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Research Article

A New Scheme on Synchronization of Commensurate Fractional-Order Chaotic Systems Based on Lyapunov Equation

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This paper proposes a new fractional-order approach for synchronization of a class of fractional-order chaotic systems in the presence of model uncertainties and external disturbances. A simple but practical method to synchronize many familiar fractional-order chaotic systems has been put forward. A new theorem is proposed for a class of cascade fractional-order systems and it is applied in chaos synchronization. Combined with the fact that the states of the fractional chaotic systems are bounded, many coupled items can be taken as zero items. Then, the whole system can be simplified greatly and a simpler controller can be derived. Finally, the validity of the presented scheme is illustrated by numerical simulations of the fractional-order unified system.

1. Introduction

Fractional calculus, with more than 300-year-old history, is generalization of ordinary differentiation and integration to arbitrary order. Until the recent decades, the fractional calculus attracted attention of researchers in various fields [1–5]. Many systems in physics and engineering have been found to be effectively modeled as fractional-order systems [6–8]. Fractional-order systems could also behave chaotically. Recently, it has been demonstrated that some fractional-order differential systems, including fractional-order Lü system, fractional-order Chen system, fractional-order coupled dynamos system, fractional-order Liu system, and fractional-order unified system [9–13] exhibit chaotic behaviors.

Integer order chaotic systems have been extensively studied for many decades [14, 15], whereas their fractional-order counterparts have just recently been investigated. An extensive survey of various fractional-order chaotic systems can be found in [4]. Chaos control of fractional-order chaotic systems has received a lot of considerations of researchers. However, compared to integer order chaotic systems, the research results in this field are relatively rare. For the limitation of the available theoretical tools, a number of research works mainly focus on integer order chaotic systems.

Chaos control based on stability theories for linear fractional-order systems has been extensively proposed in [16, 17]. However, this approach provides only local stability. Active control is another approach that can be found in the literature [18, 19]. The approach made use of a nonlinear control law to cancel nonlinearities in the control system. Based on the stability theory for linear fractional-order systems, a linear control law was designed to stabilize the linearized system. Reference [20] presented LMI-based stabilization method for fractional-order chaotic systems. The method required that the systems must be transformable into a linear interval fractional-order system. In [21, 22], the authors proposed a sliding mode control approach to realize chaos control. By adopting a fractional-order sliding surface, the stability can be obtained via the Lyapunov stability method. However, the fractional-order sliding surface might be difficult to implement. Moreover, discontinuous control signals could induce some undesired behavior.

Many of the above controllers are nonlinear. From a practical point of view, to design a simple linear controller by a special strategy is more valuable. However, to our best knowledge, there is a limitation of the available theoretical tools that can be used for nonlinear fractional-order systems. Therefore, designing a linear controller for fractional-order chaotic systems has still remained as an open and challenging problem.

Motivated by the above discussions, in this paper, based on fractional-order Lyapunov stability theory, a new strategy is proposed to synchronize a class of fractional-order chaotic systems in the presence of model uncertainties and external disturbances. Then, a simple linear controller with low dimensions is designed via the proposed new strategy. Compared with the control approach in [20], only two simple and linear feedback controllers are designed to achieve synchronization. Compared with the control approach in [21, 22], it can be implemented easily.

The rest of this paper is organized as follows: in Section 2, some preliminaries and main results are presented. In Section 3, we give the famous fractional-order unified chaotic systems to illustrate that it can be transformed to our cascade form for further simple design. Experimental analyses are presented in Section 4. Finally, conclusions are given in Section 5.

2. Preliminaries and Main Results

In this section, first some basic definitions and lemmas are briefly presented. Then a useful theorem is put forward to be utilized later to design a robust linear control law.

Definition 1. The definition of fractional integral is described by

$$ {}_{t_0}D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} f(\tau)\, d\tau, \quad \alpha > 0, \qquad (1)$$

where $\Gamma(\cdot)$ is the well-known Gamma function.

Definition 2. The Riemann-Liouville derivative is defined as

$$ D_t^q f(t) = D_{t_0}^m D_t^{q-m} f(t), \quad q \in [m-1, m), \qquad (2)$$

where $m \in Z^+$, D_t^m is the classical m-order derivative.

In the rest of the paper, D_t^q is used to denote the Riemann-Liouville derivative of order q.

