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Every n-dimensional normed space is the space $\mathbb{R}^n$ endowed with a normal norm

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Abstract

Recently, Alonso showed that every two-dimensional normed space is isometrically isomorphic to a generalized Day-James space introduced by Nilsrakoo and Saejung. In this paper, we consider the result of Alonso for n-dimensional normed spaces.

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A norm $\|\cdot\|$ on $\mathbb{R}^2$ is said to be absolute if $\|(x,y)\| = \|(|x|,|y|)\|$ for all $(x,y) \in \mathbb{R}^2$, and normalized if $\|(1,0)\| = \|(0,1)\| = 1$. The set of all absolute normalized norms on $\mathbb{R}^2$ is denoted by AN_2. Bonsall and Duncan [1] showed the following characterization of absolute normalized norms on $\mathbb{R}^2$. Namely, the set AN_2 of all absolute normalized norms on $\mathbb{R}^2$ is in a one-to-one correspondence with the set Ψ_2 of all convex functions ψ on $[0,1]$ satisfying $\max\{1-t,t\} \le \psi(t) \le 1$ for all $t \in [0,1]$ (*cf.* [2]). The correspondence is given by the equation $\psi(t) = \|(1-t,t)\|$ for all $t \in [0,1]$. Note that the norm $\|\cdot\|_\psi$ associated with the function $\psi \in \Psi_2$ is given by

$$\|(x,y)\|_\psi = \begin{cases} (|x|+|y|)\psi(\frac{|y|}{|x|+|y|}), & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

The Day-James space ℓ_p-ℓ_q is defined for $1 \le p, q \le \infty$ as the space $\mathbb{R}^2$ endowed with the norm

$$\|(x,y)\|_{p,q} = \begin{cases} \|(x,y)\|_p, & \text{if } xy \ge 0, \\ \|(x,y)\|_q, & \text{if } xy \le 0. \end{cases}$$

James [3] considered the space ℓ_p-ℓ_q with $p^{-1}+q^{-1}=1$ as an example of a two-dimensional normed space where Birkhoff orthogonality is symmetric. Recall that if x, y are elements of a real normed space X, then x is said to be Birkhoff-orthogonal to y, denoted by $x \perp_B y$, if $\|x+\lambda y\| \ge \|x\|$ for all $\lambda \in \mathbb{R}$. Birkhoff orthogonality is *homogeneous*, that is, $x \perp_B y$ implies $\alpha x \perp_B \beta y$ for any real numbers α and β. However, Birkhoff orthogonality is not *symmetric* in general, that is, $x \perp_B y$ does not imply $y \perp_B x$. More details about Birkhoff orthogonality can be found in Birkhoff [4], Day [5, 6] and James [3, 7, 8].

In 2006, Nilsrakoo and Saejung [9] introduced and studied generalized Day-James spaces ℓ_φ-ℓ_ψ, where ℓ_φ-ℓ_ψ is defined for $\varphi, \psi \in \Psi_2$ as the space $\mathbb{R}^2$ endowed with the norm

$$\|(x,y)\|_{\varphi,\psi} = \begin{cases} \|(x,y)\|_{\varphi}, & \text{if } xy \geq 0, \\ \|(x,y)\|_{\psi}, & \text{if } xy \leq 0. \end{cases}$$

Recently, Alonso [10] showed that every two-dimensional normed space is isometrically isomorphic to a generalized Day-James space. In this paper, we consider the result of Alonso for n-dimensional spaces.

First, we give a characterization of generalized Day-James spaces.

Proposition 1 *Let $\|\cdot\|$ be a norm on $\mathbb{R}^2$. Then the space $(\mathbb{R}^2, \|\cdot\|)$ is a generalized Day-James space if and only if $\|\cdot\|_\infty \leq \|\cdot\| \leq \|\cdot\|_1$.*

Proof If $(\mathbb{R}^2, \|\cdot\|)$ is a generalized Day-James space, then one can easily have $\|\cdot\|_\infty \leq \|\cdot\| \leq \|\cdot\|_1$. So, we assume that $\|\cdot\|_\infty \leq \|\cdot\| \leq \|\cdot\|_1$. Let

$$\varphi(t) = \|(1-t,t)\| \quad \text{and} \quad \psi(t) = \|(1-t,-t)\|$$

for all $t \in [0,1]$, respectively. Then, clearly, we have $\varphi, \psi \in \Psi_2$ and $\|\cdot\| = \|\cdot\|_{\varphi,\psi}$. Hence, the space $(\mathbb{R}^2, \|\cdot\|)$ is a generalized Day-James space. □

Motivated by this fact, we consider the following

Definition 2 A norm $\|\cdot\|$ on $\mathbb{R}^n$ is said to be normal if it satisfies $\|\cdot\|_\infty \leq \|\cdot\| \leq \|\cdot\|_1$.

