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Research Article

Existence of Nontrivial Solutions for Generalized Quasilinear Schrödinger Equations with Critical Growth

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We study the following generalized quasilinear Schrödinger equations with critical growth $-\mathrm{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = \lambda f(x,u) + g(u)|G(u)|^{2^*-2}G(u), x \in \mathbb{R}^N$, where $\lambda > 0$, $N \geq 3$, $g(s) : \mathbb{R} \to \mathbb{R}^+$ is a C^1 even function, $g(0) = 1$, and $g'(s) \geq 0$ for all $s \geq 0$, where $G(u) := \int_0^u g(t)dt$. Under some suitable conditions, we prove that the equation has a nontrivial solution by variational method.

1. Introduction and Preliminaries

Consider the following generalized quasilinear Schrödinger equations with critical growth:

$$\begin{aligned} &-\mathrm{div}\left(g^2(u)\nabla u\right) + g(u)g'(u)|\nabla u|^2 + V(x)u \\ &\quad = \lambda f(x,u) + g(u)|G(u)|^{2^*-2}G(u), \quad x \in \mathbb{R}^N, \end{aligned} \tag{1}$$

where $\lambda > 0$, $N \geq 3$, $g(s) : \mathbb{R} \to \mathbb{R}^+$ is a C^1 even function, $g(0) = 1$, and $g'(s) \geq 0$ for all $s \geq 0$.

The equations are related to the existence of solitary wave solutions for quasilinear Schrödinger equations

$$\begin{aligned} iz_t &= -\Delta z + W(x)z - k(x,|z|)z \\ &\quad - \Delta l\left(|z|^2\right)l'\left(|z|^2\right)z, \end{aligned} \tag{2}$$

where $z : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$, $W : \mathbb{R}^N \to \mathbb{R}$ is a given potential, $l : \mathbb{R} \to \mathbb{R}$, and $k : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ are suitable functions. The form of (2) has been derived as models of several physical phenomena corresponding to various types of $l(s)$. For instance, the case $l(s) = s$ models the time evolution of the condensate wave function in superfluid film [1, 2] and is called the superfluid film equation in fluid mechanics by Kurihara [1]. In the case $l(s) = (1+s)^{1/2}$, problem (2) models the self-channeling of a high-power ultra short laser in matter, the propagation of a high-irradiance laser in a plasma creates an optical index depending nonlinearly on the light intensity, and this leads to interesting new nonlinear wave equations; see [3–6]. For more physical motivations and more references dealing with applications, we can refer to [7–14] and references therein.

Set $z(t,x) = \exp(-iEt)u(x)$, where $E \in \mathbb{R}$ and u is a real function. Then (2) can be reduced to the corresponding equation of elliptic type (see [15]):

$$-\Delta u + V(x)u - \Delta l\left(u^2\right)l'\left(u^2\right)u = f(x,u), \quad x \in \mathbb{R}^N, \tag{3}$$

where $f(x,u) = k(x,|u|)u$. If we take

$$g^2(u) = 1 + \frac{\left[\left(l\left(u^2\right)\right)'\right]^2}{2}, \tag{4}$$

then (1) turns into (3) (see [16]).

Moreover, problem (3) also arises in biological models and propagation of laser beams when $g(u)$ is a positive constant (see [17, 18]). In (3), if we set $l(u) = u$, that is,

$g^2(u) = 1 + 2u^2$, then we get the superfluid film equation in plasma physics:

$$-\Delta u + V(x)u - \Delta(u^2)u = f(x,u), \quad x \in \mathbb{R}^N; \tag{5}$$

if we set $l(u) = (1+u)^{1/2}$, that is, $g^2(u) = 1 + u^2/2(1+u^2)$, then we get the equation

$$\begin{aligned} &-\Delta u + V(x)u - \left[\Delta\left(1+u^2\right)^{1/2}\right]\frac{u}{2\left(1+u^2\right)^{1/2}} \\ &\quad = f(x,u), \quad x \in \mathbb{R}^N, \end{aligned} \tag{6}$$

which models the self-channeling of a high-power ultrashort laser in matter.

In the past, the research on the existence of solitary wave solutions of Schrödinger equations (2) is for some given special function $l(s)$. In this paper, we will use a unified new variable replacement to study (2), constructed by Shen and Wang in [16]. Define the energy functional associated with (1) by

$$\begin{aligned} I_\lambda(u) = &\frac{1}{2}\int_{\mathbb{R}^N} g^2(u)|\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2 dx \\ &- \lambda\int_{\mathbb{R}^N} F(x,u)\,dx - \frac{1}{2^*}\int_{\mathbb{R}^N} |G(u)|^{2^*} dx, \end{aligned} \tag{7}$$

where $F(x,u) := \int_0^u f(x,t)dt$. However, I_λ is not well defined in $H^1(\mathbb{R}^N)$ because of the term $\int_{\mathbb{R}^N} g^2(u)|\nabla u|^2 dx$. To overcome this difficulty, we make a change of variable constructed by Shen and Wang in [16]: $v := G(u) := \int_0^u g(t)dt$. Then we obtain

$$\begin{aligned} J_\lambda(v) = &\frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)G^{-1}(v)^2 dx \\ &- \lambda\int_{\mathbb{R}^N} F\left(x, G^{-1}(v)\right)dx - \frac{1}{2^*}\int_{\mathbb{R}^N} |v|^{2^*} dx. \end{aligned} \tag{8}$$

If u is a nontrivial solution of (1), then

$$\begin{aligned} \left\langle I_\lambda'(u), \varphi\right\rangle = &\int_{\mathbb{R}^N}\Big[g^2(u)\nabla u\nabla\varphi + g(u)g'(u)|\nabla u|^2\varphi \\ &+ V(x)u\varphi - \lambda f(x,u)\varphi \\ &- g(u)|G(u)|^{2^*-2}G(u)\varphi\Big]dx = 0 \end{aligned} \tag{9}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^N)$. Let $\varphi = (1/g(u))\psi$. By [16] we know that (9) is equivalent to

$$\begin{aligned} \left\langle J_\lambda'(v), \psi\right\rangle = &\int_{\mathbb{R}^N}\left[\nabla v\nabla\psi + V(x)\frac{G^{-1}(v)}{g\left(G^{-1}(v)\right)}\psi\right. \\ &\left.- \lambda\frac{f\left(x, G^{-1}(v)\right)}{g\left(G^{-1}(v)\right)}\psi - |v|^{2^*-2}v\psi\right]dx = 0 \end{aligned} \tag{10}$$

for all $\psi \in C_0^\infty(\mathbb{R}^N)$. Therefore, in order to find the nontrivial solution of (1), it suffices to study the existence of the nontrivial solutions of the following equations:

$$\begin{aligned} &-\Delta v + V(x)\frac{G^{-1}(v)}{g\left(G^{-1}(v)\right)} - \lambda\frac{f\left(x, G^{-1}(v)\right)}{g\left(G^{-1}(v)\right)} - |v|^{2^*-2}v \\ &\quad = 0. \end{aligned} \tag{11}$$

Recently, the authors studied generalized quasilinear Schrödinger equations with subcritical growth [19, 20], critical growth [21], and supercritical growth [22].

In order to reduce the statements for main results, we list the assumptions as follows:

(V_1) $V(x) \geq V_0 := \inf_{x\in\mathbb{R}^N} V(x) > 0$ for all $x \in \mathbb{R}^N$.

(V_2) $\lim_{|x|\to\infty} V(x) = V_\infty < +\infty$ and $V(x) \leq V_\infty$ for all $x \in \mathbb{R}^N$.

(f_1) $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and there exists $2 < p < 2^*$ such that

$$\left|f(x,t)\right| \leq C\left(1 + g(t)|G(t)|^{p-1}\right) \tag{12}$$

for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$.

(f_2) $f(x,t) = o(|t|)$ uniformly in $x \in \mathbb{R}^N$ as $|t| \to 0$.

(f_3) $(f(x,G^{-1}(t))/g(G^{-1}(t)))t - 2F(x,G^{-1}(t)) \geq (f(x,G^{-1}(st))/g(G^{-1}(st)))st - 2F(x,G^{-1}(st))$ for all $t \in \mathbb{R}$ and $s \in [0,1]$.