Lemma 3. *Assume $f(t)$ is bounded and $\lim_{t\to\infty} g(t) = 0$; then $\lim_{t\to\infty}[f(t)g(t)] = 0$.*

Proof. Since $f(t)$ is bounded, then there exists a number $M > 0$ such that $|f(t)| < M$ for all t.

$\forall \varepsilon > 0$, since $\lim_{t\to\infty} g(t) = 0$, then, for $\varepsilon_1 = \varepsilon/M > 0$, $\exists \delta$ such that

$$ |g(t)| < \varepsilon_1 = \frac{\varepsilon}{M}, \quad \text{when } t > \delta. \qquad (3)$$

Then, when $t > \delta$

$$ |f(t)\, g(t)| \le |f(t)|\, |g(t)| < M * \varepsilon_1 = \varepsilon. \qquad (4)$$

This means $\lim_{t\to\infty}[f(t)g(t)] = 0$. Here the proof is complete. □

Lemma 4 (see [23]). *Let $A \in R^{n\times n}$ be a real matrix. Then, a necessary and sufficient condition for the asymptotical stability of $D^\alpha x(t) = Ax(t)$ is*

$$ |\arg(\lambda(A))| > \frac{\alpha\pi}{2}, \qquad (5)$$

where $0 < \alpha < 1$, $\lambda(A)$ is the spectrum of all eigenvalues of A.

In this paper, we consider the case of the synchronization of two commensurate fractional-order chaotic systems. The master system is as follows:

$$ D^q x = A(x)\, x, \qquad (6)$$

where $x = (x_1, x_2, \ldots, x_n)^T \in R^n$ is the state vector, $A(x)$ is the parametric coefficient matrix of states x, and $0 < q \le 1$ is the fractional commensurate order. The slave system is

$$ D^q \hat{x} = A(\hat{x})\, \hat{x} + u(t), \qquad (7)$$

where $u(t)$ is the controller to be designed. Define $e(t) = \hat{x}(t) - x(t)$. We can get the error system:

$$ D^q e = F(e, x) + u(e, x), \qquad (8)$$

where $F(e, x) = A(\hat{x})\hat{x} - A(x)x$. The synchronization problem can be transformed to design a controller u such that $\lim_{t\to\infty}\|e(t)\| = 0$.

For further discussion, a useful theorem is presented. Consider a class of cascade-connected system described by

$$ \begin{aligned} D^q x_1 &= f(x_1) \\ D^q(x_1, x_2) &= A(x_1, x_2)\, x_2 + B(x_1, x_2)\, g(x_1), \end{aligned} \qquad (9)$$

where $x_1 \in R^n$, $x_2 \in R^m$, $f(0) = 0$, and $g(0) = 0$. $f(x_1)$ and $g(x_1)$ are both C^1 vector fields. $A(x_1, x_2)$ and $B(x_1, x_2)$ are C^m and C^n coefficient matrix, respectively.

Theorem 5. *If*

(1) *the subsystem $D^q x_1 = f(x_1)$ is globally asymptotically stable at $x_1 = 0$,*

(2) *$B(x_1, x_2)$ is a bounded matrix and $\lim_{t\to\infty} g(x_1) = 0$,*

(3) *$A(x_1, x_2)$ is a matrix with the following structure:*

$$ A(x_1, x_2) = \begin{pmatrix} A_{11}(\cdot) & A_{12}(\cdot) & \cdots & A_{1n}(\cdot) \\ A_{21}(\cdot) & A_{22}(\cdot) & \ddots & \vdots \\ \vdots & \vdots & \ddots & A_{n-1n}(\cdot) \\ A_{n1}(\cdot) & \cdots & A_{nn-1}(\cdot) & A_{nn}(\cdot) \end{pmatrix}, \qquad (10)$$

where $(A_{ij}(\cdot) \le 0)$, $i = j$, $(A_{ij}(\cdot) = -A_{ji}(\cdot))$, $i \ne j$, then, system (9) is globally asymptotically stable at the equilibrium $(x_1, x_2) = (0, 0)$.

