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A note on regularity criterion for 3D compressible nematic liquid crystal flows

Xiaochun Chen[1*] and Jishan Fan[2]

* Correspondence: xiaochunchen1@gmail.com
[1]College of Mathematics and Statistics, Chongqing Three Gorges University, Wanzhou, Chongqing 404000, P.R. China
Full list of author information is available at the end of the article

Abstract

In this article, we prove a regularity criterion for the local strong solutions to a simplified hydrodynamic flow modeling the compressible, nematic liquid crystal materials.

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1 Introduction

In this article, we consider the following simplified version of Ericksen-Leslie system modeling the hydrodynamic flow of compressible, nematic liquid crystals (see: [1,2])

$$\partial_t \rho + \text{div}(\rho u) = 0, \tag{1.1}$$

$$\partial_t(\rho u) + \text{div}(\rho u \otimes u) + \nabla p(\rho) - \mu \Delta u - (\lambda + \mu \nabla \text{div} u = -\Delta d \cdot \nabla d, \tag{1.2}$$

$$\partial_t d + u \cdot \nabla d = \Delta d + |\nabla d|^2 d, \tag{1.3}$$

$$(\rho, u, d)(x, 0) = (\rho_0, u_0, d_0)(x), |d_0| = 1, x \in \mathbb{R}^3. \tag{1.4}$$

Here ρ is the density, u is the fluid velocity and d represents the macroscopic average of the nematic liquid crystal orientation field, $p(\rho) := a\rho^\gamma$ is the pressure with positive constants $a > 0$ and $\gamma \geq 1$. μ and λ are the shear viscosity and the bulk viscosity coefficients of the fluid respectively, which are assumed to satisfy the following physical condition:

$$\mu > 0, \quad 3\lambda + 2\mu \geq 0.$$

(1.1) and (1.2) is the well-known compressible Navier-Stokes system with the external force -$\Delta d \cdot \nabla d$. (1.3) is the well-known heat flow of harmonic map when $u = 0$.

Very recently, Ericksen [3] proved the following local-in-time well-posedness:

Proposition 1.1. *Let* $\rho_0 \in W^{1,q} \cap H^1 \cap L^1$ *for some* $q \in (3, 6]$ *and* $\rho_0 \geq 0$ *in* $\mathbb{R}^3$, $\nabla u_0 \in H^1$, $\nabla d_0 \in H^2$ *and* $|d_0| = 1$ *in* $\mathbb{R}^3$. *If, in additions, the following compatibility condition*

$$-\mu \Delta u_0 - (\lambda + \mu)\nabla \text{div} u_0 - \nabla p(\rho_0) - \Delta d_0 \cdot d_0 = \sqrt{\rho_0} g \quad \text{for some } g \in L^2(\mathbb{R}^3) \tag{1.5}$$

holds, then there exist $T_0 > 0$ *and a unique strong solution* (ρ, u, d) *to the problem (1.1)-(1.4).*

Based on the above Proposition 1.1, Huang et al. [4] proved the following regularity criterion:

$$\int_0^T \left\|\mathcal{D}(u)\right\|_{L^\infty} + \|\nabla d\|_{L^\infty}^2 \, dt < \infty, \tag{1.6}$$

where $\mathcal{D}(u) := \frac{1}{2}\left(\nabla u + {}^t\nabla u\right)$.

The aim of this note is to refine (1.6) as follows.

Theorem 1.2. *Let the assumptions in Proposition 1.1 holds true. If*

$$\int_0^T \left\|\mathcal{D}(u)\right\|_{L^\infty} + \|\nabla d\|_{BMO}^2 dt < \infty, \tag{1.7}$$

then the solution (ρ, u, d) *can be extended beyond* $T > 0$.

Here BMO denotes the space of functions of bounded mean oscillations.