(f_4) $f(x,t)t > 0$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}\backslash\{0\}$.

(f_5) $\lim_{|t|\to+\infty}(F(x,G^{-1}(t))/t^2) = +\infty$ uniformly in $x \in \mathbb{R}^N$.

Set $E = H^1(\mathbb{R}^N)$ with the norm

$$\|u\|_E = \left[\int_{\mathbb{R}^N}\left(|\nabla u|^2 + u^2\right)dx\right]^{1/2}. \tag{13}$$

It is easy to prove that J_λ is well defined on E and $J_\lambda \in C^1(E,\mathbb{R})$ under our assumptions and its Gateaux derivative is given by

$$\begin{aligned} \left\langle J_\lambda'(v), \varphi\right\rangle = &\int_{\mathbb{R}^N}\left[\nabla v\nabla\varphi + V(x)\frac{G^{-1}(v)}{g\left(G^{-1}(v)\right)}\varphi\right. \\ &\left.- \lambda\frac{f\left(x, G^{-1}(v)\right)}{g\left(G^{-1}(v)\right)}\varphi - |v|^{2^*-2}v\varphi\right]dx \end{aligned} \tag{14}$$

for all $v, \varphi \in E$.

Our main result of this paper is as follows.

Theorem 1. *Suppose that* (V_1), (V_2), *and* (f_1)–(f_5) *are satisfied. Then if* $N \geq 5$, *(1) admits a nontrivial solution for all* $\lambda > 0$; *if* $N = 3, 4$, *(1) admits a nontrivial solution for large* λ.

Remark 2. Condition (f_3) is weaker than the following condition (f_6).

(f_6) $f(x,G^{-1}(t))/g(G^{-1}(t))t$ is nonincreasing on $(-\infty,0)$ and nondecreasing on $(0,+\infty)$.

Indeed, set $l(s) = s^2t(f(x,G^{-1}(t))/g(G^{-1}(t))) - 2F(x,G^{-1}(st))$, $\forall s\in[0,1]$. Then

$$\begin{aligned} l'(s) &= 2st\frac{f(x,G^{-1}(t))}{g(G^{-1}(t))} - 2\frac{f(x,G^{-1}(st))}{g(G^{-1}(st))}t \\ &= 2st\frac{f(x,G^{-1}(t))}{g(G^{-1}(t))} - 2t\frac{f(x,G^{-1}(st))}{g(G^{-1}(st))\,st}(st). \end{aligned} \tag{15}$$

If (f_6) holds, then

$$l'(s) \geq 2st\frac{f(x,G^{-1}(t))}{g(G^{-1}(t))} - 2t\frac{f(x,G^{-1}(t))}{g(G^{-1}(t))\,t}(st) = 0 \tag{16}$$

whenever $t>0$ or $t<0$. Hence $l(s)$ is nondecreasing on $[0,1]$, and hence $l(1)\geq l(s)$ for all $s\in[0,1]$. Consequently, (f_6) implies that

$$\begin{aligned} &t\frac{f(x,G^{-1}(t))}{g(G^{-1}(t))} - 2F(x,G^{-1}(t)) \\ &\quad\geq s^2t\frac{f(x,G^{-1}(t))}{g(G^{-1}(t))} - 2F(x,G^{-1}(st)) \\ &\quad= s^2t|t|\frac{f(x,G^{-1}(t))}{g(G^{-1}(t))|t|} - 2F(x,G^{-1}(st)) \\ &\quad\geq s^2t|t|\frac{f(x,G^{-1}(st))}{g(G^{-1}(st))|st|} - 2F(x,G^{-1}(st)) \\ &\quad= st\frac{f(x,G^{-1}(st))}{g(G^{-1}(st))} - 2F(x,G^{-1}(st)) \end{aligned} \tag{17}$$

for all $s\in[0,1]$; that is, the condition (f_3) holds.

From Remark 2 we obtain Corollary 3.

Corollary 3. *Suppose that (V_1), (V_2), (f_1)-(f_2), (f_4)-(f_5), and (f_6) are satisfied. Then if $N\geq 5$, (1) admits a nontrivial solution for all $\lambda>0$; if $N=3,4$, (1) admits a nontrivial solution for large λ.*

Remark 4. In [16], Shen and Wang studied the existence of nontrivial solutions for generalized quasilinear Schrödinger equations

$$\begin{aligned} &-\operatorname{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u \\ &\quad= h(u), \quad x\in\mathbb{R}^N, \end{aligned} \tag{18}$$

where h is a subcritical nonlinearity satisfying the following conditions:

(h_0) $h(t)=0$ if $t\leq 0$.

(h_1) $h(t)=o(t)$ as $t\to 0^+$.

(h_2) There exists $2<p<2^*$ such that

$$|h(t)| \leq C\left(1+g(t)|G(t)|^{p-1}\right) \tag{19}$$

for all $t>0$.

(h_3) There exists $\mu>2$ such that, for any $t>0$, there holds

$$0<\mu g(G^{-1}(t))H(G^{-1}(t)) \leq h(G^{-1}(t))t. \tag{20}$$

As mentioned above, if we set $g^2(u)=1+2u^2$, then we get the superfluid film equation in plasma physics

$$-\Delta u + V(x)u - \Delta(u^2)u = h(u), \quad x\in\mathbb{R}^N, \tag{21}$$

whose nontrivial solutions were studied in [23]. But our problem (1) is elliptic problem involving the critical exponent, so our result extends the results of the work [16, 23] to a critical setting. Moreover, the assumptions about the nonlinearity in this paper are different from the assumptions about the nonlinearity in [16, 23].

Remark 5. In [24], Deng et al. studied problem (1) and their result based on more harsh conditions:

$(f_1)^*$ $f(x,t)\geq 0$ is differentiable with respect to $t\in[0,+\infty)$ for all $x\in\mathbb{R}^N$ and continuous with respect to $x\in\mathbb{R}^N$ for all $t\in[0,+\infty)$. Moreover, $f(x,t)\equiv 0$ for all $(x,t)\in\mathbb{R}^N\times\mathbb{R}^-$.

$(f_3)^*$ There exists $\delta\in(0,2^*-2)$ such that, for any $t>0$, there holds $(1+\delta)f(x,t)\leq G(t)[f(x,t)/g(t)]'$, which implies that there exists $\mu\in(2,2^*)$ such that $f(x,t)G(t)\geq\mu g(t)F(x,t)$ for all $(x,t)\in\mathbb{R}^N\times\mathbb{R}$.

In this paper, we just assume that f is a continuous function. Moreover, there are functionals $f(x,t)$ satisfying (f_3) but not satisfying the above Ambrosetti-Rabinowitz type condition (see Remark 1.2 in [25]). Hence, our result is different from the result there.

2. Proof of Theorem 1

To begin with, we give some lemmas.

Lemma 6. *For the functions g, G, and G^{-1}, the following properties hold:*

(1) *the functions $G(\cdot)$ and $G^{-1}(\cdot)$ are strictly increasing and odd;*

(2) *$G(s)\leq g(s)s$ for all $s\geq 0$; $G(s)\geq g(s)s$ for all $s\leq 0$;*

(3) *$g(G^{-1}(s))\geq g(0)=1$ for all $s\in\mathbb{R}$;*

(4) *$G^{-1}(s)/s$ is decreasing on $(0,+\infty)$ and increasing on $(-\infty,0)$;*

(5) *$|G^{-1}(s)|\leq(1/g(0))|s|=|s|$ for all $s\in\mathbb{R}$;*

(6) *$|G^{-1}(s)|/g(G^{-1}(s))\leq(1/g^2(0))|s|=|s|$ for all $s\in\mathbb{R}$;*

(7) *$G^{-1}(s)s/g(G^{-1}(s))\leq|G^{-1}(s)|^2$ for all $s\in\mathbb{R}$;*

(8) $\lim_{|s|\to 0}(G^{-1}(s)/s) = 1/g(0) = 1$ *and*

$$\lim_{|s|\to\infty}\frac{G^{-1}(s)}{s} = \begin{cases}\dfrac{1}{g(\infty)}, & \text{if } g \text{ is bounded},\\ 0, & \text{if } g \text{ is unbounded}.\end{cases} \tag{22}$$