Proof. Since the subsystem

$$D^q x_1 = f(x_1) \tag{11}$$

is already globally asymptotically stable at $x_1 = 0$, we only need to consider the subsystem

$$D^q(x_1, x_2) = A(x_1, x_2) x_2 + B(x_1, x_2) g(x_1). \tag{12}$$

From assumption (2) we know that $\lim_{t\to\infty} g(x_1) = 0$. Consider the fact that $B(x_1, x_2)$ is a bounded matrix. From Lemma 3 we know that

$$\lim_{t\to\infty} [B(x_1, x_2) g(x_1)] = 0. \tag{13}$$

This means that the second item in the system

$$D^q(x_1, x_2) = A(x_1, x_2) x_2 + B(x_1, x_2) g(x_1) \tag{14}$$

can be neglected when $t \to \infty$. To study the asymptotical stability problem of system (9), we only need to consider the following system:

$$D^q(x_1, x_2) = A(x_1, x_2) x_2. \tag{15}$$

Here, we take three steps to illustrate that system (15) is globally asymptotically stable at the equilibrium $(x_1, x_2) = (0, 0)$.

Firstly, let us consider the following equation:

$$A(x_1, x_2)\xi = \lambda\xi, \tag{16}$$

where λ is one of the eigenvalues of $A(x_1, x_2)$ and ξ is a nonzero eigenvector of λ. Take the conjugate transpose on either side of (16) and we can get

$$\overline{(A(x_1, x_2)\xi)}^T = \bar{\lambda}\xi^H. \tag{17}$$

Combine (16) with (17) and we can get

$$\xi^H \left(PA(x_1, x_2) + (A(x_1, x_2))^H P\right)\xi = \left(\lambda + \bar{\lambda}\right)\xi^H P\xi. \tag{18}$$

Secondly, because $A(x_1, x_2)$ has the structure

$$A(x_1, x_2) = \begin{pmatrix} A_{11}(\cdot) & A_{12}(\cdot) & \cdots & A_{1n}(\cdot) \\ A_{21}(\cdot) & A_{22}(\cdot) & \ddots & \vdots \\ \vdots & \vdots & \ddots & A_{n-1n}(\cdot) \\ A_{n1}(\cdot) & \cdots & A_{nn-1}(\cdot) & A_{nn}(\cdot) \end{pmatrix} = \begin{pmatrix} A_{11}(\cdot) & A_{12}(\cdot) & \cdots & A_{1n}(\cdot) \\ -A_{12}(\cdot) & A_{22}(\cdot) & \ddots & \vdots \\ \vdots & \ddots & \ddots & A_{n-1n}(\cdot) \\ -A_{1n}(\cdot) & \cdots & -A_{n-1n}(\cdot) & A_{nn}(\cdot) \end{pmatrix}, \tag{19}$$

we can know that the matrix $A(x_1, x_2)$ satisfies the continuous Lyapunov equation

$$A(x_1, x_2) P + PA(x_1, x_2)^H = -Q, \tag{20}$$

where $P = I$ is the real symmetric identity matrix and

$$Q = \begin{pmatrix} -2A_{11} & & & \\ & -2A_{22} & & \\ & & \ddots & \\ & & & -2A_{nn} \end{pmatrix} \tag{21}$$

is a Hermitian matrix. Moreover, we can have

$$\begin{aligned} &A(x_1, x_2) P + PA(x_1, x_2)^H \\ &\quad = \left(A(x_1, x_2) P + PA(x_1, x_2)^H\right)^H. \end{aligned} \tag{22}$$

Namely, $A(x_1, x_2)P + P(A(x_1, x_2))^H$ is also a Hermitian matrix.

Lastly, according to the properties of positive definite and negative semidefinite matrix, we can get two inequalities

$$\begin{gathered} \xi^H \left(PA(x_1, x_2) + (A(x_1, x_2))^H P\right)\xi = \xi^H(-Q)\xi \le 0 \\ \xi^H P\xi > 0. \end{gathered} \tag{23}$$

Consequently, combining (18), we can have

$$\left(\lambda + \bar{\lambda}\right) = \frac{\xi^H(-Q)\xi}{(\xi^H P\xi)} \le 0. \tag{24}$$

Obviously, any eigenvalue λ of the coefficient matrix $A(x_1, x_2)$ satisfies the following inequality:

$$|\arg(\lambda)| \ge \frac{\pi}{2} > \frac{\alpha\pi}{2} \quad (\alpha < 1). \tag{25}$$

From Lemma 4, we know that system (15) is globally asymptotically stable at the equilibrium $(x_1, x_2) = (0, 0)$. This completes the proof. □

3. Illustrative Example

In the first part of section, the fractional-order unified chaotic system and its synchronization are transformed to our cascade form and a simpler controller is derived. Then its robustness analysis will be introduced.

