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Inequalities for a class of multivariate operators

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Abstract

This paper introduces and studies a class of generalized multivariate Bernstein operators defined on the simplex. By means of the modulus of continuity and so-called Ditzian-Totik's modulus of function, the direct and inverse inequalities for the operators approximating multivariate continuous functions are simultaneously established. From these inequalities, the characterization of approximation of the operators follows. The obtained results include the corresponding ones of the classical Bernstein operators.

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Keywords: generalized Bernstein operators; direct and inverse inequalities; characterization of approximation

1 Introduction

Let \mathbb{N} be the set of natural numbers, and $\{s_n\}_{n=1}^{\infty}$ ($s_n \geq 1$, $s_n \in \mathbb{N}$) be a sequence. In [3], Cao introduced the following generalized Bernstein operators defined on $[0, 1]$:

$$(\mathcal{L}_n f)(x) := \frac{1}{s_n} \sum_{k=0}^n \left(\sum_{j=0}^{s_n-1} f\left(\frac{k+j}{n+s_n-1}\right) \right) P_{n,k}(x), \quad (1)$$

where $x \in [0, 1]$, $f \in C[0, 1]$, and

$$P_{n,k}(x) := \binom{n}{k} x^k (1-x)^{n-k}. \quad (2)$$

Clearly, when $s_n = 1$, $\mathcal{L}_n f$ reduce to the classical Bernstein operators, $\mathcal{B}_n f$, given by

$$(\mathcal{B}_n f)(x) := \sum_{k=0}^n f\left(\frac{k}{n}\right) P_{n,k}(x). \quad (3)$$

Furthermore, Cao [3] proved that the necessary and sufficient condition of convergence for the operators is $\lim_{n \rightarrow \infty} (s_n/n) = 0$, and he also proved that for $n \in Q = \{n : n \in \mathbb{N}, \text{ and } 0 < (s_n - 1)/n + 1/\sqrt{n} \leq 1\}$ the following estimate of approximation degree holds:

$$\|\mathcal{L}_n f - f\| \leq 4\omega\left(f, \frac{s_n-1}{n} + \frac{1}{\sqrt{n}}\right). \quad (4)$$

Here, $\omega(f, t)$ is the modulus of continuity of first order of the function f . In [4], some approximation properties for the operators were further investigated.

In this paper, we will introduce and study the multivariate version defined on the simplex of the generalized Bernstein operators given by (1). The main aim is to establish the direct and inverse inequalities of approximation, which will imply the characterization of approximation of the operators.

For convenience, we denote by bold letter the vector in \mathbb{R}^d . Let

$$\mathbf{e}_i := (0, 0, \dots, 0, 1, 0, \dots, 0)$$

denote the canonical unit vector in \mathbb{R}^d , i.e., its i th component is 1 and the others are 0, and let

$$T := T_d := \left\{ \mathbf{x} = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d : x_i \geq 0, i = 1, 2, \dots, d, \sum_{i=1}^d x_i \leq 1 \right\} \quad (5)$$

be the simplex in \mathbb{R}^d . For $\mathbf{x} \in T$, $\mathbf{k} := (k_1, k_2, \dots, k_d) \in \mathbb{N}_0^d$, we denote as usual

$$|\mathbf{x}| := \sum_{i=1}^d x_i, \quad \mathbf{x}^{\mathbf{k}} := x_1^{k_1} x_2^{k_2} \dots x_d^{k_d}, \quad |\mathbf{k}| := \sum_{i=1}^d k_i, \quad \mathbf{k}! := k_1! k_2! \dots k_d!. \quad (6)$$

Then the well-known Bernstein basis function on T is given by

$$P_{n,\mathbf{k}}(\mathbf{x}) := \frac{n!}{\mathbf{k}!(n - |\mathbf{k}|)!} \mathbf{x}^{\mathbf{k}} (1 - |\mathbf{x}|)^{n - |\mathbf{k}|}, \quad \mathbf{x} \in T. \quad (7)$$

By means of the basis function, we define the multivariate generalized Bernstein operators on the simplex T as

$$(\mathcal{L}_{n,d}f)(\mathbf{x}) := \frac{1}{s_n} \sum_{|\mathbf{k}| \leq n} P_{n,\mathbf{k}}(\mathbf{x}) \left(\sum_{|\mathbf{j}| \leq s_n - 1} f\left(\frac{\mathbf{k} + \mathbf{j}}{n + s_n - 1}\right) \right). \quad (8)$$

Obviously, when $d = 1$, these operators reduce to the univariate operators defined by (1), and when $s_n = 1$ they are just the well-known multivariate Bernstein operators on the simplex T , $\mathcal{B}_{n,d}$, defined by

$$(\mathcal{B}_{n,d}f)(\mathbf{x}) := \sum_{|\mathbf{k}| \leq n} P_{n,\mathbf{k}}(\mathbf{x}) f\left(\frac{\mathbf{k}}{n}\right). \quad (9)$$