Proof. Properties (1)–(3) are obvious. By (2), we have

$$\left(\frac{G^{-1}(s)}{s}\right)' = \frac{s - G^{-1}(s)\,g\left(G^{-1}(s)\right)}{g\left(G^{-1}(s)\right)s^2} \leq 0 \tag{23}$$

for all $s > 0$ and

$$\left(\frac{G^{-1}(s)}{s}\right)' = \frac{s - G^{-1}(s)\,g\left(G^{-1}(s)\right)}{g\left(G^{-1}(s)\right)s^2} \geq 0 \tag{24}$$

for all $s < 0$. Consequently, we obtain (4). By mean value theorem and (3), one has

$$\begin{aligned}\left|G^{-1}(s)\right| &= \left|G^{-1}(s) - G^{-1}(0)\right| = \frac{1}{g\left(G^{-1}(\theta s)\right)}|s| \\ &\leq \frac{1}{g(0)}|s|\end{aligned} \tag{25}$$

for all $s \in \mathbb{R}$, where $\theta \in (0,1)$; that is, (5) is proved. Obviously, (6) is a consequence of (3) and (5). Moreover, (7) is a consequence of (2). Finally, using L' Hospital's rule, we know that (8) is satisfied. This completes the proof. □

Denote

$$\begin{aligned}h_\lambda(x,s) &= V(x)\,s - V(x)\frac{G^{-1}(s)}{g\left(G^{-1}(s)\right)} \\ &\quad + \lambda\frac{f\left(x, G^{-1}(s)\right)}{g\left(G^{-1}(s)\right)}.\end{aligned} \tag{26}$$

Then

$$\begin{aligned}H_\lambda(x,s) &:= \int_0^s h_\lambda(x,t)\,dt \\ &= \frac{1}{2}V(x)\left[s^2 - G^{-1}(s)^2\right] + \lambda F\left(x, G^{-1}(s)\right).\end{aligned} \tag{27}$$

Consequently,

$$\begin{aligned}J_\lambda(v) &= \frac{1}{2}\int_{\mathbb{R}^N}\left[|\nabla v|^2 + V(x)\,v^2\right]dx \\ &\quad - \int_{\mathbb{R}^N} H_\lambda(x,v)\,dx - \frac{1}{2^*}\int_{\mathbb{R}^N}|v|^{2^*}dx.\end{aligned} \tag{28}$$

Lemma 7. *The functions $h_\lambda(x,s)$ and $H_\lambda(x,s)$ enjoy the following properties under (f_1)-(f_5):*

(1) $\lim_{|s|\to 0}(h_\lambda(x,s)/s) = 0$ *and* $\lim_{|s|\to 0}(H_\lambda(x,s)/s^2) = 0$ *uniformly in* $x \in \mathbb{R}^N$;

(2) $\lim_{|s|\to\infty}(h_\lambda(x,s)/|s|^{2^*-1}) = 0$ *and* $\lim_{|s|\to\infty}(H_\lambda(x,s)/|s|^{2^*}) = 0$ *uniformly in* $x \in \mathbb{R}^N$;

(3) $th_\lambda(x,t) - 2H_\lambda(x,t) \geq sth_\lambda(x,st) - 2H_\lambda(x,st)$ *for all* $t \in \mathbb{R}$ *and* $s \in [0,1]$;

(4) $H_\lambda(x,s) \geq 0$ *for all* $(x,s) \in \mathbb{R}^N \times \mathbb{R}$;

(5) $\lim_{|s|\to+\infty}(H_\lambda(x,s)/s^2) = +\infty$ *uniformly in* $x \in \mathbb{R}^N$.

Proof. By (f_1)-(f_2), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\left|\frac{f\left(x, G^{-1}(s)\right)}{g\left(G^{-1}(s)\right)}\right| \leq \varepsilon|s| + C_\varepsilon|s|^{p-1} \tag{29}$$

for all $(x,s) \in \mathbb{R}^N \times \mathbb{R}$. Set $G^{-1}(s) = t$. Then Lemma 6(8) implies that

$$\begin{aligned}\lim_{|s|\to 0}\frac{h_\lambda(x,s)}{s} &= V(x)\left[1 - \frac{1}{g^2(0)}\right] + \lambda\lim_{|t|\to 0}\frac{f(x,t)}{g(t)\,G(t)} \\ &= 0\end{aligned} \tag{30}$$

uniformly in $x \in \mathbb{R}^N$. Moreover, by Lemma 6(6) one has

$$\begin{aligned}\lim_{|s|\to\infty}\frac{h_\lambda(x,s)}{|s|^{2^*-1}} &= -V(x)\lim_{|s|\to\infty}\frac{G^{-1}(s)}{s g\left(G^{-1}(s)\right)}\frac{s}{|s|^{2^*-1}} \\ &\quad + \lambda\lim_{|t|\to\infty}\frac{f(x,t)}{g(t)\,|G(t)|^{2^*-1}} = 0\end{aligned} \tag{31}$$

uniformly in $x \in \mathbb{R}^N$. Similarly, we have

$$\lim_{|s|\to 0}\frac{H_\lambda(x,s)}{s^2} = 0 \tag{32}$$

uniformly in $x \in \mathbb{R}^N$ and

$$\lim_{|s|\to\infty}\frac{H_\lambda(x,s)}{|s|^{2^*}} = 0 \tag{33}$$

uniformly in $x \in \mathbb{R}^N$. Hence, (1) and (2) hold.

In the following, we set $l(t) = G^{-1}(t)^2 - G^{-1}(t)t/g(G^{-1}(t))$, $\forall t \in \mathbb{R}$. If $t \geq 0$, by Lemma 6(2) and $g'(t) \geq 0$ for $t \geq 0$, we have

$$G(t)\left[\frac{1}{g^2(t)}\left(g(t) - g'(t)\,t\right)\right] \leq t \tag{34}$$

for $t \geq 0$, which implies that

$$G(t)\left(\frac{t}{g(t)}\right)'\frac{1}{g(t)} \leq \frac{t}{g(t)} \tag{35}$$

for all $t \geq 0$. Let $r = G(t)$. Then

$$G(t)\frac{d}{dr}\left(\frac{t}{g(t)}\right) \leq \frac{t}{g(t)} \tag{36}$$

and hence

$$r\left[\frac{G^{-1}(r)}{g\left(G^{-1}(r)\right)}\right]' \leq \frac{G^{-1}(r)}{g\left(G^{-1}(r)\right)} \tag{37}$$

for $r \geq 0$. Consequently,

$$l'(t) = \frac{2G^{-1}(t)}{g(G^{-1}(t))} - \left[\frac{G^{-1}(t)}{g(G^{-1}(t))}\right]' t - \frac{G^{-1}(t)}{g(G^{-1}(t))} = \frac{G^{-1}(t)}{g(G^{-1}(t))} - \left[\frac{G^{-1}(t)}{g(G^{-1}(t))}\right]' t \geq 0 \quad (38)$$

for all $t \geq 0$, that is, $l(t)$ is increasing with respect to $t \geq 0$. Hence $l(st) \leq l(t)$ for all $s \in [0,1]$ and $t \geq 0$; that is,

$$G^{-1}(st)^2 - \frac{G^{-1}(st)\,st}{g(G^{-1}(st))} \leq G^{-1}(t)^2 - \frac{G^{-1}(t)\,t}{g(G^{-1}(t))} \quad (39)$$

for all $s \in [0,1]$ and $t \geq 0$. Note that Lemma 6(1) implies that $l(t)$ is an even function. Therefore, if $t < 0$, we easily obtain that $l(st) \leq l(t)$ for all $s \in [0,1]$ and $t < 0$. Consequently,

$$G^{-1}(st)^2 - \frac{G^{-1}(st)\,st}{g(G^{-1}(st))} \leq G^{-1}(t)^2 - \frac{G^{-1}(t)\,t}{g(G^{-1}(t))} \quad (40)$$

for all $s \in [0,1]$ and $t \in \mathbb{R}$. Combining with (f_3), we can conclude (3). Moreover, (f_4) and Lemma 6(5) imply that $H(x,s) \geq 0$ for all $(x,s) \in \mathbb{R}^N \times \mathbb{R}$. Clearly, (f_5) and Lemma 6(5) imply that (5) is satisfied. This completes the proof. □

