Hindawi Publishing Corporation
Advances in Operations Research
Volume 2014, Article ID 985453, 12 pages
http://dx.doi.org/10.1155/2014/985453

Research Article

Insensitive Bounds for the Stationary Distribution of a Single Server Retrial Queue with Server Subject to Active Breakdowns

Mohamed Boualem

Laboratory of Modeling and Optimization of Systems (LaMOS), University of Bejaia, 06000 Bejaia, Algeria

Correspondence should be addressed to Mohamed Boualem; robertt15dz@yahoo.fr

Received 25 March 2013; Revised 27 November 2013; Accepted 16 February 2014; Published 24 March 2014

Academic Editor: Khosrow Moshirvaziri

The paper addresses monotonicity properties of the single server retrial queue with no waiting room and server subject to active breakdowns. The obtained results allow us to place in a prominent position the insensitive bounds for the stationary distribution of the embedded Markov chain related to the model in the study. Numerical illustrations are provided to support the results.

1. Introduction

Queueing systems with repeated attempts have been widely used to model many problems in telecommunication and computer systems [1–3]. The essential feature of a retrial queue is that arriving customers who find all servers busy are obliged to abandon the service area and join a retrial group, called orbit, in order to try their luck again after some random time. For a detailed review of the main results and the literature on this topic the reader is referred to the monographs [4, 5]. In recent years, there has been an increasing interest in the investigation of the retrial phenomenon in cellular mobile network, see [6–10] and the references therein, and in many other telecommunication systems including starlike local area networks [11], wavelength-routed optical networks [12], circuit-switched systems with hybrid fiber-coax architecture [13], and wireless sensor networks [14].

On the other hand, in most of the queueing literature, the server is assumed to be always available, although this assumption is evidently unrealistic. In fact, queueing systems with server breakdowns are very common in communication systems and manufacturing systems, the machine may break down due to the machine or job related problems. This results in a period of unavailable time until the servers are repaired. Such a system with repairable server has been studied as a queueing model and a reliability model by many authors. Aissani [15] studies the influence of the reliability of the communication line on the distribution of the number of customers in the $M/G/1/1$ retrial queues. A generalization of the well-known Pollaczek-Khinchin formula is given for this case. Aissani [16] considers a retrial queue with redundancy and unreliable server. Dudin [17] treats a problem similar to [16], and the problem of redundancy and related control problem are also discussed. Djellab [18] studies a system with breakdowns in heavy traffic. Kumar et al. [19] consider an $M/G/1$ retrial queue with feedback and starting failure, which occurs in the startup period and its repair can be interpreted as a warm up period (the server is unavailable to customers). Retrial queues with a server subject to breakdowns and repairs are investigated in [20], where the limiting behavior of two models is considered by using the tools of Markov regenerative processes. Aissani and Artalejo [21] deal with a single server retrial queueing system subject to active and independent breakdowns. Wang et al. [22] study the active breakdowns model from the viewpoint of reliability and some main reliability indices are obtained along with queueing characteristics. Atencia et al. [23] analysed a retrial queue with active breakdowns where the interrupted customers have the option of joining the orbit or remaining in the server for the repair in order to conclude their remaining service. Djellab [24] considered an approximation method for the study of queue size distribution of an unreliable $M/G/1$ with general retrial distribution based on the stochastic decomposition property. Wang and Li [25] investigated a repairable $M/G/1$ retrial queue with Bernoulli vacation, setup times and two-phase service allowing balking of new arriving customers and reneging of customers in the retrial queue. Sherman et al. [26] study an $M/G/1$ retrial queue in which the server is subject to failures and repairs. Only customers who are interrupted through the server failures enter into a retrial orbit of infinite size. Assuming the service times to be generally distributed and all other times to be exponentially distributed, the authors provide conditions for the system stability. By applying the supplementary variables method, Taleb et al. [27] investigate an $M/G/1$ retrial queue with unreliable server in which an arriving customer decides to leave without service or enter into a retrial orbit of infinite size according to a Bernoulli trial when the server is busy. For a review of main results and methods, the reader is referred to the survey paper by Krishnamoorthy et al. [28] and references therein.

An examination of the literature reveals the remarkable fact that the nonhomogeneity caused by the flow of repeated attempts is the key to understand most analytical difficulties arising in the study of retrial queues. Many efforts have been devoted to deriving performance measures such as queue length, waiting time, and busy period distributions. However, these performance characteristics have been provided through transform methods which have made the expressions cumbersome and the obtained results cannot be put into practice. In the last decade, there has been a tendency towards the research of approximations and bounds. Qualitative properties of stochastic models constitute an important theoretical basis for approximation methods. One of the important qualitative properties and approximation methods is monotonicity which can be studied using the general theory of stochastic ordering [29].

There is a significant body of literature on monotonicity results in retrial queues and networks. Liang and Kulkarni [30] study the monotonicity properties of retrial queues in order to investigate how the retrial time distribution affects the behavior of the system. They assume that retrial times have phase type distributions and show that systems with longer retrial times, with respect to the K-dominance, create more customers in the system and in the orbit. From these results, they derive monotonicity properties of several performance measures of interest. Liang [31] shows that if the hazard rate function of the retrial time distribution is decreasing, then stochastically longer service time or less servers will result in more customers in the system. Khalil and Falin [32] investigate some monotonicity properties of an $M/G/1$ retrial queue with exponential retrial times and linear retrial rate. They show that the number of customers in steady state stochastically decreases when the arrival rate decreases with increasing retrial rate and decreasing service time either stochastically or in the convex ordering. Inequalities are derived for the mean characteristics of the busy period and the number of customers served during a busy period. Boualem et al. [33] investigate some monotonicity properties of an $M/G/1$ queue with constant retrial policy in which the server operates under a general exhaustive service and multiple vacation policy relative to strong stochastic ordering and convex ordering. Taleb and Aissani [34] show that if the distribution of the retrial time is close to the exponential distribution in Laplace transform, then the exponential bound is closer to the exact value than the deterministic bound. Otherwise, the deterministic bound is better. More recently, Boualem et al. [35] use the tools of a qualitative analysis to investigate various monotonicity properties for an $M/G/1$ retrial queue with classical retrial policy and Bernoulli feedback. The obtained results allow to place in a prominent position the insensitive bounds for both the stationary distribution and the conditional distribution of the stationary queue of the considered model. Mokdad and Castel-Taleb [36] propose to use a mathematical method based on stochastic comparisons of Markov chains in order to derive performance indices bounds of fixed and mobile networks. Their main objective consists in finding Markovian bounding models with reduced state spaces, which are easier to solve. They apply the methodology to performance evaluation of complex telecommunication systems modeled by large size Markov chains which cannot be solved by exact methods. Using stochastic comparisons methods, they prove that the new systems represent bounds for the exact ones. To validate their approach and illustrate its interest, they present some numerical results. Bušić and Fourneau [37] illustrate through examples how monotonicity may help for performance evaluation of mobile networks, by considering two different applications. In the first one, they assume that a Markov chain of the model depends on a parameter that can be estimated only up to a certain level and they have only an interval that contains the exact value of the parameter. Instead of taking an approximated value for the unknown parameter, they show how monotonicity properties of the Markov chain can be used to take into account the error bound from the measurements. In the second application, they consider a well-known approximation method: the decomposition into Markovian submodels. They show that the monotonicity property may help to derive bounds for Markovian submodels and are sufficient conditions for convergence of iterative algorithms which are often designed to give approximations.