In this note, we will use the following inequality [5]:

$$\|u\|_{Lp} \le C \|u\|_{BMO}^{1-\frac{q}{p}} \|u\|_{L^q}^{\frac{q}{p}} \ (1 < q < p < \infty). \tag{1.8}$$

For the standard nematic liquid crystal flows, we refer to recent studies in [6,7].

2 Proof of Theorem 1.2

Since (ρ, u, d) is the local strong solution, we only need to prove

$$\nabla d \in L^2(0, T; L^\infty). \tag{2.1}$$

By the same calculations as that in [4], it is easy to show that

$$\begin{gathered} \|\rho\|_{L^\infty(0,T;L^\infty)} \le C, \\ \int \rho |u|^2 + |\nabla d|^2 dx + \int_0^T \int |\nabla u|^2 + \left|\Delta d + |\nabla d|^2 d\right|^2 dx dt \le C. \end{gathered} \tag{2.2}$$

Using (1.8), we see that

$$\begin{aligned} \int_0^T \int |\nabla d|^4 dx dt &\le C \int_0^T \|\nabla d\|_{BMO}^2 \|\nabla d\|_{L^2}^2 \, dt \\ &\le \max_t \|\nabla d\|_{L^2}^2 \int_0^T \|\nabla d\|_{BMO}^2 \, dt \le C, \end{aligned}$$

from which and (2.2), we get

$$\int_0^T \int |\Delta d|^2 dx dt \le 2 \int_0^T \int \left|\Delta d + |\nabla d|^2 d\right|^2 dx dt + 2 \int_0^T \int |\nabla d|^4 dx dt \le C. \tag{2.3}$$

Applying ∇ to (1.3), testing by $r|\nabla d|^{r-2}\nabla d$ ($r \geq 2$), using (1.8), we infer that

$$\begin{aligned}
&\frac{d}{dt}\int |\nabla d|^r dx + r\int |\nabla d|^{r-2}\left|\nabla^2 d\right|^2 + (r-2)|\nabla d|^{r-2}|\nabla|\nabla d||^2 dx \\
&= r\int \nabla\left(|\nabla d|^2 d\right)|\nabla d|^{r-2}\nabla d dx - r\int \nabla(u\cdot\nabla d)|\nabla d|^{r-2}\nabla d dx \\
&= r\int |\nabla d|^{r+2}dx - r\int |\nabla d|^{r-2}\nabla_i u^j < \nabla_j d, \nabla_i, d > dx - \int u\cdot\nabla|\nabla d|^r dx \\
&= r\int |\nabla d|^{r+2}dx - r\int |\nabla d|^{r-2}\mathcal{D}(u) : \nabla d\otimes\nabla d dx + \int (\mathrm{div}\, u)|\nabla d|^r dx \\
&\leq C\|\nabla d\|_{BMO}^2\|\nabla d\|_{L^r}^r + C\left\|\mathcal{D}(u)\right\|_{L^\infty}\|\nabla d\|_{L^r}^r,
\end{aligned}$$

which yields

$$\sup_{0\leq t<T}\|\nabla d\|_{L^r} + \int_0^T\int |\nabla d|^{r-2}\left|\nabla^2 d\right|^2 dxdt \leq C. \tag{2.4}$$

Let

$$\dot{f} := f_t + u\cdot\nabla f$$

denotes the material derivative of f.

Testing (1.2) by $\dot{u}$, we derive

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \mu|\nabla u|^2 + (\lambda+\mu)\left|\mathrm{div}\, u\right|^2 dx + \int \rho\,|\dot{u}|^2 dx \\
&= \int < (u\cdot\nabla)u, -\mu\Delta u - (\lambda+\mu)\nabla\mathrm{div}\, u > dx \\
&\quad - \int u\cdot\nabla u\cdot\nabla p(\rho)dx - \int u_t\cdot\nabla p(\rho)dx \\
&\quad - \int u\cdot\nabla u\cdot < \Delta d, \nabla d > dx - \int u_t < \Delta d, \nabla d > dx \\
&=: I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned} \tag{2.5}$$