Lemma 8. *Suppose that (V_1), (V_2), and (f_1)-(f_2) are satisfied. Then the energy functional J_λ satisfies the following conditions:*

(i) *There exist $\beta, \rho > 0$ such that $J_\lambda(v) \geq \beta$ for $\|v\|_E = \rho$.*

(ii) *There exists $e \in E$ with $\|e\|_E > \rho$ such that $J_\lambda(e) < 0$.*

Proof. (i) Set $S_\rho := \{u \in E : \|u\|_E = \rho\}$. By (f_1)-(f_2), Lemmas 6(6) and 7(1), and (2), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|H_\lambda(x,s)| \leq \varepsilon\left(|s|^2 + |s|^{2^*}\right) + C_\varepsilon |s|^p \quad (41)$$

for all $(x,s) \in \mathbb{R}^N \times \mathbb{R}$. Consequently, for $v \in S_\rho$, we have

$$\begin{aligned} J_\lambda(v) &\geq \frac{1}{2}C_1\|v\|_E^2 - C_2\varepsilon\|v\|_E^2 - C_3\varepsilon\|v\|_E^{2^*} - C_4C_\varepsilon\|v\|_E^p \\ &\geq \frac{1}{2}C_1\rho^2 - C_2\varepsilon\rho^2 - C_3\varepsilon\rho^{2^*} - C_4C_\varepsilon\rho^p := \beta > 0 \end{aligned} \quad (42)$$

for small $\varepsilon > 0$ and $\rho > 0$.

(ii) Take $v^* \in E\backslash\{0\}$. Then

$$\begin{aligned} J_\lambda(tv^*) \leq{}& \frac{1}{2}C_5t^2\|v^*\|_E^2 - \frac{1}{2^*}t^{2^*}\int_{\mathbb{R}^N}|v^*|^{2^*}dx \\ &+ \varepsilon t^2\int_{\mathbb{R}^N}|v^*|^2dx + \varepsilon t^{2^*}\int_{\mathbb{R}^N}|v^*|^{2^*}dx \\ &+ C_\varepsilon t^p\int_{\mathbb{R}^N}|v^*|^p dx < 0 \end{aligned} \quad (43)$$

for large $t > 0$ and small $\varepsilon > 0$. Consequently, we can take $e := t^*v^*$ for some large $t^* > 0$ such that (ii) holds. This completes the proof. □

Lemma 9. *Suppose that (V_1), (V_2), and (f_1)–(f_4) are satisfied. Then there exists a bounded Cerami sequence $\{v_n\} \subset E$ for J_λ with $J_\lambda(v_n) \to c_\lambda \geq \beta > 0$, where*

$$c_\lambda := \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]} J_\lambda(\gamma(t)), \quad \Gamma := \{\gamma \in C([0,1],E) : \gamma(0) = 0,\ J_\lambda(\gamma(1)) < 0\}, \quad (44)$$

β is the constant appearing in Lemma 8.

Proof. By Lemma 8 and the mountain pass theorem without (PS) condition (see Theorem 4.1 in [26]), there exists a Cerami sequence $\{v_n\} \subset E$ satisfying

$$J_\lambda(v_n) \longrightarrow c_\lambda \geq \beta > 0, \quad (1 + \|v_n\|_E)\|J'_\lambda(v_n)\|_{E^*} \longrightarrow 0, \quad (45)$$

where

$$c_\lambda := \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]} J_\lambda(\gamma(t)), \quad \Gamma := \{\gamma \in C([0,1],E) : \gamma(0) = 0,\ J_\lambda(\gamma(1)) < 0\}, \quad (46)$$

β is the constant appearing in Lemma 8.

Let $t_n \in [0,1]$ be such that $J_\lambda(t_nv_n) = \max_{t\in[0,1]}J_\lambda(tv_n)$. Then $\{J_\lambda(t_nv_n)\}$ is bounded from above. Indeed, without loss of the generality, we may assume that $t_n \in (0,1)$ for all $n \in \mathbb{N}$. Hence, by Lemma 7(3) we have

$$\begin{aligned} J_\lambda(t_nv_n) &= J_\lambda(t_nv_n) - \frac{1}{2}\left\langle J'_\lambda(t_nv_n), t_nv_n\right\rangle \\ &= \left(\frac{1}{2} - \frac{1}{2^*}\right)t_n^{2^*}\int_{\mathbb{R}^N}|v_n|^{2^*}dx \\ &\quad + \int_{\mathbb{R}^N}\left[\frac{1}{2}t_nv_nh_\lambda(x,t_nv_n) - H_\lambda(x,t_nv_n)\right]dx \\ &\leq \left(\frac{1}{2} - \frac{1}{2^*}\right)\int_{\mathbb{R}^N}|v_n|^{2^*}dx \\ &\quad + \int_{\mathbb{R}^N}\left[\frac{1}{2}v_nh_\lambda(x,v_n) - H_\lambda(x,v_n)\right]dx \\ &= J_\lambda(v_n) - \frac{1}{2}\left\langle J'_\lambda(v_n), v_n\right\rangle = c_\lambda + o(1). \end{aligned} \quad (47)$$

This shows that $\{J_\lambda(t_nv_n)\}$ is bounded from above.

Now, we prove that $\{v_n\}$ is bounded in E. Otherwise, if $\|v_n\|_E$ is unbounded, then, up to a subsequence, we may assume that $\|v_n\|_E \to +\infty$. Set $w_n = v_n/\|v_n\|_E$. Then there exists $w \in E$ such that $w_n \rightharpoonup w$ in E. By $J_\lambda(v_n) \to c_\lambda$, we have

$$o(1) + \frac{1}{2}\max\{1, V_\infty\} \geq \frac{1}{2^*}\frac{\|v_n\|_{2^*}^{2^*}}{\|v_n\|_E^2} + \int_{\mathbb{R}^N}\frac{H_\lambda(x,v_n)}{\|v_n\|_E^2}dx. \quad (48)$$

Set $\Omega = \{x \in \mathbb{R}^N : w(x) \neq 0\}$. If $\text{meas}(\Omega) > 0$, then by Lemma 7(4) and Fatou Lemma, one has

$$\begin{aligned} o(1) &+ \frac{1}{2}\max\{1, V_\infty\} \\ &\geq \frac{1}{2^*}\frac{\|v_n\|_{2^*}^{2^*}}{\|v_n\|_E^2} + \int_{\mathbb{R}^N}\frac{H_\lambda(x, v_n)}{\|v_n\|_E^2}dx \\ &\geq \frac{1}{2^*}\int_\Omega w_n^2|v_n|^{2^*-2}dx \longrightarrow +\infty \end{aligned} \tag{49}$$

as $n \to \infty$. This is a contradiction. Hence $|\Omega| = 0$, that is, $w = 0$ a.e. on $\mathbb{R}^N$. For any $B > 0$, by $\|v_n\|_E \to +\infty$ we have

$$\begin{aligned} J_\lambda(t_n v_n) &\geq J_\lambda\left(\frac{B}{\|v_n\|_E}v_n\right) = J_\lambda(Bw_n) \\ &\geq \frac{B^2}{2}\min\{1, V_0\} - \int_{\mathbb{R}^N}H_\lambda(x, Bw_n)dx \\ &\quad - \frac{B^{2^*}}{2^*}\int_{\mathbb{R}^N}|w_n|^{2^*}dx \end{aligned} \tag{50}$$

for n sufficiently large. By (29), Lemmas 6(6) and 7(1), and (2), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|h_\lambda(x, s)s| \leq \varepsilon\left(|s|^2 + |s|^{2^*}\right) + C_\varepsilon|s|^p \tag{51}$$

for all $(x, s) \in \mathbb{R}^N \times \mathbb{R}$. Consequently,

$$\begin{aligned} \int_{\mathbb{R}^N}|w_n|^{2^*}dx &\leq \frac{\max\{1, V_\infty\}}{\|v_n\|_E^{2^*-2}} \\ &\quad - \frac{1}{\|v_n\|_E^{2^*}}\int_{\mathbb{R}^N}h_\lambda(x, v_n)v_n dx + o(1) \\ &\longrightarrow 0 \end{aligned} \tag{52}$$

as $n \to \infty$ and so $\int_{\mathbb{R}^N}|w_n|^p dx \to 0$ as $n \to \infty$ by using interpolation inequality. Moreover, (41) implies that

$$\begin{aligned} \left|\int_{\mathbb{R}^N}H_\lambda(x, Bw_n)dx\right| &\leq \varepsilon B^2\int_{\mathbb{R}^N}w_n^2 dx \\ &\quad + \varepsilon B^{2^*}\int_{\mathbb{R}^N}|w_n|^{2^*}dx \\ &\quad + C_\varepsilon B^p\int_{\mathbb{R}^N}|w_n|^p dx. \end{aligned} \tag{53}$$

