 3.16

(i) $\lim f_k(x) = f(x)$ *uniformly on* $A \Rightarrow \lim f_k(x) = f(x)$ *on* $A \Rightarrow S^\alpha - \lim f_k(x) = f(x)$ *pointwise on* A.

(ii) $S^\alpha - \lim f_k(x) = f(x)$ *uniformly on* $A \Rightarrow S^\alpha - \lim f_k(x) = f(x)$ *pointwise on* A.

(iii) *If* $0 < \alpha \leq \beta \leq 1$, *then* $S_u^\alpha(f) \subseteq S_u^\beta(f)$.

Definition 3.17 Let α be any real number such that $0 < \alpha \leq 1$ and let $\{f_k\}$ be a sequence of functions on a set A. The sequence $\{f_k\}$ is a uniformly statistically Cauchy sequence of order α (or uniformly α-statistically Cauchy sequence) provided that for every $\varepsilon > 0$, there exists a number N $(= N(\varepsilon))$ such that

$$\left|f_k(x) - f_N(x)\right| < \varepsilon \quad a.a.k\ (\alpha) \text{ for all } x \in A,$$

i.e.,

$$\lim_{n\to\infty} \frac{1}{n^\alpha} \left| \left\{ k \leq n : \left|f_k(x) - f_N(x)\right| \geq \varepsilon \text{ for all } x \in A \right\} \right| = 0.$$

The proofs of the following two theorems are similar to those of Theorem 3.2 and Theorem 3.4, therefore we give them without proof.

Theorem 3.18 *Let* $0 < \alpha \leq 1$ *and* $\{f_k\}$, $\{g_k\}$ *be sequences of real-valued functions defined on a set* A.

(i) *If* $S_u^\alpha - \lim f_k(x) = f(x)$ *and* $c \in R$, *then* $S_u^\alpha - \lim cf_k(x) = cf(x)$.

(ii) *If* $S_u^\alpha - \lim f_k(x) = f(x)$ *and* $S_u^\alpha - \lim g_k(x) = g(x)$, *then* $S_u^\alpha - \lim(f_k(x) + g_k(x)) = f(x) + g(x)$.

Theorem 3.19 *Let* α *be any real number such that* $0 < \alpha \leq 1$ *and let* $\{f_k\}$ *be a sequence of functions on a set* A. *The following statements are equivalent*:

(i) $\{f_k\}$ *is a uniformly* α*-statistically convergent sequence on* A;

(ii) $\{f_k\}$ *is a uniformly* α*-statistically Cauchy sequence on* A;

(iii) $\{f_k\}$ *is a sequence of functions for which there is a uniformly convergent sequence of order* α, *a sequence of functions* $\{g_k\}$ *such that* $f_k(x) = g_k(x)$ *a.a.k* (α) *for all* $x \in A$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

MC, MK, and ME have contributed to all parts of the article. All authors read and approved the final manuscript.

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Received: 14 September 2012 Accepted: 30 January 2013 Published: 18 February 2013

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doi:10.1186/1687-1812-2013-33
Cite this article as: Çinar et al.: On pointwise and uniform statistical convergence of order α for sequences of functions. *Fixed Point Theory and Applications* 2013 **2013**:33.