Let $C(T)$ denote the space of continuous functions on T with the norm defined by $\|f\| := \max_{\mathbf{x} \in T} |f(\mathbf{x})|$, $f \in C(T)$. For arbitrary vector $\mathbf{e} \in \mathbb{R}^d$, we write for the r th symmetric difference of a function f in the direction of \mathbf{e}

$$\Delta_{h\mathbf{e}}^r f(\mathbf{x}) := \begin{cases} \sum_{i=0}^r (-1)^i \binom{r}{i} f\left(\mathbf{x} + \left(\frac{r}{2} - i\right)h\mathbf{e}\right), & \mathbf{x} \pm \frac{r h \mathbf{e}}{2} \in T, \\ 0, & \text{otherwise.} \end{cases}$$

Then the Ditzian-Totik's modulus of function $f \in C(T)$ is defined by (see [1])

$$\omega_\varphi^r(f, t) := \sup_{0 < h \leq t} \sum_{1 \leq i \leq j \leq d} \|\Delta_{h\varphi_{ij}\mathbf{e}_{ij}}^r f\|,$$

where the weighted functions

$$\varphi_{ii}(\mathbf{x}) := \sqrt{x_i(1 - |\mathbf{x}|)}, \quad 1 \leq i \leq d; \quad \varphi_{ij}(\mathbf{x}) := \sqrt{x_i x_j}, \quad 1 \leq i < j \leq d,$$

and

$$\mathbf{e}_{ii} := \mathbf{e}_i, \quad 1 \leq i \leq d; \quad \mathbf{e}_{ij} := \mathbf{e}_i - \mathbf{e}_j, \quad 1 \leq i < j \leq d.$$

Define differential operators:

$$D_i := D_{ii} := \frac{\partial}{\partial x_i}, \quad 1 \leq i \leq d; \quad D_{ij} := D_i - D_j, \quad 1 \leq i < j \leq d;$$

$$D_{ij}^r := D_{ij}(D_{ij}^{r-1}), \quad 1 \leq i \leq j \leq d, r \in \mathbb{N},$$

then the weighted Sobolev space can be defined by

$$D_r(T_d) := \{g \in C(T) : g \in C^r(\overset{\circ}{T}), \text{ and } \varphi_{ij}^r D_{ij}^r g \in C(T), 1 \leq i \leq j \leq d\},$$

where $\overset{\circ}{T}$ is inner of T , and the Peetre K -functional on $C(T)$ is given by

$$\mathcal{K}_\varphi^r(f, t^r) := \inf_{g \in D_r(T_d)} \left\{ \|f - g\| + t^r \sum_{1 \leq i \leq j \leq d} \|\varphi_{ij}^2 D_{ij}^2 f\| \right\}, \quad t > 0.$$

Berens and Xu [1] proved that $\mathcal{K}_\varphi^r(f, t^r)$ is equivalent to $\omega_\varphi^r(f, t)$, i.e.,

$$C^{-1} \omega_\varphi^r(f, t) \leq \mathcal{K}_\varphi^r(f, t^r) \leq C \omega_\varphi^r(f, t), \tag{10}$$

here and in the following C denotes a positive constant independent of f and n , but its value may be different at a different occurrence.

We also need the usual modulus of continuity of function $f \in C(T)$ defined by (see [8])

$$\omega(f, t) := \sup_{0 < |\mathbf{h}|_2 \leq t} \|f(\cdot + \mathbf{h}) - f(\cdot)\|,$$

where $\mathbf{h} = (h_1, h_2, \dots, h_d) \in \mathbb{R}^d$ and $|\mathbf{h}|_2 := (\sum_{i=1}^d h_i^2)^{1/2}$, and another \mathcal{K} -functional given by (see [8])

$$\mathcal{K}(f, t) := \inf_{g \in C^1(T)} \left\{ \|f - g\| + t \sum_{i=1}^d \|D_i g\| \right\}.$$

It is shown in [8] that

$$C^{-1} \omega(f, t) \leq \mathcal{K}(f, t) \leq C \omega(f, t). \tag{11}$$

Now we state the main results of this paper as follows.