By the arbitrariness of ε, we obtain $\int_{\mathbb{R}^N}H_\lambda(x, Bw_n)dx \to 0$ as $n \to \infty$. Hence

$$\liminf_{n\to\infty}J_\lambda(t_n v_n) \geq \frac{B^2}{2}\min\{1, V_0\}, \quad \forall B > 0. \tag{54}$$

This contradicts the fact that $\{J_\lambda(t_n v_n)\}$ is bounded from above. Consequently, $\{v_n\}$ is bounded in E. This completes the proof of Lemma 9. □

Lemma 10. *Suppose that* (V_1), (V_2), *and* (f_1)–(f_5) *are satisfied. Then if* $N \geq 5$, *the minimax level* c_λ *satisfies* $c_\lambda < (1/N)S^{N/2}$ *for all* $\lambda > 0$; *if* $N = 3, 4$, *the minimax level* c_λ *satisfies* $c_\lambda < (1/N)S^{N/2}$ *for large* λ, *where* S *is the best constant of the embedding* $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$.

Proof. From the minimax characterization of c_λ we see that it is sufficient to show that there exists $v_0 \in E\backslash\{0\}$ such that $\sup_{t\geq 0}J_\lambda(tv_0) < (1/N)S^{N/2}$.

We follow the strategy used in [24] but need to modify some process. Given $\varepsilon > 0$, we consider the function

$$w_\varepsilon(x) = \frac{[N(N-2)\varepsilon]^{(N-2)/4}}{\left(\varepsilon + |x|^2\right)^{(N-2)/2}}, \tag{55}$$

which satisfies the following equations:

$$\begin{aligned} -\Delta u &= u^{2^*-1}, \quad \text{in } \mathbb{R}^N, \\ u &\in D^{1,2}\left(\mathbb{R}^N\right), \\ u(x) &> 0, \quad \text{in } \mathbb{R}^N. \end{aligned} \tag{56}$$

Moreover, $w_\varepsilon(x)$ satisfies

$$|\nabla w_\varepsilon|_2^2 = |w_\varepsilon|_{2^*}^{2^*} = S^{N/2}. \tag{57}$$

Let $\varphi \in C_0^\infty(\mathbb{R}^N, [0, 1])$ be such that $\varphi(x) \equiv 1$ for $|x| \leq \rho_\varepsilon$ and $\varphi(x) \equiv 0$ for $|x| \geq 2\rho_\varepsilon$, where $\rho_\varepsilon := \varepsilon^\tau$ with $\tau \in (1/4, 1/2)$. Set $\psi_\varepsilon(x) = \varphi(x)w_\varepsilon(x)$. Then

$$\begin{aligned} &\int_{\mathbb{R}^N}|\nabla\psi_\varepsilon|^2 dx = S^{N/2} + O\left(\varepsilon^{(N-2)/2}\right), \\ &\int_{\mathbb{R}^N}|\psi_\varepsilon|^{2^*}dx = S^{N/2} + O\left(\varepsilon^{N/2}\right), \\ &\int_{\mathbb{R}^N}|\psi_\varepsilon|dx \leq C\varepsilon^{(N-2)/4}, \\ &\int_{\mathbb{R}^N}|\psi_\varepsilon|^{2^*-1}dx \leq C\varepsilon^{(N-2)/4}, \\ &\int_{\mathbb{R}^N}|\nabla\psi_\varepsilon|dx \leq C\varepsilon^{(N-2)/4}, \\ &\int_{\mathbb{R}^N}|\psi_\varepsilon|^2 dx = \begin{cases} C\varepsilon + O\left(\varepsilon^{(N-2)/2}\right), & \text{if } N \geq 5, \\ C\varepsilon|\ln\varepsilon| + O(\varepsilon), & \text{if } N = 4, \\ O\left(\varepsilon^{1/2}\right), & \text{if } N = 3. \end{cases} \end{aligned} \tag{58}$$

Since $J_\lambda(0) = 0$ and $\lim_{t\to\infty}J_\lambda(t\psi_\varepsilon) = -\infty$, there exists $t_\varepsilon > 0$ such that $J_\lambda(t_\varepsilon\psi_\varepsilon) = \max_{t\geq 0}J_\lambda(t\psi_\varepsilon)$. We claim that there exist two positive constants t_1, t_2 independent of ε such that

$$t_1 \leq t_\varepsilon \leq t_2 \tag{59}$$

for small $\varepsilon > 0$. Indeed, by $\langle J_\lambda'(t_\varepsilon\psi_\varepsilon), \psi_\varepsilon\rangle = 0$ we have

$$\frac{\int_{\mathbb{R}^N}\left[|\nabla\psi_\varepsilon|^2 + V(x)\psi_\varepsilon^2\right]dx}{|\psi_\varepsilon|_{2^*}^{2^*}} - t_\varepsilon^{2^*-2} - \frac{\int_{\mathbb{R}^N} h_\lambda(x, t_\varepsilon\psi_\varepsilon)\, t_\varepsilon\psi_\varepsilon dx}{t_\varepsilon^2 |\psi_\varepsilon|_{2^*}^{2^*}} = 0. \tag{60}$$

By (29), Lemmas 6(6) and 7(1), and (2), for any $\delta > 0$, there exists $C_\delta > 0$ such that

$$|h_\lambda(x,s)\,s| \le \delta |s|^{2^*} + C_\delta |s|^2 \tag{61}$$

for all $(x, s) \in \mathbb{R}^N \times \mathbb{R}$. Consequently,

$$\begin{aligned}
\left|\frac{\int_{\mathbb{R}^N} h_\lambda(x, t_\varepsilon\psi_\varepsilon)\, t_\varepsilon\psi_\varepsilon dx}{t_\varepsilon^2 |\psi_\varepsilon|_{2^*}^{2^*}}\right|
&\le \frac{\int_{\mathbb{R}^N}\left[\delta t_\varepsilon^{2^*}\psi_\varepsilon^{2^*} + C_\delta t_\varepsilon^2\psi_\varepsilon^2\right]dx}{t_\varepsilon^2|\psi_\varepsilon|_{2^*}^{2^*}} \\
&= \delta t_\varepsilon^{2^*-2} + C_\delta \frac{|\psi_\varepsilon|_2^2}{|\psi_\varepsilon|_{2^*}^{2^*}} \\
&= \delta t_\varepsilon^{2^*-2} + C_\delta\left[S^{N/2} + O\left(\varepsilon^{N/2}\right)\right]^{-1}|\psi_\varepsilon|_2^2 \\
&\le \delta t_\varepsilon^{2^*-2} + CS^{-N/2}|\psi_\varepsilon|_2^2 \\
&= \delta t_\varepsilon^{2^*-2} + CS^{-N/2}\begin{cases} C\varepsilon + O\left(\varepsilon^{(N-2)/2}\right), & \text{if } N \ge 5, \\ C\varepsilon|\ln\varepsilon| + O(\varepsilon), & \text{if } N = 4, \\ O\left(\varepsilon^{1/2}\right), & \text{if } N = 3 \end{cases} \\
&= \delta t_\varepsilon^{2^*-2} + o(1)
\end{aligned} \tag{62}$$