In this paper, we use the general theory of stochastic ordering to study monotonicity properties similar to that of Boualem et al. [35], for a single server retrial queue with server subject to active breakdowns, that is, the service station can fail only during the service period, relative to the strong stochastic ordering, convex ordering, and Laplace ordering. The obtained results give insensitive bounds for the stationary distribution of the considered embedded Markov chain. The rest of the paper is organized as follows. In the next section, we describe the mathematical model. The embedded Markov chain at departure epochs are investigated in Section 3. The monotonicity properties of the latter are discussed in Section 4, and the stochastic inequalities for its stationary distribution are given in Section 5. The last Section is devoted to the practical aspect.

2. The Mathematical Model

We consider a single server queueing system in which new customers (primary calls) arrival in a Poisson process with rate λ. We assume that there is no waiting space and therefore if an arriving customer finds the server idle, the customer

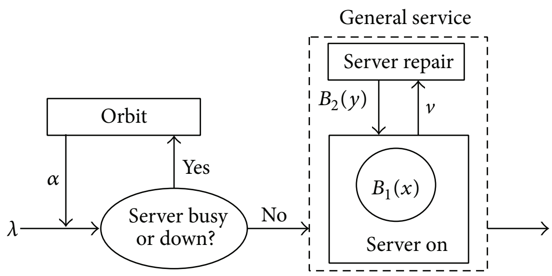

FIGURE 1: Dynamics of the queueing system.

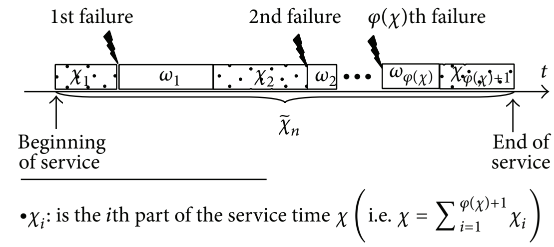

FIGURE 2: The generalized service time of the nth customer.

obtains service immediately and leaves the system after service completion. Otherwise, if the server is found busy or down, the customer makes a retrial at a later time and then the arriving customer becomes a source of repeated calls (a customer in retrial group). The pool of sources of repeated calls may be viewed as a sort of queue with infinite capacity. It is assumed that the retrial times for any repeated customer are exponentially distributed with rate α/n given that there are n customers in orbit. In this case, the retrial rate diminishes as more customers unite to the retrial group. Farahmand [38] calls the retrial queue with this retrial rate control policy a retrial queue with discouraged repeated demands.

The service times are independent and identically distributed with common distribution function $B_1(x)$, Laplace-Stieltjes transform $L_{B_1}(s)$ and nth moments $\beta_{1,n}$. Customers leave the system forever after service completion.

The server may breakdown when serving customers, and when the server fails it is sent to repair directly. The customer just being served before server failure waits for the server to complete his remaining service. We suppose that the server lifetime has exponential distribution with rate ν; that is, the server fails after an exponential time with mean $1/\nu$. It is assumed that the service time for a customer is cumulative, and after repair, the server is as good as new. The repair times follow a general distribution $B_2(y)$ with Laplace-Stieltjes transform $L_{B_2}(s)$ and nth moments $\beta_{2,n}$.

As usual, we suppose that interarrival periods, retrial times, service times, server lifetimes, and repair times are mutually independent. Figure 1 illustrates the dynamics of the queueing system.

At an arbitrary time t, the system can be described by

$$X(t) = (C(t), N(t), \xi_1(t), \xi_2(t)), \tag{1}$$

where $C(t)$ denotes the server state (0, 1, or 2, depending if the server is free, busy, or down) and $N(t)$ is the number of repeated customers at time t. If $C(t) = 1$, then $\xi_1(t)$ represents the elapsed service time of the customer currently being served. If $C(t) = 2$, then $\xi_1(t)$ means the elapsed service time for the customer under service and $\xi_2(t)$ symbolizes the elapsed repair time.

From this description, it is clear that the evolution of our retrial queue can be described in terms of an alternating sequence of idle and busy periods for the server. After each service, the next customer to be served is determined by a competition between two exponential laws of rates λ and $(1-\delta_{0,n})\alpha$, given that the previous service time left n customers in orbit. This is the main difference with classical waiting lines without retrials.

We define $\tilde{\chi}_n$ to be the generalized service time of the nth customer; that is, the length of time since the nth customer begins to be served until the service is completed, where $\tilde{\chi}_n$ includes the service time χ_n and some eventual repair times (ω_i). It is obvious that $\tilde{\chi}_n$ is independent of n, $n = 1, 2, \ldots$. Figure 2 illustrates the details of the general service presented on Figure 1 to locate the different variables in the model.

By assumption, repeated calls have no effects on $\tilde{\chi}_n$. Hence, some results obtained in [39], where the classical $M/G/1$ queueing system with repairable server was studied, can be used here. In order to obtain the distribution function of $\tilde{\chi}_n$, $n = 1, 2, \ldots$, define

$$\begin{aligned}
&\tilde{B}_n^{(l)}(t) \\
&\quad = \Pr\{\tilde{\chi}_n \le t \text{ and server just fails } l \text{ times during} \\
&\qquad \text{the interval since the } n\text{th customer begins} \\
&\qquad \text{to be served until the service is completed}\}, \\
&\qquad\qquad n \ge 1, \quad l \ge 0, \quad t \ge 0.
\end{aligned} \tag{2}$$

Then, it can be shown in [39] that the generalized successive service times $\tilde{\chi}_n$ are identically distributed, independent random variables with distribution function:

$$\begin{aligned}
\tilde{B}(t) &\triangleq \tilde{B}_n(t) = \Pr[\tilde{\chi}_n \leqslant t] \\
&= \sum_{l=0}^{\infty} \int_0^t B_2^{(l)}(x-u)\, e^{-\nu u} \frac{(\nu u)^l}{l!} dB_1(u),
\end{aligned} \tag{3}$$

which is independent of n. Its Laplace-Stieltjes transform is

$$L_{\tilde{B}}(s) = \int_0^{\infty} e^{-st} \tilde{B}(t) = L_{B_1}\left(s + \nu - \nu L_{B_2}(s)\right), \quad \operatorname{Re}(s) > 0, \tag{4}$$

and its expected value is given by

$$E(\tilde{\chi}_n) = -\left.\frac{dL_{\tilde{B}}(s)}{ds}\right|_{s=0} = \beta_{1,1}(1 + \nu\beta_{2,1}). \tag{5}$$

We denote by $\widetilde{\chi}$ the generic random variable corresponding to the sequence $\{\widetilde{\chi}_n\}$ of independent, identically distributed random variables with common distribution $\widetilde{B}(x)$ and Laplace-Stieltjes transform $L_{\widetilde{B}}(s)$. According to these assumptions, we have

$$\widetilde{\chi} = \chi + \sum_{i=1}^{\varphi(\chi)} \omega_i, \tag{6}$$

where $\varphi(\chi)$ is the number of failures during the interval since the customer in service begins to be served until the service is completed.