By the same calculations as that in [4], we have

$$\begin{aligned}
I_1 &= \mu\int \mathcal{D}(u) : \mathrm{curl}\, u\otimes\mathrm{curl}\, u - \frac{1}{2}\mathrm{div}\, u(\mathrm{curl}\, u)^2 dx \\
&\quad - (2\mu+\lambda)\int (\nabla u{:}^t\nabla u)\mathrm{div}\, u - \frac{1}{2}(\mathrm{div}\, u)^3 dx \\
&\leq C\left\|\mathcal{D}(u)\right\|_{L^\infty}\|\nabla u\|_{L^2}^2, \\
I_2 &= \int p(\rho)(\nabla u{:}^t\nabla u - (\mathrm{div}\, u)^2)dx - \int u\,\mathrm{div}\, u\cdot\nabla p(\rho)dx \\
&\leq C\|\nabla u\|_{L^2}^2 + C\int \left|\nabla\rho\,\|u\|\,\mathrm{div}\, u\right| dx \\
&\leq C\|\nabla u\|_{L^2}^2 + C\|u\|_{L^6}\|\nabla\rho\|_{L^2}\left\|\mathrm{div}\, u\right\|_{L^3} \\
&\leq C\|\nabla u\|_{L^2}^2 + C\|\nabla u\|_{L^2}\|\nabla\rho\|_{L^2}\left\|\mathcal{D}(u)\right\|_{L^\infty}^{\frac{1}{3}}\left\|\mathcal{D}(u)\right\|_{L^2}^{\frac{2}{3}} \\
&\leq C\|\nabla u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2\|\nabla\rho\|_{L^2}^2 + C\left\|\mathcal{D}(u)\right\|_{L^\infty}^{\frac{2}{3}}\left\|\mathcal{D}(u)\right\|_{L^2}^{\frac{4}{3}} \\
&\leq C\|\nabla u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2\|\nabla\rho\|_{L^2}^2 + C\left(1+\left\|\mathcal{D}(u)\right\|_{L^\infty}\right)\|\nabla u\|_{L^2}^2, \\
I_3 &= \frac{d}{dt}\int p(\rho)\mathrm{div}\, u dx - \int \mathrm{div}\, u\partial_t p(\rho)dx \\
&\leq \frac{d}{dt}\int p(\rho)\mathrm{div}\, u dx + C + C\left(1+\left\|\mathcal{D}(u)\right\|_{L^\infty}\right)\|\nabla u\|_{L^2}^2 \\
&\quad + C\|\nabla u\|_{L^2}^2\|\nabla\rho\|_{L^2}^2, \\
I_4 &\leq \|u\|_{L^6}\|\nabla u\|_{L^6}\|\Delta d\|_{L^2}\|\nabla d\|_{L^6} \\
&\leq C\|\nabla u\|_{L^2}\|\Delta u\|_{L^2}\|\Delta d\|_{L^2}(\text{by (2.4) for } r=6) \\
&\leq \varepsilon\|\Delta u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2\|\Delta d\|_{L^2}^2,
\end{aligned}$$

for any $\epsilon > 0$.

We denote $M(d) := \nabla d \otimes \nabla d - \frac{1}{2}|\nabla d|^2 I_3$, I_5 is estimated as follows, which is slightly different from that in [4]:

$$\begin{aligned}
I_5 &= \frac{d}{dt}\int M(d) : \nabla u dx - \int \partial_t M(d) : \nabla u dx \\
&\leq \frac{d}{dt}\int M(d) : \nabla u dx + C\int |\nabla d_t| |\nabla d| |\nabla u| \, dx \\
&\leq \frac{d}{dt}\int M(d) : \nabla u dx + C\|\nabla d\|_{L^6}\|\nabla u\|_{L^3}\|\nabla d_t\|_{L^2} \\
&\leq \frac{d}{dt}\int M(d) : \nabla u dx + C\|\nabla u\|_{L^3}\|\nabla d_t\|_{L^2} \quad (\text{by } (2.4) \text{ for } r = 6) \\
&\leq \frac{d}{dt}\int M(d) : \nabla u dx + \varepsilon\|\nabla d_t\|_{L^2}^2 + \varepsilon\|\Delta u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2 .
\end{aligned}$$