as $\varepsilon \to 0$. Note that

$$\begin{aligned}
\frac{\|\psi_\varepsilon\|_E^2}{|\psi_\varepsilon|_{2^*}^{2^*}} &= \frac{|\nabla\psi_\varepsilon|_2^2 + |\psi_\varepsilon|_2^2}{|\psi_\varepsilon|_{2^*}^{2^*}} = \frac{1}{S^{N/2} + O(\varepsilon^{N/2})} \\
&\cdot \begin{cases} S^{N/2} + O\left(\varepsilon^{(N-2)/2}\right) + C\varepsilon + O\left(\varepsilon^{(N-2)/2}\right), & \text{if } N \ge 5, \\ S^{N/2} + O\left(\varepsilon^{(N-2)/2}\right) + C\varepsilon|\ln\varepsilon| + O(\varepsilon), & \text{if } N = 4, \\ S^{N/2} + O\left(\varepsilon^{(N-2)/2}\right) + O\left(\varepsilon^{1/2}\right), & \text{if } N = 3 \end{cases} \\
&\longrightarrow 1
\end{aligned} \tag{63}$$

as $\varepsilon \to 0$. Hence by (60) one has

$$0 \ge \min\{1, V_0\}(1 + o(1)) - t_\varepsilon^{2^*-2} - \delta t_\varepsilon^{2^*-2} + o(1) \tag{64}$$

as $\varepsilon \to 0$, which implies that

$$t_\varepsilon \ge \left[\frac{\min\{1, V_0\}}{2(1+\delta)}\right]^{1/(2^*-2)} := t_1 > 0 \tag{65}$$

for $\varepsilon > 0$ small enough. On the other hand, (60) leads to

$$\begin{aligned}
t_\varepsilon^{2^*-2} &\le \max\{1, V_\infty\}\frac{\|\psi_\varepsilon\|_E^2}{|\psi_\varepsilon|_{2^*}^{2^*}} + \left|\frac{\int_{\mathbb{R}^N} h_\lambda(x, t_\varepsilon\psi_\varepsilon)\, t_\varepsilon\psi_\varepsilon dx}{t_\varepsilon^2|\psi_\varepsilon|_{2^*}^{2^*}}\right| \\
&\le \max\{1, V_\infty\}(1 + o(1)) + \delta t_\varepsilon^{2^*-2} + o(1)
\end{aligned} \tag{66}$$

as $\varepsilon \to 0$, which implies that

$$t_\varepsilon \le \left[\frac{2\max\{1, V_\infty\}}{1-\delta}\right]^{1/(2^*-2)} := t_2 < +\infty \tag{67}$$

for $\delta > 0$ and $\varepsilon > 0$ small enough.

Since $Q(t) := t^2/2 - t^{2^*}/2^*$ has only maximum at $t = 1$, one has

$$\begin{aligned}
J_\lambda(t_\varepsilon\psi_\varepsilon) &= \frac{1}{2}t_\varepsilon^2\int_{\mathbb{R}^N}|\nabla\psi_\varepsilon|^2 dx + \frac{1}{2}t_\varepsilon^2\int_{\mathbb{R}^N}V(x)\psi_\varepsilon^2 dx \\
&\quad - \int_{\mathbb{R}^N} H_\lambda(x, t_\varepsilon\psi_\varepsilon)\,dx \\
&\quad - \frac{1}{2^*}t_\varepsilon^{2^*}\int_{\mathbb{R}^N}\psi_\varepsilon^{2^*}dx \\
&= \left(\frac{t_\varepsilon^2}{2} - \frac{t_\varepsilon^{2^*}}{2^*}\right)S^{N/2} + O\left(\varepsilon^{(N-2)/2}\right) \\
&\quad + \frac{1}{2}t_\varepsilon^2\int_{\mathbb{R}^N}V(x)\psi_\varepsilon^2 dx \\
&\quad - \int_{\mathbb{R}^N} H_\lambda(x, t_\varepsilon\psi_\varepsilon)\,dx \\
&\le \frac{1}{N}S^{N/2} + O\left(\varepsilon^{(N-2)/2}\right) \\
&\quad + \frac{1}{2}t_2^2 V_\infty\int_{\mathbb{R}^N}\psi_\varepsilon^2 dx \\
&\quad - \int_{\mathbb{R}^N} H_\lambda(x, t_\varepsilon\psi_\varepsilon)\,dx.
\end{aligned} \tag{68}$$

Notice that, for $x \in B_{\rho_\varepsilon}$, we have

$$\begin{aligned}
t_\varepsilon\psi_\varepsilon = t_\varepsilon w_\varepsilon &= t_\varepsilon\frac{[N(N-2)\varepsilon]^{(N-2)/4}}{\left(\varepsilon + |x|^2\right)^{(N-2)/2}} \\
&\ge Ct_\varepsilon\frac{[N(N-2)]^{(N-2)/4}\varepsilon^{(N-2)/4}}{\varepsilon^{\tau(N-2)}} \\
&\ge Ct_1[N(N-2)]^{(N-2)/4}\varepsilon^{(N-2)(1/4-\tau)} \longrightarrow +\infty
\end{aligned} \tag{69}$$

as $\varepsilon \to 0$, which combining with Lemma 7(4) and (5) implies that for any $M > 0$

$$\int_{\mathbb{R}^N} H_\lambda(x, t_\varepsilon\psi_\varepsilon)\,dx \ge Mt_\varepsilon^2\int_{B_{\rho_\varepsilon}}\psi_\varepsilon^2 dx \tag{70}$$

for $\varepsilon > 0$ small enough. Note that

$$\int_{B_{\rho_\varepsilon}} \psi_\varepsilon^2 dx = [N(N-2)]^{(N-2)/2} N\omega_N \varepsilon \int_0^{\rho_\varepsilon/\sqrt{\varepsilon}} \frac{s^{N-1}}{(1+s^2)^{N-2}} ds,$$
$$\int_0^{+\infty} \frac{s^{N-1}}{(1+s^2)^{N-2}} ds \geq \int_0^1 \frac{s^{N-1}}{(1+s^2)^{N-2}} ds \geq \frac{1}{N} \cdot \frac{1}{2^{N-2}} \coloneqq C > 0. \tag{71}$$

Consequently,

$$\int_{\mathbb{R}^N} H_\lambda(x, t_\varepsilon \psi_\varepsilon) dx \geq MC\varepsilon \tag{72}$$

for $\varepsilon > 0$ small enough. Hence by (68)

$$\begin{aligned} J_\lambda(t_\varepsilon \psi_\varepsilon) &\leq \frac{1}{N} S^{N/2} + O\left(\varepsilon^{(N-2)/2}\right) + \frac{1}{2} t_2^2 V_\infty \int_{\mathbb{R}^N} \psi_\varepsilon^2 dx - \int_{\mathbb{R}^N} H_\lambda(x, t_\varepsilon \psi_\varepsilon) dx \\ &\leq \frac{1}{N} S^{N/2} + O\left(\varepsilon^{(N-2)/2}\right) - MC\varepsilon + C \begin{cases} \varepsilon + O\left(\varepsilon^{(N-2)/2}\right), & \text{if } N \geq 5, \\ \varepsilon |\ln \varepsilon| + O(\varepsilon), & \text{if } N = 4, \\ O\left(\varepsilon^{1/2}\right), & \text{if } N = 3. \end{cases} \end{aligned} \tag{73}$$

From this, we see that $J_\lambda(t_\varepsilon \psi_\varepsilon) < (1/N) S^{N/2}$ for $\varepsilon > 0$ small enough and M big enough if $N \geq 5$. Consequently, $c_\lambda < (1/N) S^{N/2}$ for all $\lambda > 0$ if $N \geq 5$.