Theorem 1.1 *Let $f \in C(T)$, then for $n \in Q = \{n : n \in \mathbb{N}, \text{ and } 0 < \frac{s_n-1}{n} + \frac{1}{\sqrt{n}} \leq 1\}$, there holds*

$$\|\mathcal{L}_{n,d}f - f\| \leq 4d\omega\left(f, \frac{s_n-1}{n} + \frac{1}{\sqrt{n}}\right).$$

Theorem 1.2 *If $f \in C(T)$ and $\lim_{n \rightarrow \infty}(s_n/n) = 0$, then*

$$\|\mathcal{L}_{n,d}f - f\| \leq C\left(\omega_\varphi^2\left(f, \frac{1}{\sqrt{n}}\right) + \omega\left(f, \frac{s_n-1}{n+s_n-1}\right) + \frac{1}{n}\|f\|\right).$$

Theorem 1.3 *If $f \in C(T)$ and $\lim_{n \rightarrow \infty}(s_n/n) = 0$, then there hold*

$$\omega_\varphi^2\left(f, \frac{1}{\sqrt{n}}\right) \leq Cn^{-1} \sum_{k=1}^n \|\mathcal{L}_{k,d}f - f\|$$

and

$$\omega\left(f, \frac{1}{n}\right) \leq Cn^{-1} \left(\sum_{k=1}^n \|\mathcal{L}_{k,d}f - f\| + \|f\|\right).$$

From Theorem 1.2 and Theorem 1.3, we easily obtain the following corollaries, which characterize the approximation feature of the multivariate operators $\mathcal{L}_{n,d}$ given by (8).

Corollary 1.1 *Let $f \in C(T)$, $0 < \alpha < 1$. Then, for the Bernstein operators given by (9), the necessary and sufficient condition for which*

$$\|\mathcal{B}_{n,d}f - f\| = \mathcal{O}\left(\frac{1}{n^\alpha}\right), \quad n \rightarrow \infty$$

is $\omega_\varphi^2(f, t) = \mathcal{O}(t^{2\alpha})$ ($t \rightarrow 0$).

Corollary 1.2 *If $f \in C(T)$, $0 < \alpha < 1$, $s_n > 1$ and $\lim_{n \rightarrow \infty}(s_n/n) = 0$, then $\omega_\varphi^2(f, t) = \mathcal{O}(t^{2\alpha})$ and $\omega(f, t) = \mathcal{O}(t^\alpha)$ ($t \rightarrow 0$) imply*

$$\|\mathcal{L}_{n,d}f - f\| = \mathcal{O}\left(\left(\frac{s_n}{n+s_n-1}\right)^\alpha\right), \quad n \rightarrow \infty.$$

Corollary 1.3 *If $s_n > 1$ and $s_n = \mathcal{O}(n^{1-\epsilon})$ ($n \rightarrow \infty$), $0 < \epsilon \leq 1$, then for any $f \in C(T)$ and $0 < \alpha < 1$, the statement*

$$\|\mathcal{L}_{n,d}f - f\| = \mathcal{O}(n^{-\epsilon\alpha}), \quad n \rightarrow \infty$$

implies that $\omega_\varphi^2(f, t) = \mathcal{O}(t^{2\epsilon\alpha})$ and $\omega(f, t) = \mathcal{O}(t^{\epsilon\alpha})$ ($t \rightarrow 0$).

From Corollary 1.2 and Corollary 1.3, we have the following.

Corollary 1.4 *Let $0 < \alpha < 1$ and $1 < s_n = \mathcal{O}(1)$ ($n \rightarrow \infty$), then, for any $f \in C(T)$ the necessary and sufficient condition for which*

$$\|\mathcal{L}_{n,d}f - f\| = \mathcal{O}\left(\frac{1}{n^\alpha}\right), \quad n \rightarrow \infty$$

is $\omega_\varphi^2(f, t) = \mathcal{O}(t^{2\alpha})$ and $\omega(f, t) = \mathcal{O}(t^{\epsilon\alpha})$ ($t \rightarrow 0$).

2 Some lemmas

In this section, we prove some lemmas.

Defining the transformation \mathcal{T}_i ($i = 1, 2, \dots, d$) from T to itself, i.e.,

$$\mathcal{T}_i(\mathbf{x}) := \mathbf{u}, \quad \mathbf{u} = (x_1, \dots, x_{i-1}, 1 - |\mathbf{x}|, x_{i+1}, \dots, x_d), \quad \mathbf{x} \in T,$$

we have the following symmetric property for the operators $\mathcal{L}_{n,d}$, which is similar to the known one of the multivariate Bernstein operators (see [6, 7]).

Lemma 2.1 *For the above transformation \mathcal{T}_i , $i = 1, 2, \dots, d$, there holds*

$$(\mathcal{L}_{n,d}f)(\mathbf{x}) = (\mathcal{L}_{n,d}f_i)(\mathbf{u}),$$

where $f_i(\mathbf{x}) = f(\mathcal{T}_i(\mathbf{x}))$, $\mathbf{u} = \mathcal{T}_i(\mathbf{x})$.