3.1. Fractional-Order Unified Chaotic System and Its Synchronization. Similar to the classical unified chaotic system, the fractional-order unified system could be considered as the system that bridges the gap among the fractional-order Lorenz system, the fractional-order Lü system, and

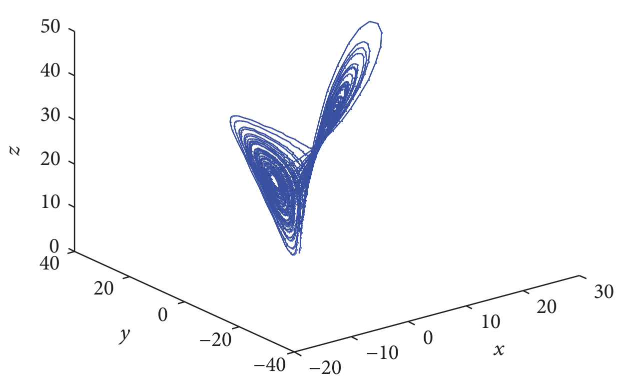

Figure 1: The fractional-order unified chaotic system with $q_1 = q_2 = q_3 = 0.95$ and $\alpha = 0.2$.

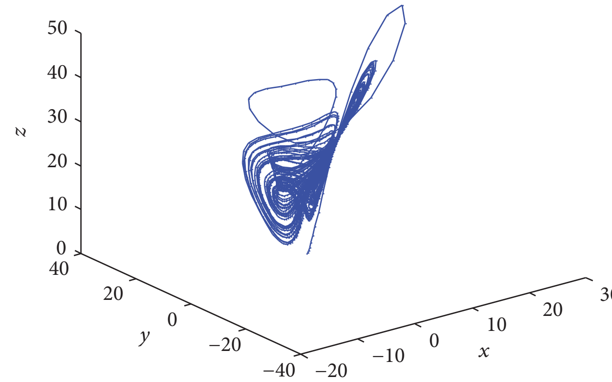

Figure 2: The fractional-order unified chaotic system with $q_1 = q_2 = q_3 = 0.95$ and $\alpha = 1$.

the fractional-order Chen system. The fractional-order unified chaotic system [13] is described by

$$\begin{aligned} \frac{d^{q_1}x}{dt^{q_1}} &= (25\alpha + 10)(y - x) \\ \frac{d^{q_2}y}{dt^{q_2}} &= (28 - 35\alpha)x - xz + (29\alpha - 1)y \\ \frac{d^{q_3}z}{dt^{q_3}} &= xy - \frac{(8+\alpha)z}{3}, \end{aligned} \tag{26}$$

where $d^{q_i}/dt^{q_i} = D_*^{q_i}$ $(i = 1, 2, 3)$. Its order is denoted by $q = (q_1, q_2, q_3)$ subject to $0 < q_1, q_2, q_3 \leq 1$, and $\alpha \in [0, 1]$. Some examples of the chaotic attractor of system (26) with $q_1 = q_2 = q_3 = 0.95$ are shown in Figures 1 and 2.

Two systems in synchronization are called the master system and the slave system, respectively. From (26), the master system and the slave system can be expressed, respectively, as

$$\begin{aligned} \frac{d^{q_1}x_1}{dt^{q_1}} &= (25\alpha + 10)(x_2 - x_1) \\ \frac{d^{q_2}x_2}{dt^{q_2}} &= (28 - 35\alpha)x_1 - x_1x_3 + (29\alpha - 1)x_2 \longrightarrow \\ &\text{master system} \\ \frac{d^{q_3}x_3}{dt^{q_3}} &= x_1x_2 - \frac{(8+\alpha)x_3}{3}, \end{aligned} \tag{27}$$

$$\begin{aligned} \frac{d^{q_1}y_1}{dt^{q_1}} &= (25\alpha + 10)(y_2 - y_1) + u_1 \\ \frac{d^{q_2}y_2}{dt^{q_2}} &= (28 - 35\alpha)y_1 - y_1y_3 + (29\alpha - 1)y_2 + u_2 \longrightarrow \\ &\text{slave system} \\ \frac{d^{q_3}y_3}{dt^{q_3}} &= y_1y_2 - \frac{(8+\alpha)y_3}{3}, \end{aligned} \tag{28}$$

where u_1, u_2 are the control signals used to drive the slave system to follow the master system.