3. Embedded Markov Chain

Let τ_i be the time of the ith departure and $N_i = N(\tau_i+)$ the number of repeated customers just after the time τ_i. It is not difficult to see the following recursive equation:

$$N_i = N_{i-1} - W_i + \eta_i, \tag{7}$$

where $W_i = 1$ or 0, depending on whether the customer who leaves the system at time τ_i proceeds from the orbit or not, and η_i represents the number of customers who enter the system during the generalized service time of the ith customer.

The random variable W_i depends on the history of the system before the time τ_{i-1} only through the variable N_{i-1} and its conditional distribution is given by

$$\begin{aligned} P\left(W_i = 1 \mid N_{i-1} = n\right) &= \frac{\left(1-\delta_{0,n}\right)\alpha}{\lambda + \left(1-\delta_{0,n}\right)\alpha}, \\ P\left(W_i = 0 \mid N_{i-1} = n\right) &= \frac{\lambda}{\lambda + \left(1-\delta_{0,n}\right)\alpha}. \end{aligned} \tag{8}$$

The distribution of the random variable η_i is given by the following formula:

$$b_n = P\left[\eta_i = n\right] = \int_0^\infty \frac{(\lambda x)^n}{n!} e^{-\lambda x} d\widetilde{B}(x), \quad n \geq 0, \tag{9}$$

where $\widetilde{B}(x)$ is the distribution function of the time a customer remains in the server. It is easy to show that

$$\begin{aligned} \sum_{n=0}^{\infty} b_n z^n &= L_{\widetilde{B}}(\lambda - \lambda z), \\ E\left(\eta_i\right) &= \sum_{n=0}^{\infty} n b_n = \rho\left(1 + \nu\beta_{2,1}\right), \end{aligned} \tag{10}$$

where $L_{\widetilde{B}}(s) = L_{B_1}(s + \nu - \nu L_{B_2}(s))$ is the Laplace-Stieltjes transform of the time a customer stays in the service station and $\rho = \lambda\beta_{1,1}$ is the load of the system.

The previous comments imply that the sequence of random variables $\{N_i\}_{i=0}^{\infty}$ forms a Markov chain with $\{0, 1, 2, \ldots\}$ as state space, which is the embedded Markov chain for our queueing system. It is not difficult to see that $\{N_i,\ i \in \mathbb{N}\}$ is irreducible and aperiodic (see (7)).

The one step transition probabilities of $\{N_i,\ i \in \mathbb{N}\}$ are defined by the following manner:

$$p_{n,m} = \begin{cases} b_m, & \text{if } n = 0, \\ \dfrac{\alpha}{\lambda+\alpha} b_0, & \text{if } n = m+1, \\ \dfrac{\lambda}{\lambda+\alpha} b_{m-n} + \dfrac{\alpha}{\lambda+\alpha} b_{m-n+1}, & \text{if } 1 \leq n \leq m, \\ 0, & \text{otherwise}. \end{cases} \tag{11}$$

From (5), we can see that in order to complete the service of one customer, the server must spend on average $\beta_{1,1}(1 + \nu\beta_{2,1})$ units of time during which $\lambda\beta_{1,1}(1 + \nu\beta_{2,1})$ more customers will arrive on average. Therefore, for the system to be stable, we must have $\rho(1+\nu\beta_{2,1}) < 1 - (\lambda/(\lambda+\alpha))$. Indeed, we will use Foster's criterion (see [40]). The mean drifts are given by

$$\begin{aligned} x_n &= E\left[N_{i+1} - N_i \mid N_i = n\right] \\ &= E\left[\eta_{i+1} \mid N_i = n\right] - E\left[W_{i+1} \mid N_i = n\right] \\ &= \lambda\beta_{1,1}\left(1 + \nu\beta_{2,1}\right) - \frac{\alpha}{\lambda+\alpha}. \end{aligned} \tag{12}$$

Suppose that $\lambda\beta_{1,1}(1 + \nu\beta_{2,1}) < \alpha/(\lambda+\alpha) = 1 - (\lambda/(\lambda+\alpha))$.

Then $\varepsilon = (1/2)[1 - (\lambda/(\lambda+\alpha)) - \lambda\beta_{1,1}(1+\nu\beta_{2,1})]$ is positive and there exists

$$\lim_{n\to\infty} x_n = -1 + \frac{\lambda}{\lambda+\alpha} + \lambda\beta_{1,1}\left(1 + \nu\beta_{2,1}\right) = -2\varepsilon < -\varepsilon. \tag{13}$$

Hence, $x_n < -\varepsilon$ for all the states except a finite number. Therefore,

$$\lambda\beta_{1,1}\left(1 + \nu\beta_{2,1}\right) < 1 - \frac{\lambda}{\lambda+\alpha} \tag{14}$$

is a sufficient condition for the ergodicity of the embedded Markov chain.

To prove that the previous condition is also a necessary condition for ergodicity of our embedded Markov chain, we apply Kaplan's condition: $x_i < \infty$, for all $i \geq 0$, and there is an i_0 such that $x_i \geq 0$, for $i \geq i_0$. In our case, this condition is verified because $p_{ij} = 0$ for $j < i - 1$ and $i > 0$ (see (11)).

Remark 1. (i) When $\nu = 0$ (without breakdowns), our system reduces to the $M/G/1$ retrial queue with reliable server and constant repeated attempts [41].

(ii) When the discipline of retrials is considered classical policy, our system becomes the $M/G/1$ retrial queue with active breakdowns, where the interrupted customer stays at the server waiting for the repair in order to complete his remaining service [22].

(iii) It should be also pointed out that our system can be considered as a queueing model with server vacations where the server commences the vacations whenever a service finishes. The length of the vacations depends on the arrival process, the number of repeated customers and the interretrial times; thus, the vacations conclude when an external customer arrives or the server randomly selects one of the repeated customers. In this way, the end of the vacations is determined by a competition between two exponential laws of rates λ and $(1-\delta_{0,n})\alpha$, given that at the beginning of the vacations there were n customers in the orbit. With this definition, it is not surprising that our system verifies the property stated by Fuhrmann and Cooper [42].

4. Monotonicity Properties of the Embedded Markov Chain

Stochastic orders lead to powerful approximation methods and bounds in situations where realistic stochastic models are too complex for rigorous treatment. They are also helpful in situations where fundamental model distributions are only known partially. In economics, there are valuable tools in the theory of individual decision under risk, where a decision maker has to compare actions leading to different uncertain payments. Important fields of application are, amongst others, queueing theory, reliability theory, statistical physics, epidemiology, and insurance mathematics [29, 43].

We first introduce the following notions on stochastic orderings. Here, we only note specifically the following fact, which will be used later on.

Let $F_1(x)$ and $F_2(x)$ be two distribution functions of nonnegative random variables. Then

(a) $F_1 \leq_{st} F_2$ if and only if $F_1(x) \geq F_2(x)$ for all $x \geq 0$,

(b) $F_1 \leq_v F_2$ if and only if $\int_x^{+\infty}(1-F_1(u))d(u) \leq \int_x^{+\infty}(1-F_2(u))d(u)$, for all $x \geq 0$,

(c) $F_1 \leq_L F_2$ if and only if $\int_0^{+\infty} e^{-sx}dF_1(x) \geq \int_0^{+\infty} e^{-sx}dF_2(x)$, for all $s \geq 0$.