Substituting the above estimates into (2.5), we deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \mu+|\nabla u|^2(\lambda+\mu)\left|\operatorname{div} u\right|^2 dx + \int \rho|\dot{u}|^2 dx \\
&\leq C\|\nabla u\|_{L^2}^2\|\nabla \rho\|_{L^2}^2 + C\left(1+\left\|\mathcal{D}(u)\right\|_{L^\infty}\right)\|\nabla u\|_{L^2}^2 \\
&\quad + C + \frac{d}{dt}\int p(\rho)\operatorname{div} u + M(d) : \nabla u dx \\
&\quad + C\|\nabla u\|_{L^2}^2\|\Delta d\|_{L^2}^2 + 2\varepsilon\left(\|\nabla d_t\|_{L^2}^2 + \|\Delta u\|_{L^2}^2\right)
\end{aligned} \tag{2.6}$$

for any $0 < \epsilon < 1$.

By the same calculations as that in [4], we write

$$\frac{d}{dt}\|\nabla\rho\|_{L^2}^2 \leq C\left(1+\left\|\mathcal{D}(u)\right\|_{L^\infty}\right)\|\nabla\rho\|_{L^2}^2 + \varepsilon\|\Delta u\|_{L^2}^2 . \tag{2.7}$$

Testing (1.3) by Δd_t, using (2.4), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |\Delta d|^2 dx + \int |\nabla d_t|^2 dx = \int \left(u\cdot\nabla d - |\nabla d|^2 d\right)\Delta d_t dx \\
&= -\int \nabla\left(u\cdot\nabla d - |\nabla d|^2 d\right)\cdot\nabla d_t dx \\
&\leq \left(\|\nabla u\|_{L^3}\|\nabla d\|_{L^6} + \|u\|_{L^6}\|\Delta d\|_{L^3} + \|\nabla d\|_{L^6}^3 + \|\nabla d\|_{L^6}\|\Delta d\|_{L^3}\right)\|\nabla d_t\|_{L^2} \\
&\leq C\left(\|\nabla u\|_{L^3} + \|\nabla u\|_{L^2}\|\Delta d\|_{L^3} + 1 + \|\Delta d\|_{L^3}\right)\|\nabla d_t\|_{L^2} \\
&\leq C\left(\|\nabla u\|_{L^2}^{1/2}\|\Delta u\|_{L^2}^{1/2} + \|\nabla u\|_{L^2}\|\nabla d\|_{L^6}^{1/2}\|\nabla\Delta d\|_{L^2}^{1/2} + 1 + \|\nabla d\|_{L^6}^{1/2}\|\nabla\Delta d\|_{L^2}^{1/2}\|\nabla d_t\|_{L^2}\right) \\
&\leq C\left(\|\nabla u\|_{L^2}^{1/2}\|\Delta u\|_{L^2}^{1/2} + \|\nabla u\|_{L^2}\|\nabla\Delta d\|_{L^2}^{1/2} + 1 + \|\nabla\Delta d\|_{L^2}^{1/2}\|\nabla d_t\|_{L^2}\right) \\
&\leq \varepsilon\|\nabla d_t\|_{L^2}^2 + \varepsilon\|\Delta u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^4 + C + \varepsilon\|\nabla\Delta d\|_{L^2}^2 .
\end{aligned} \tag{2.8}$$