Denote the synchronization error as $e = y - x$. Our aim is to design a controller $u(t) = (u_1, u_2)^T$ such that controlled system (28) asymptotically synchronizes master system (27) in the sense that

$$\lim_{t\to\infty} \|e\| = \lim_{t\to\infty} \|y(t, y_0) - x(t, x_0)\| = 0. \tag{29}$$

Subtracting (27) from (28), then the synchronization error equation can be obtained as

$$\begin{aligned} \frac{d^{q_1}e_1}{dt^{q_1}} &= (25\alpha + 10)(e_2 - e_1) + u_1 \\ \frac{d^{q_2}e_2}{dt^{q_2}} &= (28 - 35\alpha)e_1 - e_1e_3 - e_1x_3 - e_3x_1 \\ &\quad + (29\alpha - 1)e_2 + u_2 \\ \frac{d^{q_3}e_3}{dt^{q_3}} &= e_1e_2 + e_1x_2 + e_2x_1 - \frac{(8+\alpha)e_3}{3}. \end{aligned} \tag{30}$$

Here, we take two steps to design a linear controller to globally asymptotically stabilize error system (30).

Step 1. From the proof process of Theorem 5 we know that if the matrix $A(\cdot)$ satisfies assumption (3) in Theorem 5, system (15) is asymptotically stable. If we can design a controller u_1 such that it has a similar form to (15), e_1 will be stable. Let $u_1 = -(25\alpha + 10)e_2$ and the first subsystem of (28) becomes

$$\frac{d^{q_1}e_1}{dt^{q_1}} = -(25\alpha + 10)e_1. \tag{31}$$

Obviously, for each $\alpha \in [0, 1]$, it is globally asymptotically stable at $e_1 = 0$.

Step 2. A similar idea to u_1 is used in the design of the controller u_2. Let $u_2 = -ke_2$ $(k > 29\alpha - 1)$. Consider the remaining subsystem of (30) and substitute $u_2 = -ke_2$ ($k > 29\alpha - 1$) into the remaining subsystem. Then we can have

$$\begin{aligned}\frac{d^{q_2}e_2}{dt^{q_2}} &= (28-35\alpha)e_1 - e_1e_3 - e_1x_3 - e_3x_1 \\ &\quad + (29\alpha-1)e_2 + ke_2 \\ \frac{d^{q_3}e_3}{dt^{q_3}} &= e_1e_2 + e_1x_2 + e_2x_1 - \frac{(8+\alpha)e_3}{3}.\end{aligned} \quad (32)$$

Equation (32) can be rewritten by the matrix form as

$$\begin{bmatrix} d^{q_2}e_2 \\ d^{q_3}e_3 \end{bmatrix} = A\begin{bmatrix} e_2 \\ e_3 \end{bmatrix} + Be_1, \quad (33)$$

where

$$A = \begin{bmatrix} k+29\alpha-1 & -e_1-x_1 \\ e_1+x_1 & -\dfrac{(8+\alpha)}{3} \end{bmatrix}, \qquad B = \begin{bmatrix} 28-35\alpha \\ 0 \end{bmatrix} + \begin{bmatrix} -x_3 \\ x_2 \end{bmatrix}. \quad (34)$$

Because master system (27) is a chaotic system, its states are bounded. This means that there exists a positive constant $\lambda > 0$ such that

$$|x_i| \le \lambda \quad (i = 1,2,3). \quad (35)$$

So the matrix B is also a bounded matrix. Obviously, $A(x)$ has a special form that satisfies assumption (3) in Theorem 5. Form Theorem 5, error system (30) is globally asymptotically stable at $E(0,0,0)$, and slave system (28) will synchronize with master system (27).

3.2. Robustness Analysis. In order to verify the robustness of the linear chaos controller, the disturbance signal is added to the fractional-order unified chaotic system. Master system (27) and slave system (28) can be rewritten as the following forms:

$$\begin{aligned}\frac{d^{q_1}x_1}{dt^{q_1}} &= (25\alpha+10)x_2 - (25\alpha+10+\Delta_1)x_1 \\ \frac{d^{q_2}x_2}{dt^{q_2}} &= (28-35\alpha+\Delta_2)x_1 - (x_1+\Delta_3)x_3 \\ &\quad + (29\alpha-1+\Delta_4)x_2 \\ \frac{d^{q_3}x_3}{dt^{q_3}} &= (x_1+\Delta_3)x_2 + \Delta_5x_1 - \frac{(8+\alpha+\Delta_6)x_3}{3},\end{aligned} \quad (36)$$