Proof It is sufficient to prove the case $i = 1$. Let

$$\begin{aligned} \mathbf{l} &= (l_1, l_2, \dots, l_d), & \mathbf{l}^* &= (l_2, \dots, l_d), & l_1 &= n - |\mathbf{k}|, & l_i &= k_i, & i &= 2, 3, \dots, d, \\ \mathbf{t} &= (t_1, t_2, \dots, t_d), & \mathbf{t}^* &= (t_2, \dots, t_d), & t_1 &= s_n - 1 - |\mathbf{j}|, & t_i &= j_i, & i &= 2, 3, \dots, d, \end{aligned}$$

and $\mathbf{x} = (x_1, \mathbf{x}^*) \in T$, $\mathbf{x}^* = (x_2, x_3, \dots, x_d)$. Then, from definition (8), it follows that

$$\begin{aligned} (\mathcal{L}_{n,d}f)(\mathbf{x}) &= \frac{1}{s_n} \sum_{|\mathbf{k}| \leq n} P_{n,\mathbf{k}}(\mathbf{x}) \left\{ \sum_{|\mathbf{j}| \leq s_n - 1} f\left(\frac{\mathbf{k} + \mathbf{j}}{n + s_n - 1}\right) \right\} \\ &= \frac{1}{s_n} \sum_{|\mathbf{l}| \leq n} \frac{n!}{(n - |\mathbf{l}|)! \mathbf{l}^*! l_1!} x_1^{n - |\mathbf{l}|} (\mathbf{x}^*)^{\mathbf{l}^*} (1 - |\mathbf{x}|)^{l_1} \\ &\quad \times \left\{ \sum_{|\mathbf{t}| \leq s_n - 1} f\left(\frac{n - |\mathbf{l}| - |\mathbf{t}|}{n + s_n - 1}, \frac{\mathbf{l}^* + \mathbf{t}^*}{n + s_n - 1}\right) \right\} \\ &= \frac{1}{s_n} \sum_{|\mathbf{l}| \leq n} P_{n,\mathbf{l}}(1 - |\mathbf{x}|, \mathbf{x}^*) \left\{ \sum_{|\mathbf{t}| \leq s_n - 1} f\left(1 - \frac{|\mathbf{l}| + |\mathbf{t}|}{n + s_n - 1}, \frac{\mathbf{l}^* + \mathbf{t}^*}{n + s_n - 1}\right) \right\} \\ &= (\mathcal{L}_{n,d}f_1)(\mathbf{u}). \end{aligned}$$

The proof of Lemma 2.1 is completed. □

To prove Theorem 1.3, we need some the following lemmas. At first, similar to the estimates for the Bernstein operators (see [2, 5, 6]), it is not difficult to derive the following Lemma 2.2.

Lemma 2.2 *The following inequalities hold:*

$$\|D_i(\mathcal{L}_{n,d}f)\| \leq \begin{cases} 2n\|f\|, & f \in C(T), \\ \|D_i f\|, & f \in C^1(T), \end{cases} \quad 1 \leq i \leq d;$$

$$\|D_i^2(\mathcal{L}_{n,d}f)\| \leq \begin{cases} 4n^2\|f\|, & f \in C(T), \\ \|D_i^2 f\|, & f \in C^2(T), \end{cases} \quad 1 \leq i \leq d.$$

Secondly, we need prove two Bernstein type inequalities.

Lemma 2.3 *Let $f \in C(T)$, $1 \leq i \leq j \leq d$. Then*

$$\|\varphi_{ij}^2 D_{ij}^2(\mathcal{L}_{n,d}f)\| \leq 2n\|f\|.$$

Proof For $d = 1$, by direct computation we have (see [9])

$$(\mathcal{L}_n f)''(x) = \frac{1}{s_n} \varphi^{-4}(x) n^2 \sum_{k=1}^n r_{n,k}(x) P_{n,k}(x) \left(\sum_{j=0}^{s_n-1} f\left(\frac{k+j}{n+s_n-1}\right) \right),$$

where

$$r_{n,k}(x) = \left(\frac{k}{n} - x\right)^2 - (1-2x)\frac{k}{n^2} - \frac{x^2}{n}.$$

Noting that

$$|r_{n,k}(x)| \leq \left(\frac{k}{n} - x\right)^2 + (1-2x)\frac{k}{n^2} + \frac{x^2}{n},$$

we obtain

$$|\varphi^2(x)(\mathcal{L}_n f)''(x)| \leq \|f\| \varphi^{-2}(x) n^2 \sum_{k=0}^n |r_{n,k}(x)| P_{n,k}(x) \leq 2n\|f\|.$$