If these random variables are of the discrete type and $p^{(1)} = (p_n^{(1)})$, $p^{(2)} = (p_n^{(2)})$ are the corresponding distributions, then the above definitions can be given in the following manner.

(a) $p^{(1)} \leq_{st} p^{(2)}$ if and only if $\sum_{n=m}^{\infty} p_n^{(1)} \leq \sum_{n=m}^{\infty} p_n^{(2)}$ for all m.

(b) $p^{(1)} \leq_v p^{(2)}$ if and only if $\sum_{n=m}^{\infty}\sum_{l=n}^{\infty} p_l^{(1)} \leq \sum_{n=m}^{\infty}\sum_{l=n}^{\infty} p_l^{(2)}$, for all m.

(c) $p^{(1)} \leq_L p^{(2)}$ if and only if $\sum_{n\geq 0} p_n^{(1)} z^n \geq \sum_{n\geq 0} p_n^{(2)} z^n$, for all $z \in [0,1]$.

Let X be a positive random variable with distribution function F and mean m.

(a) F is NBUE (New Better than Used in Expectation) if and only if $F \leq_v F^*$,

(b) F is NWUE (New Worse than Used in Expectation) if and only if $F \geq_v F^*$,

where F^* is the exponential distribution function with the same mean as F.

Now, we study monotonicity properties of our embedded Markov chain $\{N_i\}_{i=0}^{\infty}$ relative to the strong stochastic ordering $\leq_{st}$, the convex ordering $\leq_v$, and Laplace ordering $\leq_L$.

Let $\Sigma^{(1)}$ and $\Sigma^{(2)}$ be two $M/G/1$ retrial queues with server subject to active breakdowns defined by

$$\begin{gathered} \lambda^{(1)}, \alpha^{(1)}, \nu^{(1)}, B_1^{(1)}(x), B_2^{(1)}(x), b_i^{(1)}, \\ \lambda^{(2)}, \alpha^{(2)}, \nu^{(2)}, B_1^{(2)}(x), B_2^{(2)}(x), b_i^{(2)}, \end{gathered} \tag{15}$$

respectively. Let $\chi^{(i)}$ and $\tilde{\chi}^{(i)}$ be the service time and the generalized service time in the ith system, $i = 1, 2$.

Lemma 2. *(1) If $\nu^{(1)} \leq \nu^{(2)}$ and $B_1^{(1)} \leq_{st} B_1^{(2)}$, then $\varphi^{(1)}(\chi^{(1)}) \leq_{st} \varphi^{(2)}(\chi^{(2)})$.*

(2) If $\nu^{(1)} \leq \nu^{(2)}$, $B_1^{(1)} \leq_{st} B_1^{(2)}$, and $B_2^{(1)} \leq_{st} B_2^{(2)}$, then $\tilde{\chi}^{(1)} \leq_{st} \tilde{\chi}^{(2)}$.

Proof. (1) By definition,

$$\begin{aligned} \sum_{j=m}^{\infty}\left[\varphi^{(i)}\left(\chi^{(i)}\right)=j\right] &= \sum_{j=m}^{\infty}\int_0^{\infty}\frac{\left(\nu^{(i)}x\right)^j}{j!}e^{-\nu^{(i)}x}dB_1^{(i)}(x) \\ &= \int_0^{\infty}\left[\sum_{j=m}^{\infty}\frac{\left(\nu^{(i)}x\right)^j}{j!}e^{-\nu^{(i)}x}\right]dB_1^{(i)}(x). \end{aligned} \tag{16}$$

Consider $f_m(x,\nu) = \sum_{j=m}^{\infty}((\nu x)^j/j!)e^{-\nu x}$, this is an increasing function with respect to ν and x:

$$\begin{aligned} \left(\frac{\partial}{\partial x}\right) f_m(x,\nu) &= \nu\left(\frac{(\nu x)^{m-1}}{(m-1)!}\right)e^{-\nu x} > 0, \\ \left(\frac{\partial}{\partial \nu}\right) f_m(x,\nu) &= x\left(\frac{(\nu x)^{m-1}}{(m-1)!}\right)e^{-\nu x} > 0. \end{aligned} \tag{17}$$

Under the assumption that $B_1^{(1)}(x) \leq_{st} B_1^{(2)}(x)$ and with the help of Theorem 1.2.2 given in [43] and by monotonicity of $f_m(x,\nu)$ with respect to ν, one can find that

$$\begin{aligned} \int_0^{\infty} f_m\left(x,\nu^{(1)}\right)dB_1^{(1)}(x) &\leq \int_0^{\infty} f_m\left(x,\nu^{(1)}\right)dB_1^{(2)}(x) \\ &\leq \int_0^{\infty} f_m\left(x,\nu^{(2)}\right)dB_1^{(2)}(x). \end{aligned} \tag{18}$$

Therefore,

$$\begin{gathered} \sum_{j=m}^{\infty}\left[\varphi^{(1)}\left(\chi^{(1)}\right)=j\right] \leq \sum_{j=m}^{\infty}\left[\varphi^{(2)}\left(\chi^{(2)}\right)=j\right] \\ \text{or} \quad \varphi^{(1)}\left(\chi^{(1)}\right) \leq_{st} \varphi^{(2)}\left(\chi^{(2)}\right). \end{gathered} \tag{19}$$

(2) To prove that $\widetilde{\chi}^{(1)} \leq_{st} \widetilde{\chi}^{(2)}$, we have to establish the usual numerical inequality $E[g(\widetilde{\chi}^{(1)})] \leq E[g(\widetilde{\chi}^{(1)})]$ for all bounded differentiable increasing functions g. In our case,

$$E\left[g\left(\widetilde{\chi}^{(i)}\right)\right] = E\left[g\left\{\chi^{(i)} + \sum_{k=1}^{\varphi^{(i)}(\chi^{(i)})} \omega_k^{(i)}\right\}\right] = \int_0^\infty E\left[g\left\{x + \sum_{k=1}^{\varphi^{(i)}(x)} \omega_k^{(i)}\right\}\right] dB_1^{(i)}(x) = \int_0^\infty h^{(i)}(x)\, dB_1^{(i)}(x). \tag{20}$$

By direct calculation we can obtain

$$h(x) = \sum_{n=0}^{\infty} E\left[g\left\{x + \sum_{k=1}^{n} \omega_k\right\}\right] \frac{(\nu x)^n}{n!} e^{-\nu x} = \sum_{n=0}^{\infty} g_n(x) \frac{(\nu x)^n}{n!} e^{-\nu x}, \tag{21}$$

differentiating in x. Since g is increasing and (ω_k) are positive random variable, then $g_n(x)$ is increasing in n and x. Therefore, $h(x)$ is an increasing function. Finally, we obtain $h^{(1)}(x) \leq h^{(2)}(x)$ and

$$\int_0^\infty h^{(1)}(x)\, dB_1^{(1)}(x) \leq \int_0^\infty h^{(2)}(x)\, dB_1^{(1)}(x) \leq \int_0^\infty h^{(2)}(x)\, dB_1^{(2)}(x). \tag{22}$$

□

Lemma 3. *If $\lambda^{(1)} \leq \lambda^{(2)}$, $\nu^{(1)} \leq \nu^{(2)}$, $B_1^{(1)} \leq_{st} B_1^{(2)}$, and $B_2^{(1)} \leq_{st} B_2^{(2)}$, then $\{b_n^{(1)}\} \leq_{st} \{b_n^{(2)}\}$.*

Proof. We have

$$\overline{b}_n^{(i)} = \sum_{j\geq n} b_j^{(i)} = \int_0^{+\infty} \sum_{j\geq n} \frac{\left(\lambda^{(i)} x\right)^j}{j!} e^{-\lambda^{(i)} x} d\widetilde{B}^{(i)}(x), \quad i = 1, 2. \tag{23}$$

To prove that $\{b_n^{(1)}\} \leq_{st} \{b_n^{(2)}\}$, we have to establish the usual numerical inequality:

$$\overline{b}_n^{(1)} = \sum_{m\geq n} b_m^{(1)} \leq \overline{b}_n^{(2)}. \tag{24}$$

The function $f(x,\lambda) = \sum_{j\geq n}((\lambda x)^j/j!)e^{-\lambda x}$ is increasing in x and λ.