$$\begin{aligned}\frac{d^{q_1}y_1}{dt^{q_1}} &= (25\alpha+10)y_2 - (25\alpha+10+\Delta_1)y_1 + u_1 \\ \frac{d^{q_2}y_2}{dt^{q_2}} &= (28-35\alpha+\Delta_2)y_1 - (y_1+\Delta_3)y_3 \\ &\quad + (29\alpha-1+\Delta_4)y_2 + u_2 \\ \frac{d^{q_3}y_3}{dt^{q_3}} &= (y_1+\Delta_3)y_2 + \Delta_5y_1 - \frac{(8+\alpha+\Delta_6)y_3}{3},\end{aligned} \quad (37)$$

where Δ_i ($i = 1,2,\ldots,6$) express the independent uncertainties which may be time varying or state dependent but are bounded by a constant $\gamma > 0$:

$$\sup|\Delta_i| \le \gamma \quad (i = 1,2,\ldots,6). \quad (38)$$

Form system (36), we can see that the system model can denote system uncertainties and parameter uncertainties.

Subtract (36) from (37) and we can get the following error system:

$$\begin{aligned}\frac{d^{q_1}e_1}{dt^{q_1}} &= (25\alpha+10)e_2 - (25\alpha+10+\Delta_1)e_1 + u_1 \\ \frac{d^{q_2}e_2}{dt^{q_2}} &= (28-35\alpha+\Delta_2)e_1 - x_3e_1 \\ &\quad - (e_1+x_1+\Delta_3)e_3 + (29\alpha-1+\Delta_4)e_2 \\ &\quad + u_2 \\ \frac{d^{q_3}e_3}{dt^{q_3}} &= \Delta_5e_1 + x_2e_1 + (e_1+x_1+\Delta_3)e_2 \\ &\quad - \frac{(8+\alpha+\Delta_6)e_3}{3}.\end{aligned} \quad (39)$$

Substitute $u_1 = -(25\alpha+10)e_2$ $u_2 = -ke_2$ ($k > 29\alpha - 1$) into system (39) and we can get a matrix form as

$$\begin{aligned}\frac{d^{q_1}e_1}{dt} &= -(25\alpha+10+\Delta_1)e_1 \\ \begin{bmatrix} \dfrac{d^{q_2}e_2}{dt^q} \\ \dfrac{d^{q_3}e_3}{dt^q} \end{bmatrix} &= A'\begin{bmatrix} e_2 \\ e_3 \end{bmatrix} + B'e_1,\end{aligned} \quad (40)$$

where $A' = \begin{bmatrix} k+29\alpha-1+\Delta_4 & -(e_1+x_1+\Delta_3) \\ e_1+x_1+\Delta_3 & -(8+\alpha+\Delta_6)/3 \end{bmatrix}$, $B' = \begin{bmatrix} 28-35\alpha+\Delta_2 \\ \Delta_5 \end{bmatrix} + \begin{bmatrix} -x_3 \\ x_2 \end{bmatrix}$.

When $k + 29\alpha - 1 + \Delta_4 < 0$, $8 + \alpha + \Delta_6 > 0$, together with Theorem 5, the error system (39) is globally asymptotically stable at $E(0,0,0)$. Hence the chaos controller is robust when $k + 29\alpha - 1 + \Delta_4 < 0$, $8 + \alpha + \Delta_6 > 0$.

4. Experimental Analyses

In this section, the predictor-corrector method is used to obtain the solutions of fractional-order differential equations with step size 0.001. If the step size is too large, the simulation results will not converge. Many authors in other references have adopted the step size as 0.001. And we have also adopted variable step size to do the simulation and we found the simulation results were better. In the simulation process for the fractional-order unified system, we let $q_1 = q_2 = q_3 = 0.95$ and choose the initial states of the master and slave system as $(x_1(0), x_2(0), x(0))^T = (1,2,3)^T$, $(y_1(0), y_2(0), y(0))^T = (-1,-2,6)^T$.