This inequality shows that Lemma 2.3 is valid for $d = 1$. For the proof of the case $d > 1$, we use a decomposition technique and the induction. In fact, let

$$g_{k_1, j_1}(\mathbf{u}) := f\left(\frac{k_1 + j_1}{n + s_n - 1}, \left(1 - \frac{k_1 + j_1}{n + s_n - 1}\right)\mathbf{u}\right), \quad \mathbf{u} = (u_1, u_2, \dots, u_{d-1})$$

and

$$\mathbf{z} := (z_1, z_2, \dots, z_{d-1}) := \left(\frac{x_2}{1-x_1}, \frac{x_2}{1-x_1}, \dots, \frac{x_d}{1-x_1}\right),$$

$$\mathbf{k}^\circ := (k_2, k_3, \dots, k_d), \quad |\mathbf{k}^\circ| := \sum_{i=2}^d k_i,$$

$$\mathbf{j}^\circ := (j_2, j_3, \dots, j_d), \quad |\mathbf{j}^\circ| := \sum_{i=2}^d j_i,$$

then we can decompose the generalized Bernstein operators as

$$\begin{aligned}
 (\mathcal{L}_{n,d}f)(\mathbf{x}) &= \frac{1}{s_n} \sum_{k_1=0}^n P_{n,k_1}(x_1) \sum_{|\mathbf{k}^*| \leq n-k_1} P_{n,n-k_1}(\mathbf{z}) \\
 &\quad \times \left(\sum_{j_1=0}^{s_n-1} \sum_{|\mathbf{j}^*| \leq s_n-1-j_1} f\left(\frac{\mathbf{k} + \mathbf{j}}{n + s_n - 1}\right) \right) \\
 &= \sum_{k_1=0}^n P_{n,k_1}(x_1) \left(\sum_{j_1=0}^{s_n-1} \frac{s_n - j_1}{s_n} (\mathcal{L}_{n-k_1,d-1}g_{k_1,j_1})(\mathbf{z}) \right).
 \end{aligned}$$

Therefore,

$$\varphi_{22}^2(\mathbf{x}) D_{22}^2(\mathcal{L}_{n,d}f)(\mathbf{x}) = \sum_{k_1=0}^n P_{n,k_1}(x_1) \left(\sum_{j_1=0}^{s_n-1} \frac{s_n - j_1}{s_n} \varphi_{11}^2(\mathbf{z}) D_{11}^2(\mathcal{L}_{n-k_1,d-1}g_{k_1,j_1})(\mathbf{z}) \right). \tag{12}$$

Now, suppose that Lemma 2.3 is valid for $d - 1$, then from (12) it follows that

$$\left| \varphi_{22}^2(\mathbf{x}) D_{22}^2(\mathcal{L}_{n,d}f)(\mathbf{x}) \right| \leq 2 \sum_{k_1=0}^n P_{n,k_1}(x_1) \left(\sum_{j_1=0}^{s_n-1} \frac{s_n - j_1}{s_n} (n - k_1) \|g_{k_1,j_1}(\cdot)\| \right) \leq 2n\|f\|.$$

So, Lemma 2.3 is true for $i = 2$. From the symmetry, the proof of the cases $i = 1, 3, 4, \dots, d$ is the same. For the cases $1 \leq i < j \leq d$, we use Lemma 2.1 and obtain that

$$\|\varphi_{ij}^2 D_{ij}^2(\mathcal{L}_{n,d}f)\| = \|\varphi_{ii}^2 D_{ii}^2(\mathcal{L}_{n,d}f_j)\| \leq 2n\|f_j\| \leq 2n\|f\|.$$

So, the proof of Lemma 2.3 is complete. □

Lemma 2.4 For $f \in C^2(T)$, $1 \leq i \leq j \leq d$, one has

$$\|\varphi_{ij}^2 D_{ij}^2(\mathcal{L}_{n,d}f)\| \leq \|\varphi_{ij}^2 D_{ij}^2 f\| + \frac{1}{n} \|D_{ij}^2 f\|.$$

Proof We only need to prove the case $s_n > 1$ because the case $s_n = 1$ has been shown in [1]. Our approach is based on the induction. At first, for $d = 1$, let $h = (n + s_n - 1)^{-1}$, then by simple calculation we have

$$(\mathcal{L}_n f)''(x) = \frac{n(n-1)}{s_n} \sum_{k=0}^{n-2} \left(\sum_{j=0}^{s_n-1} \Delta_h^2 f\left(\frac{k+j+1}{n+s_n-1}\right) \right) P_{n-2,k}(x).$$