Here we have used the Gagliardo-Nirenberg inequality

$$\|\Delta d\|_{L^3}^2 \leq C\|\nabla d\|_{L^6}\|\nabla\Delta d\|_{L^3}. \tag{2.9}$$

Using (1.3), (2.4), and (2.9), we have

$$\begin{aligned}
\|\nabla\Delta d\|_{L^2} &\leq \left\|\nabla d_t + \nabla(u\cdot\nabla d) - \nabla\left(|\nabla d|^2 d\right)\right\|_{L^2} \\
&\leq C\|\nabla d_t\|_{L^2} + C\|u\|_{L^6}\|\Delta d\|_{L^3} + C\|\nabla u\|_{L^3}\|\nabla d\|_{L^6} \\
&\quad + C\|\nabla d\|_{L^6}^3 + C\|\nabla d\|_{L^6}\|\Delta d\|_{L^3} \\
&\leq C\|\nabla d_t\|_{L^2} + C\|\nabla u\|_{L^2}\|\nabla\Delta d\|_{L^3}^{1/2} + C\|\nabla u\|_{L^2}^{1/2}\|\Delta u\|_{L^2}^{1/2} \\
&\quad + C + C\|\nabla\Delta d\|_{L^2}^{1/2} ,
\end{aligned}$$

whence

$$\|\nabla\Delta d\|_{L^2} \le C\|\nabla d_t\|_{L^2} + C\|\nabla u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^{1/2}\|\Delta u\|_{L^2}^{1/2} + C. \tag{2.10}$$

On the other hand, it follows from (1.2), (2.4), and (2.10) that

$$\begin{aligned}\|\Delta u\|_{L^2} &\le C\|\rho\dot{u}\|_{L^2} + C\left\|\nabla p(\rho)\right\|_{L^2} + C\|\Delta d\cdot\nabla d\|_{L^2}\\ &\le C\|\rho\dot{u}\|_{L^2} + C\|\nabla\rho\|_{L^2} + C\|\nabla d\|_{L^6}\|\Delta d\|_{L^3}\\ &\le C\|\rho\dot{u}\|_{L^2} + C\|\nabla\rho\|_{L^2} + C\|\Delta d\|_{L^3}\\ &\le C\|\rho\dot{u}\|_{L^2} + C\|\nabla\rho\|_{L^2} + C\|\Delta d\|_{L^2} + C\|\nabla\Delta d\|_{L^2}\\ &\le C\|\rho\dot{u}\|_{L^2} + C\|\nabla\rho\|_{L^2} + C\|\Delta d\|_{L^2} + C\|\nabla d_t\|_{L^2}\\ &\quad + C\|\nabla u\|_{L^2}^2 + C\|\nabla u\|_{L^2} + \frac{1}{2}\|\Delta u\|_{L^2} + C,\end{aligned}$$

which implies

$$\|\Delta u\|_{L^2} \le C\|\rho\dot{u}\|_{L^2} + C\|\nabla\rho\|_{L^2} + C\|\Delta d\|_{L^2} + C\|\nabla d_t\|_{L^2} + C\|\nabla u\|_{L^2}^2 + C. \tag{2.11}$$

Combining (2.6), (2.7), (2.8), (2.10), and (2.11), taking ϵ small enough, using the Gron-wall inequality, we arrive at

$$\nabla d \in L^2(0, T; H^2),$$

whence (2.1) holds true.

This completes the proof.

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Author details
[1]College of Mathematics and Statistics, Chongqing Three Gorges University, Wanzhou, Chongqing 404000, P.R. China
[2]Department of Applied Mathematics,, Nanjing Forestry University, Nanjing 210037, P.R. China

Authors' contributions
XC wrote the manuscript and did partial computation. JF proposed the problem and did the main estimates. All authors read and approved the final manuscript.

Authors' information
Both X. Chen and J. Fan are professors. J. Fan has published more than 90 scientific papers on nonlinear partial differential equations.

Competing interests
The authors declare that they have no competing interests.

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