We recall some notions about multilinear forms. Let X be a real vector space. Then a real-valued function F on X^n is said to be an n-linear form if it is linear separately in each variable, that is,

$$\begin{aligned} &F(x_1,\ldots,x_{i-1},\alpha x_i + x_i', x_{i+1},\ldots,x_n) \\ &\quad = \alpha F(x_1,\ldots,x_{i-1},x_i,x_{i+1},\ldots,x_n) + F(x_1,\ldots,x_{i-1},x_i',x_{i+1},\ldots,x_n) \end{aligned}$$

for each $i \in \{1,2,\ldots,n\}$. If $F : X^n \to \mathbb{R}$ is an n-linear form, then F is said to be alternating if

$$F(x_1,\ldots,x_i,x_{i+1},\ldots,x_n) = -F(x_1,\ldots,x_{i+1},x_i,\ldots,x_n)$$

for each $i \in \{1,2,\ldots,n\}$ or, equivalently, $F(x_1,x_2,\ldots,x_n) = 0$ if $x_i = x_j$ for some i,j with $i \neq j$. Furthermore, F is said to be bounded if

$$\|F\| := \sup\{|F(x_1,x_2,\ldots,x_n)| : (x_1,x_2,\ldots,x_n) \in (S_X)^n\} < \infty,$$

where S_X denotes the unit sphere of X. If F is bounded, then we have

$$|F(x_1,x_2,\ldots,x_n)| \leq \|F\|\|x_1\|\|x_2\|\cdots\|x_n\|$$

for all $(x_1,x_2,\ldots,x_n) \in X^n$.

For our purpose, we give another simple proof of the following result of Day [5]. For each subset A of a normed space, let $[A]$ denote the closed linear span of A. If M, N are subspaces of a real normed space X, then M is said to be Birkhoff orthogonal to N, denoted by $M \perp_B N$, if $\|x + y\| \geq \|x\|$ for all $x \in M$ and all $y \in N$. In particular, $x \perp_B M$ denotes $[\{x\}] \perp_B M$.

Lemma 3 *Let X be an n-dimensional real normed space. Then there exists a basis $\{e_1, e_2, \ldots, e_n\}$ for X such that $\|e_i\| = 1$ and $e_i \perp_B [\{e_k\}_{k \neq i}]$ for all $i = 1, 2, \ldots, n$.*

Proof Let $\{u_1, u_2, \ldots, u_n\}$ be a basis for X. Then each vector $x \in X$ is uniquely expressed in the form $x = \sum_{k=1}^{n} \alpha_k(x) u_k$. Define the function F on X^n by

$$F(x_1, x_2, \ldots, x_n) = \begin{vmatrix} \alpha_1(x_1) & \alpha_2(x_1) & \ldots & \alpha_n(x_1) \\ \alpha_1(x_2) & \alpha_2(x_2) & \ldots & \alpha_n(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1(x_n) & \alpha_2(x_n) & \ldots & \alpha_n(x_n) \end{vmatrix}$$

for all $(x_1, x_2, \ldots, x_n) \in X^n$. Then it is easy to check that F is an alternating bounded n-linear form. Since F is jointly continuous on the compact subset $(S_X)^n$ of X^n, there exists $(e_1, e_2, \ldots, e_n) \in (S_X)^n$ such that

$$F(e_1, e_2, \ldots, e_n) = \|F\| > 0.$$

For all $i = 1, 2, \ldots, n$ and all $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n$, we have

$$\begin{aligned} \|F\| \left\| \sum_{k=1}^{n} \alpha_k e_k \right\| &\geq \left| F\left(e_1, \ldots, e_{i-1}, \sum_{k=1}^{n} \alpha_k e_k, e_{i+1}, \ldots, e_n\right) \right| \\ &= \left| \sum_{k=1}^{n} \alpha_k F(e_1, \ldots, e_{i-1}, e_k, e_{i+1}, \ldots, e_n) \right| \\ &= \left| \alpha_i F(e_1, e_2, \ldots, e_n) \right| = \|F\| |\alpha_i|. \end{aligned}$$