Therefore,

$$\begin{aligned}
 &|\varphi^2(x)(\mathcal{L}_n f)''(x)| \\
 &= \left| \frac{1}{s_n} \sum_{k=0}^{n-2} (k+1)(n-k-1) \left(\sum_{j=0}^{s_n-1} \Delta_h^2 f\left(\frac{k+j+1}{n+s_n-1}\right) \right) P_{n,k+1}(x) \right| \\
 &= \left| \frac{1}{h^2 s_n} \sum_{k=1}^{n-1} \sum_{j=0}^{s_n-1} \frac{k}{n+s_n-1} \frac{n-k}{n+s_n-1} \Delta_h^2 f\left(\frac{k+j}{n+s_n-1}\right) P_{n,k}(x) \right|
 \end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{h^2 s_n} \sum_{k=1}^{n-1} \sum_{j=0}^{s_n-1} \varphi^2\left(\frac{k+j}{n+s_n-1}\right) \left| \Delta_{hf}^2 \left(\frac{k+j}{n+s_n-1}\right) \right| P_{n,k}(x) \\ &= \frac{1}{h^2 s_n} \sum_{k=1}^{n-1} \sum_{j=0}^{s_n-1} \varphi^2\left(\frac{k+j}{n+s_n-1}\right) P_{n,k}(x) \\ &\quad \times \left| \int_{-h/2}^{h/2} \int_{-h/2}^{h/2} f''\left(\frac{k+j}{n+s_n-1} + s+t\right) ds dt \right|. \end{aligned}$$

Let $y = (k+j)/(n+s_n-1)$, then we have for $1 \leq k \leq n-1$, $0 \leq j \leq s_n-1$,

$$h = \frac{1}{n+s_n-1} \leq y \leq 1 - \frac{1}{n+s_n-1} = 1-h$$

and for $|u| \leq h$, there holds $|1-2y-u| \leq 1$. Hence,

$$\varphi^2(y) = \varphi^2(y+u) - u(1-2y-u) \leq \varphi^2(y+u) + |u| \leq \varphi^2(y+u) + h,$$

which implies

$$\begin{aligned} \varphi^2(y) |\Delta_h^2(y)| &\leq \varphi^2(y) \left| \int_{-h/2}^{h/2} \int_{-h/2}^{h/2} f''(y+s+t) ds dt \right| \\ &\leq \int_{-h/2}^{h/2} \int_{-h/2}^{h/2} (\varphi^2(y+s+t) + h) |f''(y+s+t)| ds dt \\ &\leq h^2 (\|\varphi^2 f''\| + h \|f''\|). \end{aligned}$$

So,

$$\|\varphi^2(\mathcal{L}_n f)''\| \leq \|\varphi^2 f''\| + \frac{1}{n} \|f''\|.$$

Now, assume that Lemma 2.4 is valid for $d-1$, then by (12)

$$\begin{aligned} &|\varphi_{22}^2(\mathbf{x}) D_{22}^2(\mathcal{L}_{n,d} f)(\mathbf{x})| \\ &\leq \sum_{k_1=0}^n P_{n,k_1}(x_1) \left(\sum_{j_1=0}^{s_n-1} \frac{s_n-j_1}{s_n} \left(\|\varphi_{11}^2 D_{11}^2 g_{k_1,j_1}\| + \frac{1}{n-k_1+s_n-j_1-1} \|D_{11}^2 g_{k_1,j_1}\| \right) \right). \end{aligned}$$

Also, we can check the following inequalities:

$$\begin{aligned} |D_{11}^2 g_{k_1,j_1}(\mathbf{z})| &= \left| \left(1 - \frac{k_1+j_1}{n+s_n-1}\right)^2 D_{22}^2 f\left(\frac{k_1+j_1}{n+s_n-1}, \left(1 - \frac{k_1+j_1}{n+s_n-1}\right) \mathbf{z}\right) \right| \\ &\leq \left(1 - \frac{k_1+j_1}{n+s_n-1}\right)^2 \|D_{22}^2 f\|, \end{aligned}$$

and

$$|\varphi_{11}^2(\mathbf{z}) D_{11}^2 g_{k_1,j_1}(\mathbf{z})| = \left| (\varphi_{22}^2 D_{22}^2 f) \left(\frac{k_1+j_1}{n+s_n-1}, \left(1 - \frac{k_1+j_1}{n+s_n-1}\right) \mathbf{z}\right) \right| \leq \|\varphi_{22}^2 D_{22}^2 f\|.$$

Thus,

$$\|\varphi_{22}^2 D_{22}^2(\mathcal{L}_n df)\| \leq \|\varphi_{22}^2 D_{22}^2 f\| + \frac{1}{n} \|D_{22}^2 f\|.$$

Similarly, the cases $i = 1, 3, 4, \dots, d$ can be proved. For the case $1 \leq i \leq j \leq d$, we use the transformation \mathcal{T}_i and Lemma 2.1, it is easy to verify

$$\begin{aligned} \|\varphi_{ij}^2 D_{ij}^2(\mathcal{L}_n df)\| &= \|\varphi_{ii}^2 D_{ii}^2(\mathcal{L}_n df_j)\| \\ &\leq \|\varphi_{ii}^2 D_{ii}^2 f_j\| + n^{-1} \|D_{ii}^2 f_j\| \\ &= \|\varphi_{ij}^2 D_{ij}^2 f\| + n^{-1} \|D_{ij}^2 f\|. \end{aligned}$$

Hence, the proof of Lemma 2.4 is complete. □

We also need the following two interesting results related to nonnegative numerical sequence. The proof of the first result can be found in [10], and the proof of the other is similar to Lemma 2.1 of [10] where the proof of case $v_1 = 0$ and $C = 1$ was given.