By Lemma 2, we have $\widetilde{B}^{(1)}(x) \leq_{st} \widetilde{B}^{(2)}(x)$. Then,

$$\int_0^\infty f\left(x, \lambda^{(1)}\right) d\widetilde{B}^{(1)}(x) \leq \int_0^\infty f\left(x, \lambda^{(2)}\right) d\widetilde{B}^{(1)}(x) \leq \int_0^\infty f\left(x, \lambda^{(2)}\right) d\widetilde{B}^{(2)}(x). \tag{25}$$

□

Lemma 4. *(1) If $\nu^{(1)} \leq \nu^{(2)}$ and $B_1^{(1)} \leq_{v} B_1^{(2)}$, then $\varphi^{(1)}(\chi^{(1)}) \leq_{v} \varphi^{(2)}(\chi^{(2)})$.*

(2) If $\nu^{(1)} \leq \nu^{(2)}$, $B_1^{(1)} \leq_{v} B_1^{(2)}$, and $B_2^{(1)} \leq_{v} B_2^{(2)}$, then $\widetilde{\chi}^{(1)} \leq_{v} \widetilde{\chi}^{(2)}$.

Proof. (1) Consider also $\overline{f}_m(x,\nu) = \sum_{j=m}^{\infty} f_j(x,\nu) = \sum_{j=m}^{\infty}\sum_{l=j}^{\infty}((\nu x)^l/l!)e^{-\nu x}$; this is an increasing function with respect to ν, and an increasing and convex one with respect to x:

$$\left(\frac{\partial^2}{\partial x^2}\right) \overline{f}_m(x,\nu) = \nu^2 \left(\frac{(\nu x)^{m-2}}{(m-2)!}\right) e^{-\nu x} > 0. \tag{26}$$

Similarly, with the help of Theorem 1.3.1 (see [43]) and by monotonicity of $\overline{f}_m(x,\lambda)$ with respect to λ, we obtain the result.

(2) Let g be a twice differentiable increasing convex function. To prove that $\widetilde{\chi}^{(1)} \leq_{v} \widetilde{\chi}^{(2)}$, we have to establish the usual numerical inequality:

$$E\left[g\left(\widetilde{\chi}^{(1)}\right)\right] \leq E\left[g\left(\widetilde{\chi}^{(1)}\right)\right], \quad \forall g. \tag{27}$$

In our case, $h(x) = \sum_{n=0}^{\infty} g_n(x)((\nu x)^n/n!)e^{-\nu x}$, which is increasing and convex.

The rest of demonstration is similar to that of Lemma 2.

□

Lemma 5. *If $\lambda^{(1)} \leq \lambda^{(2)}$, $\nu^{(1)} \leq \nu^{(2)}$, $B_1^{(1)} \leq_{v} B_1^{(2)}$, and $B_2^{(1)} \leq_{v} B_2^{(2)}$, then $\{b_n^{(1)}\} \leq_{v} \{b_n^{(2)}\}$.*

Proof. By definition,

$$\overline{\overline{b}}_n^{(i)} = \sum_{j\geq n} \overline{b}_j^{(i)} = \int_0^{+\infty} \sum_{j\geq n}\sum_{l\geq j} \frac{\left(\lambda^{(i)} x\right)^l}{l!} e^{-\lambda^{(i)} x} d\widetilde{B}^{(i)}(x), \quad i = 1, 2. \tag{28}$$

To prove that $\{b_n^{(1)}\} \leq_{v} \{b_n^{(2)}\}$, we have to establish the usual numerical inequality:

$$\overline{\overline{b}}_n^{(1)} = \sum_{m\geq n} \overline{b}_m^{(1)} \leq \overline{\overline{b}}_n^{(2)}. \tag{29}$$

The function $\overline{f}_n(x,\lambda) = \sum_{j\geq n}\sum_{l\geq j}((\lambda x)^l/l!)e^{-\lambda x}$, is increasing in λ and is convex in x.

By Lemma 4, we have $\widetilde{B}^{(1)}(x) \leq_{v} \widetilde{B}^{(2)}(x)$. Then,

$$\int_0^\infty f\left(x, \lambda^{(1)}\right) d\widetilde{B}^{(1)}(x) \leq \int_0^\infty f\left(x, \lambda^{(2)}\right) d\widetilde{B}^{(1)}(x) \leq \int_0^\infty f\left(x, \lambda^{(2)}\right) d\widetilde{B}^{(2)}(x). \tag{30}$$

□

Lemma 6. *(1) If $\nu^{(1)} \leq \nu^{(2)}$ and $B_1^{(1)} \leq_{L} B_1^{(2)}$, then $\varphi^{(1)}(\chi^{(1)}) \leq_{L} \varphi^{(2)}(\chi^{(2)})$.*

(2) If $\nu^{(1)} \leq \nu^{(2)}$, $B_1^{(1)} \leq_{L} B_1^{(2)}$, and $B_2^{(1)} \leq_{L} B_2^{(2)}$, then $\widetilde{\chi}^{(1)} \leq_{L} \widetilde{\chi}^{(2)}$.

Proof. (1) We have

$$\sum_{n\geq 0} P\left[\varphi^{(i)}\left(\chi^{(i)}\right)\right] z^n = L_{B_1^{(i)}}\left(\nu^{(i)}(1-z)\right), \quad i=1,2. \tag{31}$$

Let $\nu^{(1)} \leq \nu^{(2)}$ and $B_1^{(1)} \leq_L B_1^{(2)}$. To prove that $\varphi^{(1)}(\chi^{(1)}) \leq_L \varphi^{(2)}(\chi^{(2)})$, we have to establish that

$$L_{B_1^{(1)}}\left(\nu^{(1)}(1-z)\right) \geq L_{B_1^{(2)}}\left(\nu^{(2)}(1-z)\right). \tag{32}$$

The inequality $B_1^{(1)} \leq_L B_1^{(2)}$ implies that $L_{B_1^{(1)}}(s) \geq L_{B_1^{(2)}}(s)$ for all $s \geq 0$.