When $\alpha = 0$ system (26) is a fractional-order Lorenz chaotic system. When $\alpha = 0.8$ system (26) is

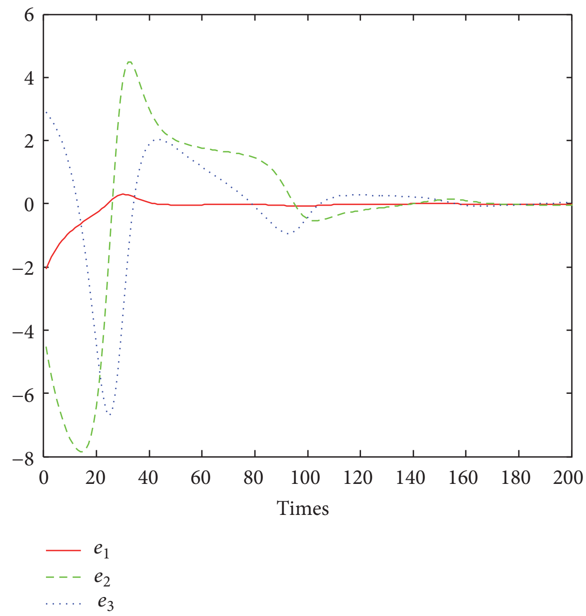

FIGURE 3: Synchronization errors of the fractional-order Lorenz system ($\alpha = 0$).

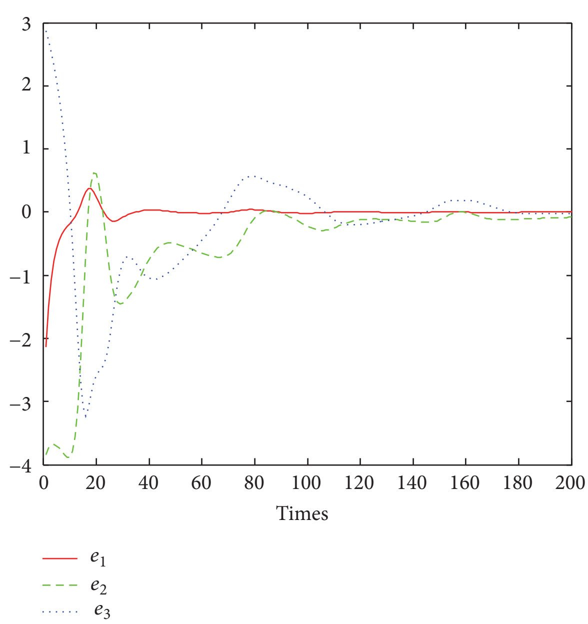

FIGURE 4: Synchronization errors of the fractional-order Lü system ($\alpha = 0.8$).

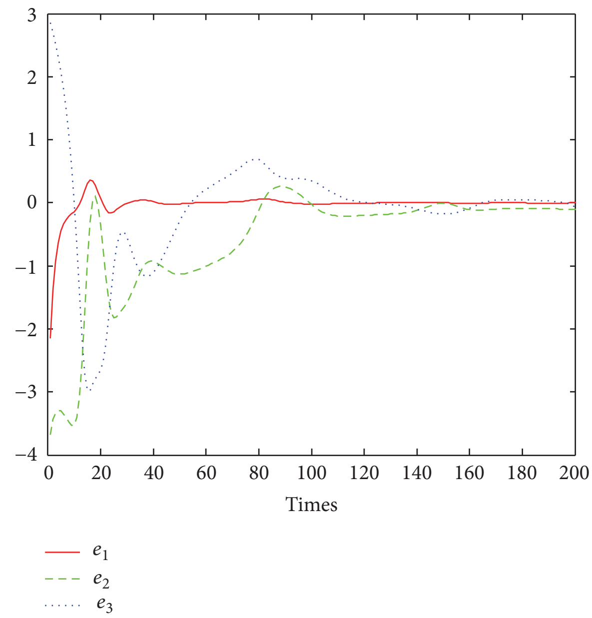

FIGURE 5: Synchronization errors of the fractional-order Chen system ($\alpha = 1$).

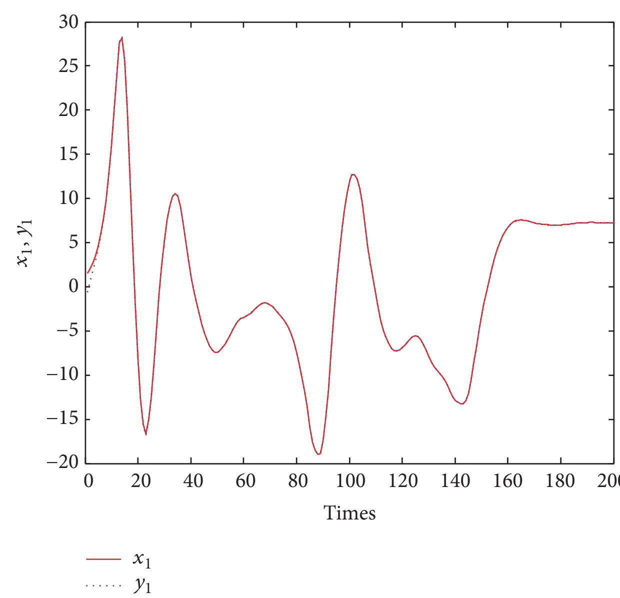

FIGURE 6: State trajectories of an uncertain fractional-order Chen system $x_1 - y_1 (\alpha = 1)$.