Lemma 2.5 *Let μ_n, v_n , and ψ_n are all nonnegative numerical sequence, and $\mu_1 = v_1 = 0$. If for $0 < r < s$ and $1 \leq k \leq n, n \in \mathbb{N}$, there holds*

$$\mu_n \leq \left(\frac{k}{n}\right)^r \mu_k + v_k + \psi_k, \quad v_n \leq \left(\frac{k}{n}\right)^s v_k + \psi_k,$$

then

$$\mu_n \leq C n^{-r} \sum_{k=1}^n k^{r-1} \psi_k.$$

Lemma 2.6 *Let v_n and ψ_n are all nonnegative numerical sequence. If for $s > 0$ and $1 \leq k \leq n, n \in \mathbb{N}$, there holds $v_n \leq \left(\frac{k}{n}\right)^s v_k + C \psi_k$, then*

$$v_n \leq C n^{-s} \left(\sum_{k=1}^n k^{s-1} \psi_k + v_1 \right).$$

3 The proof of main results

First, we prove Theorem 1.1. By straight calculation, we have (see also [3])

$$\begin{aligned} (\mathcal{L}_{n,d}(u_i - x_i))(\mathbf{x}) &= (\mathcal{L}_n(u_i - x_i))(x_i) = \frac{s_n - 1}{n + s_n - 1} (2x_i - 1), \\ (\mathcal{L}_{n,d}(|\mathbf{u} - \mathbf{x}|_2^2))(\mathbf{x}) &= \sum_{i=1}^d (\mathcal{L}_n(u_i - x_i)^2)(x_i) \\ &= \sum_{i=1}^d \frac{x_i(1 - x_i)(n - (s_n - 1)^2)}{(n + s_n - 1)^2} + \frac{(s_n - 1)(2(s_n - 1) + 1)}{6(n + s_n - 1)^2} \\ &\leq \sum_{i=1}^d \frac{4}{3} \left(\frac{(s_n - 1)^2}{n^2} + \frac{1}{n} \right). \end{aligned}$$

Then we use the same method as Theorem 2 of [3] and obtain easily

$$\|\mathcal{L}_{n,d}f - f\| \leq 4d\omega\left(f, \frac{s_n - 1}{n} + \frac{1}{\sqrt{n}}\right).$$

We now prove Theorem 1.2. We use a known estimation on Bernstein operators (see [1]) as an intermediate step to deduce the direct theorem. Since

$$\begin{aligned} \frac{1}{s_n} \sum_{|j| \leq s_n - 1} \left| f\left(\frac{\mathbf{k}}{n}\right) - f\left(\frac{\mathbf{k} + \mathbf{j}}{n + s_n - 1}\right) \right| &\leq \frac{1}{s_n} \sum_{|j| \leq s_n - 1} \omega\left(f, \frac{|\mathbf{k}(s_n - 1) - n\mathbf{j}|}{n(n + s_n - 1)}\right) \\ &\leq \omega\left(f, \frac{s_n - 1}{n + s_n + 1}\right), \quad |\mathbf{k}| \leq n \end{aligned}$$

from the fact that (see [1])

$$\|\mathcal{B}_{n,d}f - f\| \leq C\left(\omega_\varphi^2\left(f, \frac{1}{\sqrt{n}}\right) + \frac{1}{n}\|f\|\right),$$

we get

$$\begin{aligned} \|\mathcal{L}_{n,d}f - f\| &\leq \|\mathcal{B}_{n,d}f - f\| + \|\mathcal{B}_{n,d}f - \mathcal{L}_{n,d}f\| \\ &\leq C\left(\omega_\varphi^2\left(f, \frac{1}{\sqrt{n}}\right) + \frac{1}{n}\|f\|\right) \\ &\quad + \max_{\mathbf{x} \in T} \frac{1}{s_n} \sum_{k=0}^n \sum_{j=0}^{s_n - 1} \left| f\left(\frac{k}{n}\right) - f\left(\frac{k + j}{n + s_n - 1}\right) \right| P_{n,k}(\mathbf{x}) \\ &\leq C\left(\omega_\varphi^2\left(f, \frac{1}{\sqrt{n}}\right) + \omega\left(f, \frac{s_n - 1}{n + s_n + 1}\right) + \frac{1}{n}\|f\|\right). \end{aligned}$$

This completes the proof of Theorem 1.2.