In the following, we consider the case $N = 3, 4$. Indeed, if the conclusion is false, then there exists a sequence $\{\lambda_n\}$ with $\lambda_n \to +\infty$ such that $c_{\lambda_n} \geq (1/N) S^{N/2}$. Take $v \in E \backslash \{0\}$. Then by the proof of Lemma 8, there exists a unique $t_{\lambda_n} > 0$ such that $\max_{t>0} J_{\lambda_n}(tv) = J_{\lambda_n}(t_{\lambda_n} v)$. Hence

$$\begin{aligned} t_{\lambda_n}^2 \int_{\mathbb{R}^N} |\nabla v|^2 dx + \int_{\mathbb{R}^N} V(x) \frac{G^{-1}(t_{\lambda_n} v)}{g(G^{-1}(t_{\lambda_n} v))} t_{\lambda_n} v \, dx &= t_{\lambda_n}^{2^*} \int_{\mathbb{R}^N} |v|^{2^*} dx \\ &\quad + \lambda_n \int_{\mathbb{R}^N} \frac{f(x, G^{-1}(t_{\lambda_n} v))}{g(G^{-1}(t_{\lambda_n} v))} t_{\lambda_n} v \, dx. \end{aligned} \tag{74}$$

By Lemma 6(6) and (f_4) we get

$$\max\{1, V_\infty\} \|v\|_E^2 \geq t_{\lambda_n}^{2^*-2} \int_{\mathbb{R}^N} |v|^{2^*} dx, \tag{75}$$

which implies that $\{t_{\lambda_n}\}$ is bounded. Hence, up to a subsequence, there exists $t_0 \geq 0$ such that $t_{\lambda_n} \to t_0$ as $n \to \infty$. If $t_0 > 0$, then by (f_4) and Fatou lemma we have

$$\lim_{n\to\infty} \left[t_{\lambda_n}^{2^*} \int_{\mathbb{R}^N} |v|^{2^*} dx + \lambda_n \int_{\mathbb{R}^N} \frac{f(x, G^{-1}(t_{\lambda_n} v))}{g(G^{-1}(t_{\lambda_n} v))} t_{\lambda_n} v \, dx \right] = +\infty. \tag{76}$$

But, on the other hand, by Lemma 6(6) one has

$$\begin{aligned} t_{\lambda_n}^{2^*} \int_{\mathbb{R}^N} |v|^{2^*} dx + \lambda_n \int_{\mathbb{R}^N} \frac{f(x, G^{-1}(t_{\lambda_n} v))}{g(G^{-1}(t_{\lambda_n} v))} t_{\lambda_n} v \, dx \\ \leq \max\{1, V_\infty\} t_{\lambda_n}^2 \|v\|_E^2 \longrightarrow \max\{1, V_\infty\} t_0^2 \|v\|_E^2, \end{aligned} \tag{77}$$

a contradiction. Hence $t_0 = 0$ and by Lemma 7(4) we know that

$$\begin{aligned} \max_{t>0} J_{\lambda_n}(tv) = J_{\lambda_n}(t_{\lambda_n} v) &\leq \frac{1}{2} \max\{1, V_\infty\} t_{\lambda_n}^2 \|v\|_E^2 \\ &\quad - \frac{1}{2^*} t_{\lambda_n}^{2^*} \int_{\mathbb{R}^N} |v|^{2^*} dx \longrightarrow 0 \end{aligned} \tag{78}$$

as $n \to \infty$. Consequently,

$$0 < \frac{1}{N} S^{N/2} \leq c_{\lambda_n} \leq \inf_{u \in E\backslash\{0\}} \max_{t>0} J_{\lambda_n}(tu) \leq \max_{t>0} J_{\lambda_n}(tv) \longrightarrow 0, \tag{79}$$

a contradiction. This completes the proof. □

Proof of Theorem 1. Since $\{v_n\} \subset E$ is a bounded Cerami sequence for J_λ at the level $c_\lambda > 0$, there exists $v \in E$ such that

$$\begin{aligned} &v_n \rightharpoonup v \quad \text{in } E, \\ &v_n \longrightarrow v \quad \text{in } L^q_{\text{loc}}(\mathbb{R}^N) \text{ for } 1 \leq q < 2^*, \\ &v_n(x) \longrightarrow v(x) \quad \text{a.e. on } \mathbb{R}^N. \end{aligned} \tag{80}$$

Using a standard argument, we know that $J'_\lambda(v) = 0$, that is, v is a weak solution of (11). Indeed, for any $\psi \in C_0^\infty(\mathbb{R}^N)$, we have

$$\begin{aligned} o(1) = \langle J'_\lambda(v_n), \psi \rangle &= \int_{\mathbb{R}^N} \nabla v_n \nabla \psi \, dx + \int_{\mathbb{R}^N} V(x) v_n \psi \, dx \\ &\quad - \int_{\mathbb{R}^N} |v_n|^{2^*-2} v_n \psi \, dx - \int_{\mathbb{R}^N} h_\lambda(x, v_n) \psi \, dx. \end{aligned} \tag{81}$$

Since $v_n \rightharpoonup v$ in E, one has

$$\begin{aligned}\int_{\mathbb{R}^N} \nabla v_n \nabla \psi \, dx &\longrightarrow \int_{\mathbb{R}^N} \nabla v \nabla \psi \, dx, \\ \int_{\mathbb{R}^N} V(x) v_n \psi \, dx &\longrightarrow \int_{\mathbb{R}^N} V(x) v \psi \, dx, \\ \int_{\mathbb{R}^N} |v_n|^{2^*-2} v_n \psi \, dx &\longrightarrow \int_{\mathbb{R}^N} |v|^{2^*-2} v \psi \, dx, \\ \int_{\mathbb{R}^N} h_\lambda(x, v_n) \psi \, dx &\longrightarrow \int_{\mathbb{R}^N} h_\lambda(x, v) \psi \, dx.\end{aligned} \tag{82}$$

Consequently,

$$\begin{aligned}0 = &\int_{\mathbb{R}^N} \nabla v \nabla \psi \, dx + \int_{\mathbb{R}^N} V(x) v \psi \, dx \\ &- \int_{\mathbb{R}^N} |v|^{2^*-2} v \psi \, dx - \int_{\mathbb{R}^N} h_\lambda(x, v) \psi \, dx\end{aligned} \tag{83}$$

for all $\psi \in C_0^\infty(\mathbb{R}^N)$. For any $\varphi \in E$, there exists a sequence $\{\psi_n\} \subset C_0^\infty(\mathbb{R}^N)$ such that $\psi_n \to \varphi$ in E. Hence

$$\begin{aligned}0 = &\int_{\mathbb{R}^N} \nabla v \nabla \psi_n dx + \int_{\mathbb{R}^N} V(x) v \psi_n dx \\ &- \int_{\mathbb{R}^N} |v|^{2^*-2} v \psi_n dx - \int_{\mathbb{R}^N} h_\lambda(x, v) \psi_n dx.\end{aligned} \tag{84}$$

Let $n \to \infty$, we get

$$\begin{aligned}0 = &\int_{\mathbb{R}^N} \nabla v \nabla \varphi \, dx + \int_{\mathbb{R}^N} V(x) v \varphi \, dx \\ &- \int_{\mathbb{R}^N} |v|^{2^*-2} v \varphi \, dx - \int_{\mathbb{R}^N} h_\lambda(x, v) \varphi \, dx;\end{aligned} \tag{85}$$

that is, $\langle J'_\lambda(v), \varphi \rangle = 0$ for all $\varphi \in E$. Hence $J'_\lambda(v) = 0$; that is, v is a weak solution of (11).

In the following, we prove that v is nontrivial. With the aid of Lemma 10, the proof follows essentially the proof of Theorem 1.1 in [16]. For completeness, we present the proof as follows. If the conclusion is false, we may assume $v = 0$. We divide the proof into four steps.