In particular, for $s = \nu^{(1)}(1-z)$ we have

$$L_{B_1^{(1)}}\left(\nu^{(1)}(1-z)\right) \geq L_{B_1^{(2)}}\left(\nu^{(1)}(1-z)\right). \tag{33}$$

Since any Laplace transform is a decreasing function, $\nu^{(1)} \leq \nu^{(2)}$ implies that

$$L_{B_1^{(2)}}\left(\nu^{(1)}(1-z)\right) \geq L_{B_1^{(2)}}\left(\nu^{(2)}(1-z)\right). \tag{34}$$

By transitivity, (33) and (34) give (32).

(2) For the generalized service time, we have

$$L_{\widetilde{B}^{(i)}} = L_{B_1^{(i)}}\left(s + \nu^{(i)} - \nu^{(i)} L_{B_2^{(i)}}(s)\right), \quad i=1,2. \tag{35}$$

The function $s + \nu - \nu L_{B_2}(s)$ is increasing in ν and decreasing in $L_{B_2}(s)$.

By hypothesis, we have $\nu^{(1)} \leq \nu^{(2)}$ and $B_2^{(1)} \leq_L B_2^{(2)}$, then

$$s + \nu^{(1)} - \nu^{(1)} L_{B_2^{(1)}}(s) \leq s + \nu^{(2)} - \nu^{(2)} L_{B_2^{(2)}}(s). \tag{36}$$

Finally, $B_1^{(1)} \leq_L B_1^{(2)}$ yields

$$\begin{aligned} & L_{B_1^{(1)}}\left(s + \nu^{(1)} - \nu^{(1)} L_{B_2^{(1)}}(s)\right) \\ & \quad \geq L_{B_1^{(2)}}\left(s + \nu^{(2)} - \nu^{(2)} L_{B_2^{(2)}}(s)\right). \end{aligned} \tag{37}$$

□

Lemma 7. *If $\lambda^{(1)} \leq \lambda^{(2)}$, $\nu^{(1)} \leq \nu^{(2)}$, $B_1^{(1)} \leq_L B_1^{(2)}$ and $B_2^{(1)} \leq_L B_2^{(2)}$, then $\{b_n^{(1)}\} \leq_L \{b_n^{(2)}\}$.*

Proof. We have

$$\sum_{n\geq 0} b_n^{(i)} z^n = L_{\widetilde{B}^{(i)}}\left(\lambda^{(i)}(1-z)\right), \quad i=1,2. \tag{38}$$

Let $\lambda^{(1)} \leq \lambda^{(2)}$ and by Lemma 6, we obtain the stated result. □

Let T be the transition operator of our embedded Markov chain, which associates to every distribution $p = \{p_n\}_{n\geq 0}$, a distribution $T_p = \{q_m\}_{m\geq 0}$ such that $q_m = \sum_n p_n p_{n,m}$. From Stoyan [43], T is monotone with respect to $\leq_{\text{st}}$ if and only if

$$\overline{p}_{n-1,m} \leq \overline{p}_{n,m} \quad \forall n, m \tag{39}$$

and is monotone with respect to $\leq_\nu$ if and only if

$$2\overline{\overline{p}}_{n,m} \leq \overline{\overline{p}}_{n-1,m} + \overline{\overline{p}}_{n+1,m} \quad \forall n, m. \tag{40}$$

Here, $\overline{p}_{n,m} = \sum_{l=m}^{\infty} p_{n,l}$ and $\overline{\overline{p}}_{n,m} = \sum_{l=m}^{\infty} \overline{p}_{n,l}$.

Theorem 8. *Consider the embedded Markov chain $\{N_i,\ i \in \mathbb{N}\}$. The transition operator T is monotone with respect to the order $\leq_{st}$ (i.e., for any two distributions $p^{(1)}$ and $p^{(2)}$, the inequality $p^{(1)} \leq_{st} p^{(2)}$ implies that $Tp^{(1)} \leq_{st} Tp^{(2)}$).*

Proof. The one-step transition probabilities $p_{n,m}$ of $\{N_i,\ i \in \mathbb{N}\}$ are given by (11). Thus,

$$\begin{aligned} \overline{p}_{n,m} &= \sum_{l=m}^{\infty}\left[\frac{\lambda}{\lambda+\alpha} b_{l-n} + \frac{\alpha}{\lambda+\alpha} b_{l-n+1}\right] \\ &= \overline{b}_{m-n} - \frac{\alpha}{\lambda+\alpha} b_{m-n} \\ &= \overline{b}_{m-n+1} + \frac{\lambda}{\lambda+\alpha} b_{m-n}. \end{aligned} \tag{41}$$

Consequently,

$$\overline{p}_{n,m} - \overline{p}_{n-1,m} = \frac{\lambda}{\lambda+\alpha} b_{m-n} + \frac{\alpha}{\lambda+\alpha} b_{m-n+1} \geq 0. \tag{42}$$

Finally, T is monotone with respect to $\leq_{\text{st}}$. □

Theorem 9. *Consider the embedded Markov chain $\{N_i,\ i \in \mathbb{N}\}$. The transition operator of our embedded Markov chain $\{N_i,\ i \in \mathbb{N}\}$ is monotone with respect to $\leq_\nu$ (i.e., for any two distributions $p^{(1)}$ and $p^{(2)}$), the inequality $p^{(1)} \leq_\nu p^{(2)}$ implies that $Tp^{(1)} \leq_\nu Tp^{(2)}$.*

Proof. For the embedded Markov chain $\{N_i,\ i \in \mathbb{N}\}$, we have

$$\begin{aligned} \overline{\overline{p}}_{n,m} &= \frac{\lambda}{\lambda+\alpha} \overline{\overline{b}}_{m-n} + \frac{\alpha}{\lambda+\alpha} \overline{\overline{b}}_{m-n+1} \\ &= \overline{\overline{b}}_{m-n} + \frac{\alpha}{\lambda+\alpha} \overline{b}_{m-n} \\ &= \overline{\overline{b}}_{m-n+1} + \frac{\lambda}{\lambda+\alpha} \overline{b}_{m-n}, \end{aligned} \tag{43}$$

$$\begin{aligned} & \overline{\overline{p}}_{n-1,m} + \overline{\overline{p}}_{n+1,m} - 2\overline{\overline{p}}_{n,m} \\ & \quad = \overline{b}_{m-n} + \frac{\alpha}{\lambda+\alpha} \overline{b}_{m-n+1} + \frac{\lambda}{\lambda+\alpha} b_{m-n-1} \geq 0. \end{aligned} \tag{44}$$

Thus, T is monotone with respect to $\leq_\nu$. □

Remark 10. In particular, the above Theorems imply that if at time $t = 0$ the system was empty then the number of customers in the orbit form a monotonically increasing sequence with respect to the above orderings.

Remark 11. The operator T is not monotone with respect to the order $\leq_L$.

Now, we add the transition operators $T^{(1)}$ and $T^{(2)}$ to models $\Sigma^{(1)}$ and $\Sigma^{(2)}$, respectively.