Finally, we prove Theorem 1.3. Let

$$\mu_n = n^{-1} \|\varphi_{ij}^2 D_{ij}^2(\mathcal{L}_{n,d}f)\|, \quad 1 \leq i \leq j \leq d,$$

$$\nu_n = n^{-2} \|D_{ij}^2(\mathcal{L}_{n,d}f)\|, \quad 1 \leq i \leq j \leq d$$

and $\psi_n = 4\|(\mathcal{L}_{n,d}f) - f\|$, then $\mu_1 = \nu_1 = 0$ and from Lemma 2.2, Lemma 2.3, and Lemma 2.4, we have for $1 \leq k \leq n$

$$\begin{aligned} \mu_n &\leq n^{-1} \|\varphi_{ij}^2 D_{ij}^2(\mathcal{L}_{n,d}(\mathcal{L}_{k,d}f))\| + n^{-1} \|\varphi_{ij}^2 D_{ij}^2(\mathcal{L}_{n,d}(\mathcal{L}_{k,d}f - f))\| \\ &\leq n^{-1} \|\varphi_{ij}^2 D_{ij}^2(\mathcal{L}_{k,d}f)\| + \frac{1}{n^2} \|D_{ij}^2(\mathcal{L}_{k,d}f)\| + 2\|\mathcal{L}_{k,d}f - f\| \\ &\leq \left(\frac{k}{n}\right) \mu_k + \nu_k + \psi_k, \\ \nu_n &\leq n^{-2} \|D_{ij}^2(\mathcal{L}_{n,d}(\mathcal{L}_{k,d}f))\| + n^{-2} \|D_{ij}^2(\mathcal{L}_{n,d}(\mathcal{L}_{k,d}f - f))\| \\ &\leq n^{-2} \|D_{ij}^2(\mathcal{L}_{k,d}f)\| + 4\|\mathcal{L}_{k,d}f - f\| \\ &= \left(\frac{k}{n}\right)^2 \nu_k + \psi_k, \end{aligned}$$

which implies from Lemma 2.5 that $\mu_n \leq Cn^{-1} \sum_{k=1}^n \psi_k$, i.e.,

$$\|\varphi_{ij}^2 D_{ij}^2(\mathcal{L}_n, df)\| \leq C \sum_{k=1}^n \|\mathcal{L}_{k,df} - f\|, \quad 1 \leq i \leq j \leq d. \tag{13}$$

Let $v_n = \|D_i(\mathcal{L}_n, df)\|$ and ψ_n be the same as the above, then we have by Lemma 2.2

$$v_n \leq \left(\frac{k}{n}\right)v_k + \psi_k, \quad 1 \leq k \leq n.$$

So, using Lemma 2.6 gives $v_n \leq Cn^{-1}(\sum_{k=1}^n \psi_k + v_1)$, namely,

$$\|D_i(\mathcal{L}_n, df)\| \leq C \left(\sum_{k=0}^n \|\mathcal{L}_{k,df} - f\| + \|D_i(\mathcal{L}_1, df)\| \right) \leq C \left(\sum_{k=0}^n \|\mathcal{L}_{k,df} - f\| + \|f\| \right). \tag{14}$$

For $n \geq 2$, there is an $m \in \mathbb{N}$, such that $n/2 \leq m \leq n$ and $\|\mathcal{L}_{m,df} - f\| \leq \|\mathcal{L}_{k,df} - f\|$ hold for $1 \leq k \leq n$. Then

$$\|\mathcal{L}_{m,df} - f\| \leq \frac{4}{n} \sum_{k=n/2}^n \|\mathcal{L}_{k,df} - f\|. \tag{15}$$

So, combining (10), (13), and (15) we see

$$\begin{aligned} \omega_\varphi^2\left(f, \frac{1}{\sqrt{n}}\right) &\leq C\mathcal{K}_\varphi^2\left(f, \frac{1}{n}\right) \leq C(\|\mathcal{L}_{m,df} - f\| + n^{-1}\|\varphi_{ij}^2 D_{ij}^2(\mathcal{L}_m, df)\|) \\ &\leq Cn^{-1} \sum_{k=1}^n \|\mathcal{L}_{k,df} - f\|. \end{aligned}$$

Also, collecting (11), (13), and (15) implies

$$\begin{aligned} \omega\left(f, \frac{1}{n}\right) &\leq C\mathcal{K}\left(f, \frac{1}{n}\right) \leq C\left(\|\mathcal{L}_{m,df} - f\| + n^{-1} \sum_{1 \leq i \leq d} \|D_i(\mathcal{L}_m, df)\|\right) \\ &\leq Cn^{-1} \sum_{k=1}^n (\|\mathcal{L}_{k,df} - f\| + \|f\|). \end{aligned}$$

The proof of Theorem 1.3 is complete.

Competing interests

The authors declare that they have no competing interests.

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