Thus, we obtain

$$\left\| \sum_{k=1}^{n} \alpha_k e_k \right\| \geq |\alpha_i| = \|\alpha_i e_i\|,$$

for all $i = 1, 2, \ldots, n$ and all $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n$. This means that $e_i \perp_B [\{e_k\}_{k \neq i}]$ for all $i = 1, 2, \ldots, n$. □

Now, we state the main theorem.

Theorem 4 *Every n-dimensional normed space is isometrically isomorphic to the space $\mathbb{R}^n$ endowed with a normal norm.*

Proof By Lemma 3, there exists an n-tuple $(e_1, e_2, \ldots, e_n)$ of elements of S_X such that $e_i \perp_B [\{e_k\}_{k \neq i}]$ for all $i = 1, 2, \ldots, n$. Since $e_i \perp_B [\{e_k\}_{k \neq i}]$, we have

$$\left\| \sum_{k=1}^{n} \alpha_k e_k \right\| \geq \|\alpha_i e_i\| = |\alpha_i|,$$

for all $i = 1, 2, \ldots, n$ and all $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n$. Hence, we obtain

$$\left\| \sum_{k=1}^{n} \alpha_k e_k \right\| \geq \max\{|\alpha_1|, |\alpha_2|, \ldots, |\alpha_n|\}$$

for all $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n$. From this fact, we note that $\{e_1, e_2, \ldots, e_n\}$ is linearly independent, that is, a basis for X.

Define the norm $\|\cdot\|_0$ on $\mathbb{R}^n$ by the formula

$$\|(\alpha_1, \alpha_2, \ldots, \alpha_n)\|_0 = \left\| \sum_{k=1}^{n} \alpha_k e_k \right\|$$

for all $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n$. Then, clearly, $\|\cdot\|_0$ is normal and X is isometrically isomorphic to the space $(\mathbb{R}^n, \|\cdot\|_0)$. This completes the proof. □

Since the space $\mathbb{R}^2$ endowed with a normal norm is a generalized Day-James space by Proposition 1, we have the result of Alonso as a corollary.

Corollary 5 ([10]) *Every two-dimensional real normed space is isometrically isomorphic to a generalized Day-James space.*

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
RT conceived of the study, carried out the study of a structure of finite dimensional normed linear spaces, and drafted the manuscript. KS participated in the design of the study and helped to draft the manuscript. All authors read and approved the final manuscripts.

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References

1. Bonsall, FF, Duncan, J: Numerical Ranges II. Cambridge University Press, Cambridge (1973)
2. Saito, KS, Kato, M, Takahashi, Y: Von Neumann-Jordan constant of absolute normalized norms on $\mathbb{C}^2$. J. Math. Anal. Appl. **244**, 515-532 (2000)
3. James, RC: Inner products in normed linear spaces. Bull. Am. Math. Soc. **53**, 559-566 (1947)
4. Birkhoff, G: Orthogonality in linear metric spaces. Duke Math. J. **1**, 169-172 (1935)
5. Day, MM: Polygons circumscribed about closed convex curves. Trans. Am. Math. Soc. **62**, 315-319 (1947)
6. Day, MM: Some characterizations of inner product spaces. Trans. Am. Math. Soc. **62**, 320-337 (1947)
7. James, RC: Orthogonality in normed linear spaces. Duke Math. J. **12**, 291-302 (1945)
8. James, RC: Orthogonality and linear functionals in normed linear spaces. Trans. Am. Math. Soc. **61**, 265-292 (1947)
9. Nilsrakoo, W, Saejung, S: The James constant of normalized norms on $\mathbb{R}^2$. J. Inequal. Appl. **2006**, Art. ID 26265 (2006)
10. Alonso, J: Any two-dimensional normed space is a generalized Day-James space. J. Inequal. Appl. **2011**, 2 (2011)

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