a fractional-order Lü chaotic system and when $\alpha = 1$ system (26) is a fractional-order Chen chaotic system. The simulation results for synchronization of the fractional-order Lorenz, Lü, and Chen chaotic systems with known parameters are shown in Figures 3–5, respectively. We chose the uncertain parameters $(\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5, \Delta_6)^T = (\sin x_1, 2\cos x_2, \sin x_3, \cos y_1, \sin y_2, 1)^T$. Figures 6–8 show the synchronization of two identical Chen chaotic systems with uncertain parameters. As expected, one can see that the trajectories of the closed loop slave system can synchronize the trajectories of the master system. These results of the simulation verify the effectiveness of the proposed scheme.

5. Conclusions

This paper discusses a linear control scheme for synchronizing a class of cascade fractional-order chaotic systems. Based on continuous Lyapunov equation, the stability of the closed loop system is proved. This synchronization approach

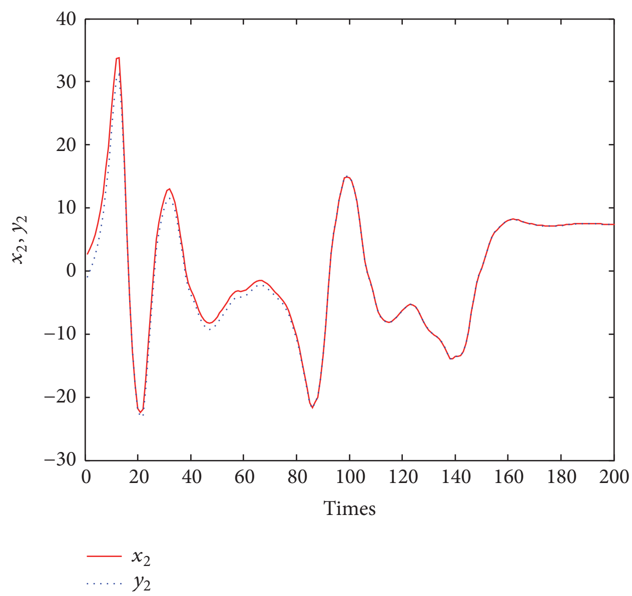

Figure 7: State trajectories of an uncertain fractional-order Chen system $x_2 - y_2 (\alpha = 1)$.

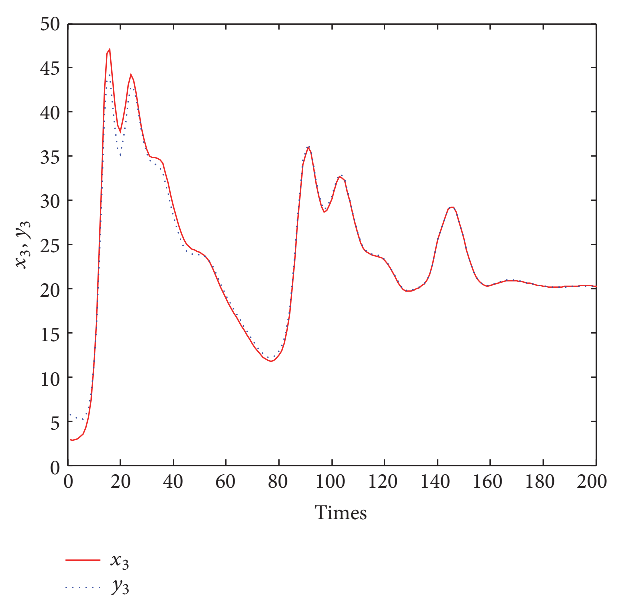

Figure 8: State trajectories of an uncertain fractional-order Chen system $x_3 - y_3 (\alpha = 1)$.

is simple, global, and theoretically rigorous. Numerical simulations have been used to clarify the effectiveness of the proposed control laws. It should be noted that the introduced fractional linear controller is applicable for a large class of commensurate fractional-order chaotic systems in the presence of model uncertainties and eternal disturbances.

Competing Interests

The authors declare that they have no competing interests.

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