Theorem 12. *If $\lambda^{(1)} \leq \lambda^{(2)}$, $\nu^{(1)} \leq \nu^{(2)}$, $\alpha^{(1)} \geq \alpha^{(2)}$, $B_1^{(1)}(x) \leq_{so} B_1^{(2)}(x)$, and $B_2^{(1)}(x) \leq_{so} B_2^{(2)}(x)$, where $\leq_{so}$ is either $\leq_{st}$ or $\leq_\nu$, then $T^{(1)} \leq_{so} T^{(2)}$; that is, for any distribution p, we have $T^{(1)}p \leq_{so} T^{(2)}p$.*

Proof. The demonstration is based on Theorem 4.2.3 given in [43]. We wish to establish that

$$\overline{p}^{(1)}_{n,m} \le \overline{p}^{(2)}_{n,m} \quad (\text{for st-ordering}), \tag{45}$$

$$\overline{\overline{p}}^{(1)}_{n,m} \le \overline{\overline{p}}^{(2)}_{n,m} \quad (\text{for } v\text{-ordering}). \tag{46}$$

Effectively, from Lemma 3, we have that $\overline{b}^{(1)}_n \le \overline{b}^{(2)}_n$ (for st-ordering). Under the hypothesis that $\lambda^{(1)} \le \lambda^{(2)}$ and $\alpha^{(1)} \ge \alpha^{(2)}$, one can obtain that $\lambda^{(1)}/\alpha^{(1)} \le \lambda^{(2)}/\alpha^{(2)}$. Moreover, the function $x/(x+n)$ is increasing. Consequently, $\lambda^{(1)}/(\lambda^{(1)} + \alpha^{(1)}) \le \alpha^{(2)}/(\lambda^{(2)} + \alpha^{(2)})$.

Finally,

$$\begin{aligned}\overline{p}^{(1)}_{n,m} &= \frac{\lambda^{(1)}}{\lambda^{(1)}+\alpha^{(1)}}\overline{b}^{(1)}_{m-n} + \frac{\alpha^{(1)}}{\lambda^{(1)}+\alpha^{(1)}}\overline{b}^{(1)}_{m-n+1} \\ &= \overline{b}^{(1)}_{m-n+1} + \frac{\lambda^{(1)}}{\lambda^{(1)}+\alpha^{(1)}}\overline{b}^{(1)}_{m-n} \\ &\le \overline{b}^{(1)}_{m-n+1} + \frac{\lambda^{(2)}}{\lambda^{(2)}+\alpha^{(2)}}\overline{b}^{(1)}_{m-n} \\ &\le \frac{\lambda^{(2)}}{\lambda^{(2)}+\alpha^{(2)}}\overline{b}^{(2)}_{m-n} + \frac{\alpha^{(2)}}{\lambda^{(2)}+\alpha^{(2)}}\overline{b}^{(1)}_{m-n+1} \\ &= \overline{p}^{(2)}_{n,m}.\end{aligned} \tag{47}$$

Following the above technique and using Lemma 5, we establish inequality (46). □

Theorem 13. *If $\lambda^{(1)} \le \lambda^{(2)}$, $v^{(1)} \le v^{(2)}$, $\alpha^{(1)} \ge \alpha^{(2)}$, $B^{(1)}_1(x) \le_L B^{(2)}_1(x)$, and $B^{(1)}_2(x) \le_L B^{(2)}_2(x)$, then $T^{(1)} \le_L T^{(2)}$; that is, for any distribution p, we have $T^{(1)}p \le_L T^{(2)}p$.*

Proof. Let $p = (p_m)$ be a distribution and $T_p = q = (q_m)$, where

$$q_m = \sum_{n\ge0} p_n p_{n,m} = p_0 b_m + \sum_{n\ge1} p_n p_{n,m}, \quad \forall m \ge 0. \tag{48}$$

Let $b(z) = \sum_{n\ge0} b_n z^n$ and $p(z) = \sum_{n\ge0} p_n z^n$ be the generating functions of (b_n) and (p_n), respectively.

The generating function of q is given by

$$\begin{aligned} q(z) &= \sum_{m\ge0} q_m z^m = \sum_{m\ge0}\left[p_0 b_m + \sum_{n\ge1} p_n p_{n,m}\right] z^m \\ &= p_0 b(z) + \frac{\lambda}{\lambda+\alpha} b(z) \sum_{n\ge1} p_n z^n \\ &\quad + \frac{\alpha}{\lambda+\alpha} b(z) \sum_{n\ge1} p_n z^{n-1} + \frac{\alpha}{\lambda+\alpha}\frac{p(z)-p_0}{z} b_0 \\ &= p_0 b(z) + \frac{\lambda}{\lambda+\alpha} b(z)(p(z)-p_0) \\ &\quad + \frac{\alpha}{\lambda+\alpha}\frac{b(z)}{z}(p(z)-p_0) + \frac{\alpha}{\lambda+\alpha}\frac{p(z)-p_0}{z} b_0 \\ &= b(z)\left[p_0 + \frac{\lambda z+\alpha}{(\lambda+\alpha)z}(p(z)-p_0)\right] \\ &\quad + \frac{\alpha}{\lambda+\alpha}\frac{p(z)-p_0}{z} b_0. \end{aligned} \tag{49}$$

By Lemma 7, we have $b^{(1)}(z) \ge b^{(2)}(z)$, for all $z \in [0,1]$ and if the conditions of Theorem 13 are fulfilled, then $q^{(1)}(z) \ge q^{(2)}(z)$. □

5. Stochastic Inequalities for the Stationary Distribution

Theorem 14. *Suppose once more that we have two models $\Sigma^{(1)}$ and $\Sigma^{(2)}$ as defined in the previous section. Let $\{N^{(1)}_i, i \in \mathbb{N}\}$, $\{N^{(2)}_i, i \in \mathbb{N}\}$ be the corresponding embedded Markov chains as well as their stationary distributions $\{\pi^{(1)}_n\}$, $\{\pi^{(2)}_n\}$, respectively. Then $\lambda^{(1)} \le \lambda^{(2)}$, $v^{(1)} \le v^{(2)}$, $\alpha^{(1)} \ge \alpha^{(2)}$, $B^{(1)}_1 \le_{so} B^{(2)}_2$, and $B^{(1)}_2(x) \le_{so} B^{(2)}_2(x)$, where $\le_{so}$ is either $\le_{st}$ or $\le_v$, imply that $\{\pi^{(1)}_n\} \le_{so} \{\pi^{(2)}_n\}$.*

Proof. By Theorem 12, the inequalities $\lambda^{(1)} \le \lambda^{(2)}$, $v^{(1)} \le v^{(2)}$, $\alpha^{(1)} \ge \alpha^{(2)}$, $B^{(1)}_1 \le_{so} B^{(2)}_2$, and $B^{(1)}_2(x) \le_{so} B^{(2)}_2(x)$ imply that $T^{(1)} \le_{so} T^{(2)}$; that is, for any distribution p, we have the following inequality:

$$T^{(1)} p \le_{so} T^{(2)} p. \tag{50}$$

According to Theorems 8 and 9, the operator $T^{(2)}$ is monotone; that is, for any two distributions $p^{(2)}_1$, $p^{(2)}_2$ such that $p^{(2)}_1 \le_{so} p^{(2)}_2$, we have

$$T^{(2)} p^{(2)}_1 \le_{so} T^{(2)} p^{(2)}_2. \tag{51}$$

Moreover, from (50), one can obtain

$$T^{(1)} p^{(1)} \le_{so} T^{(2)} p^{(1)}. \tag{52}$$

There exists a probability $p^{(2)}_1$ such that the inequality

$$T^{(2)} p^{(1)} \le_{so} T^{(2)} p^{(2)}_1, \tag{53}$$

takes place.