Step 1. We prove that $\{v_n\} \subset E$ is also a Cerami sequence for the functional $J_\lambda^\infty : E \to \mathbb{R}$, where

$$\begin{aligned}J_\lambda^\infty(v_n) = &\frac{1}{2} \int_{\mathbb{R}^N} \left[|\nabla v_n|^2 + V_\infty v_n^2 \right] dx \\ &- \int_{\mathbb{R}^N} H_\lambda(x, v_n) \, dx - \frac{1}{2^*} \int_{\mathbb{R}^N} |v_n|^{2^*} dx.\end{aligned} \tag{86}$$

By (V_2) and $v_n \rightharpoonup 0$ in E, one has

$$J_\lambda(v_n) - J_\lambda^\infty(v_n) = \frac{1}{2} \int_{\mathbb{R}^N} [V(x) - V_\infty] v_n^2 dx \longrightarrow 0 \tag{87}$$

as $n \to \infty$. Similarly, we have

$$\begin{aligned}&\left\| J'_\lambda(v_n) - (J_\lambda^\infty)'(v_n) \right\|_{E^*} \\ &= \sup_{\|\varphi\|_E \le 1} \left| \left\langle J'_\lambda(v_n) - (J_\lambda^\infty)'(v_n), \varphi \right\rangle \right| \\ &= \sup_{\|\varphi\|_E \le 1} \left| \int_{\mathbb{R}^N} [V(x) - V_\infty] v_n \varphi \, dx \right| \longrightarrow 0\end{aligned} \tag{88}$$

as $n \to \infty$. Consequently, $\{v_n\}$ is also a Cerami sequence of J_λ^∞.

Step 2. There exist $\alpha, R > 0$ and $\{y_n\} \subset \mathbb{R}^N$ such that

$$\lim_{n \to \infty} \int_{B_R(y_n)} |v_n|^2 dx \ge \alpha > 0. \tag{89}$$

Indeed, by contradiction, then by Lemma 1.21 in [27], one has $v_n \to 0$ in $L^q(\mathbb{R}^N)$ for $2 < q < 2^*$. Notice that

$$\begin{aligned}o(1) &= \left\langle J'_\lambda(v_n), v_n \right\rangle \\ &= \int_{\mathbb{R}^N} \left[|\nabla v_n|^2 + V(x) v_n^2 \right] dx \\ &\quad - \int_{\mathbb{R}^N} h_\lambda(x, v_n) v_n dx - \int_{\mathbb{R}^N} |v_n|^{2^*} dx,\end{aligned} \tag{90}$$

which combining with (51) leads to

$$\int_{\mathbb{R}^N} \left[|\nabla v_n|^2 + V(x) v_n^2 \right] dx - \int_{\mathbb{R}^N} |v_n|^{2^*} dx \longrightarrow 0 \tag{91}$$

as $n \to \infty$. Consequently, there exists a constant $l \ge 0$ such that

$$\begin{aligned}\int_{\mathbb{R}^N} \left[|\nabla v_n|^2 + V(x) v_n^2 \right] dx &\longrightarrow l, \\ \int_{\mathbb{R}^N} |v_n|^{2^*} dx &\longrightarrow l.\end{aligned} \tag{92}$$

Obviously, $l > 0$. Otherwise, $J_\lambda(v_n) \to 0$ as $n \to \infty$, which contradicts with $c_\lambda > 0$. Hence by the definition of S, we have

$$\begin{aligned}S &\le \frac{\int_{\mathbb{R}^N} |\nabla v_n|^2 dx}{\left(\int_{\mathbb{R}^N} |v_n|^{2^*} dx \right)^{2/2^*}} \\ &\le \frac{\int_{\mathbb{R}^N} \left[|\nabla v_n|^2 + V(x) v_n^2 \right] dx}{\left(\int_{\mathbb{R}^N} |v_n|^{2^*} dx \right)^{2/2^*}} \longrightarrow \frac{l}{l^{2/2^*}} = l^{2/N};\end{aligned} \tag{93}$$

that is, $l \ge S^{N/2}$. Therefore, (41) implies that

$$\begin{aligned}c_\lambda + o(1) &= J_\lambda(v_n) \\ &= \frac{1}{2} \int_{\mathbb{R}^N} \left[|\nabla v_n|^2 + V(x) v_n^2 \right] dx \\ &\quad - \int_{\mathbb{R}^N} H_\lambda(x, v_n) \, dx - \frac{1}{2^*} \int_{\mathbb{R}^N} |v_n|^{2^*} dx \\ &\longrightarrow \left(\frac{1}{2} - \frac{1}{2^*} \right) l = \frac{1}{N} l \ge \frac{1}{N} S^{N/2},\end{aligned} \tag{94}$$

as $n \to \infty$, which implies that $c_\lambda \geq (1/N)S^{N/2}$, a contradiction.

Step 3. After a translation of $\{v_n\}$ called $\{\tilde{v}_n\}$, then $\tilde{v}_n$ converges weakly to a nonzero critical point of J_λ^∞.

Set $\tilde{v}_n(x) = v_n(x+y_n)$. Since $\{v_n\} \subset E$ is a Cerami sequence of J_λ^∞ and $\|\tilde{v}_n\|_E = \|v_n\|_E$, arguing as in the case of $\{v_n\}$, we may assume $\tilde{v}_n \rightharpoonup \tilde{v}$ in E and $(J_\lambda^\infty)'(\tilde{v}) = 0$. So by Step 2 we know $\tilde{v} \neq 0$. By Lemma 7(3) and Fatou Lemma, one has

$$\begin{aligned} 2c_\lambda &= \liminf_{n\to\infty} \left[2J_\lambda^\infty(\tilde{v}_n) - \left\langle (J_\lambda^\infty)'(\tilde{v}_n), \tilde{v}_n \right\rangle\right] \\ &\geq \liminf_{n\to\infty} \int_{\mathbb{R}^N} \left[h_\lambda(x,\tilde{v}_n)\tilde{v}_n - 2H_\lambda(x,\tilde{v}_n)\right] dx \\ &\quad + \left(1 - \frac{2}{2^*}\right) \liminf_{n\to\infty} \int_{\mathbb{R}^N} |\tilde{v}_n|^{2^*} dx \\ &\geq \int_{\mathbb{R}^N} \left[h_\lambda(x,\tilde{v})\tilde{v} - 2H_\lambda(x,\tilde{v})\right] dx \\ &\quad + \left(1 - \frac{2}{2^*}\right) \int_{\mathbb{R}^N} |\tilde{v}|^{2^*} dx \\ &= 2J_\lambda^\infty(\tilde{v}) - \left\langle (J_\lambda^\infty)'(\tilde{v}), \tilde{v} \right\rangle = 2J_\lambda^\infty(\tilde{v}), \end{aligned} \tag{95}$$

which implies that $J_\lambda^\infty(\tilde{v}) \leq c_\lambda$.

Step 4. We use $\tilde{v}$ to construct a path which allows us to obtain a contradiction with the definition of mountain pass level c_λ.

Define the mountain pass level $c_\lambda^\infty := \inf_{\gamma\in\Gamma_\infty} \sup_{t\in[0,1]} J_\lambda^\infty(\gamma(t)) > 0$, where $\Gamma_\infty := \{\gamma \in C([0,1], E) : \gamma(0) = 0,\ J_\lambda^\infty(\gamma(1)) < 0\}$. It follows the arguments used in [28, 29], we can construct a path $\gamma : [0,1] \to E$ such that

$$\begin{gathered} \gamma(0) = 0, \\ J_\lambda^\infty(\gamma(1)) < 0, \\ \tilde{v} \in \gamma([0,1]), \\ \gamma(t)(x) > 0, \quad \forall x \in \mathbb{R}^N,\ t \in [0,1], \\ \max_{t\in[0,1]} J_\lambda^\infty(\gamma(t)) = J_\lambda^\infty(\tilde{v}). \end{gathered} \tag{96}$$

Then $c_\lambda^\infty \leq \max_{t\in[0,1]} J_\lambda^\infty(\gamma(t)) = J_\lambda^\infty(\tilde{v})$. If $V(x) \equiv V_\infty$, we have already proved Theorem 1. If $V(x) \leq V_\infty$ but $V(x) \not\equiv V_\infty$, we take the path γ given by above, and by $\gamma \in \Gamma_\infty \subset \Gamma$, we have

$$\begin{aligned} c_\lambda &\leq \max_{t\in[0,1]} J_\lambda(\gamma(t)) = J_\lambda(\gamma(\bar{t})) < J_\lambda^\infty(\gamma(\bar{t})) \\ &\leq \max_{t\in[0,1]} J_\lambda^\infty(\gamma(t)) = J_\lambda^\infty(\tilde{v}) \leq c_\lambda, \end{aligned} \tag{97}$$

a contradiction. Consequently, $v \not\equiv 0$. This completes the proof of Theorem 1. □

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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