From (51)–(53), for any two distributions $p^{(1)}$, $p^{(2)}$, one can obtain the following result:

$$T^{(1)} p^{(1)} \le_{so} T^{(2)} p^{(2)}. \tag{54}$$

Therefore,

$$T^{(1)} p^{(1)}_n = P\left(N^{(1)}_l = n\right) \le_{so} P\left(N^{(2)}_l = n\right) = T^{(2)} p^{(2)}_n, \tag{55}$$

when $l \to \infty$, we have $\{\pi^{(1)}_n\} \le_{so} \{\pi^{(2)}_n\}$. □

Theorem 15. *If in the $M/G/1$ retrial queue with server subject to active breakdowns, the service time distribution $B_1(x)$ and* *the repair time distribution $B_2(x)$ are NBUE (or NWUE), then the stationary distribution of the number of customers in the system is less (respectively, greater) relative to the ordering $\leq_v$ than the stationary distribution of the number of customers in the $M/M/1$ retrial queue with server subject to active breakdowns and exponential repair time.*

Proof. Denote by $\Sigma^{(1)}$ our system defined in Section 2 (i.e., a single server retrial queue with server subject to active breakdowns) with parameters:

$$B_1^{(1)} \equiv B_1, \qquad B_2^{(1)} \equiv B_2, \qquad \lambda^{(1)} = \lambda, \qquad \nu^{(1)} = \nu,$$
$$\alpha^{(1)} = \alpha, \qquad \beta_{1,1}^{(1)} = \beta_{1,1}, \qquad \beta_{2,1}^{(1)} = \beta_{2,1}. \tag{56}$$

On the other hand, let $\Sigma^{(2)}$ be an auxiliary $M/M/1$ retrial queue with server subject to active breakdowns and exponential repair times having the same arrival rate $\lambda^{(2)} = \lambda$, retrial rate $\alpha^{(2)} = \alpha$, server lifetime rate $\nu^{(2)} = \nu$, mean service $\beta_{1,1}^{(2)} = \beta_{1,1}$, and $\beta_{2,1}^{(2)} = \beta_{2,1}$ as in $\Sigma^{(1)}$ system, but with $B_1^{(2)} \equiv B_1^*$ and $B_2^{(2)} \equiv B_2^*$, where

$$B_1^*(x) = \begin{cases} 1 - e^{-x/\beta_{1,1}}, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases}$$
$$B_2^*(x) = \begin{cases} 1 - e^{-x/\beta_{2,1}}, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases} \tag{57}$$

If $B_1(x)$ and $B_2(x)$ are NBUE, then $B_1(x) \leq_v B_1^*(x)$ and $B_2(x) \leq_v B_2^*(x)$ (the inequalities are reversed if $B_1(x)$ and $B_2(x)$ are NWUE). Moreover, the following conditions of Theorem 14 are satisfied: $\lambda^{(1)} = \lambda^{(2)}$, $\nu^{(1)} = \nu^{(2)}$, $\alpha^{(1)} = \alpha^{(2)}$, $B_1^{(1)}(x) \leq_v B_1^{(2)}(x)$ (the inequality is reversed if $B_1(x)$ is NWUE) and $B_2^{(1)}(x) \leq_v B_2^{(2)}(x)$ (the inequality is reversed if $B_2(x)$ is NWUE). Thus, $\{\pi_n\}$ is less (respectively, greater if $B_1(x)$ and $B_2(x)$ are NWUE) than the corresponding distribution in the $M/M/1$ retrial queue with server subject to active breakdowns and exponential repair times. That is,

$$\left\{\pi_n^{(*)}\right\} \leq_v \{\pi_n\} \leq_v \left\{\pi_n^{(*)}\right\}. \tag{58}$$

□

Remark 16. Theorem 15 gives insensitive bounds for the stationary distribution of the number of customers in the system at departure times of the considered embedded Markov chain by using the partial information about the ageing class of the service time and repair time distributions.

6. Numerical Examples and Discussions

To illustrate the theoretical result of Theorem 15, a simulator based on the "discrete event" approach was developed under MATLAB. It reproduces the behavior of the model considered in Section 2. Indeed, the simulator can estimate the stationary distributions of such a system when the service and repair time distributions are NBUE or NWUE. The results are being compared to those of an $M/M/1$ retrial queue with server subject to active breakdowns and exponential repair time relative to the convex ordering. To do this, two probability laws of NBUE type, namely, a Weibull distribution ($Wbl(a,b)$, with $a > 1$) and Erlang distribution of order k ($E_k(\lambda)$)) and two other probability laws of NWUE type, namely, a Weibull distribution ($Wbl(a,b)$, with $a \leq 1$) and Gamma distribution ($\Gamma(a,b)$, with $0 \leq a < 1$) are chosen. Table 1 summarizes three situations for different numerical values of the laws parameters.

Table 1: Different simulation cases for fixed parameters $\lambda = 0.3$, $\alpha = 1$, and $\nu = 0.2$.

Case	Law type	$B_1(x)$	$B_2(x)$	ρ
1	NBUE	$E_2(0.50)$ $Wbl(1.1033, 4)$	$Wbl(1.1033, 4)$	0.30
	exp	$\exp(1.0)$	$\exp(1.0)$	
	NWUE	$Wbl(0.5, 0.5)$ $\Gamma(0.5, 2.0)$	$Wbl(0.5, 0.5)$	
2	NBUE	$E_2(0.75)$ $Wbl(1.6549, 4)$	$Wbl(1.1033, 4)$	0.45
	exp	$\exp(1.5)$	$\exp(1.0)$	
	NWUE	$Wbl(0.5, 0.4156)$ $\Gamma(0.5, 3.0)$	$Wbl(0.5, 0.5)$	
3	NBUE	$E_2(1.0)$ $Wbl(2.2065, 4)$	$Wbl(1.1033, 4)$	0.60
	exp	$\exp(2.0)$	$\exp(1.0)$	
	NWUE	$Wbl(0.5, 0.4156)$ $\Gamma(0.5, 3.0)$	$Wbl(0.5, 0.5)$	

For a simulation time $t_{\max} = 10000$ units and $n = 100$ (number of replications), Figure 3, reflecting the three cases studied in Table 1, shows the following.

(i) The stationary distribution of the number of customers in the $M/M/1$ retrial queue with server subject to active breakdowns and exponential repair time is greater (respectively, less) than the stationary distribution of the number of customers in the $M/G/1$ retrial queue with server subject to active breakdowns, where the service time distribution $B_1(x)$ and the repair time distribution $B_2(x)$ are NBUE (respectively, $B_1(x)$ and $B_2(x)$ are NWUE); that is, the inequality $\{\pi_n^{(\mathrm{NBUE})}\} \leq_v \{\pi_n^{(\exp)}\} \leq_v \{\pi_n^{(\mathrm{NWUE})}\}$ holds.

(ii) Figure 3 also shows that the load of the system has a significant influence on the stationary distribution of the number of customers in the system at departure times. Indeed, when ρ is close to 0, then our system tends to behave as an $M/M/1$ retrial queue with server subject to active breakdowns and exponential repair time (see Figure 3(a)). Otherwise, when, for example, ρ tends to be 0.6, our system moves away from an $M/M/1$ retrial queue with server subject to active breakdowns and exponential repair time (see Figure 3(c)).

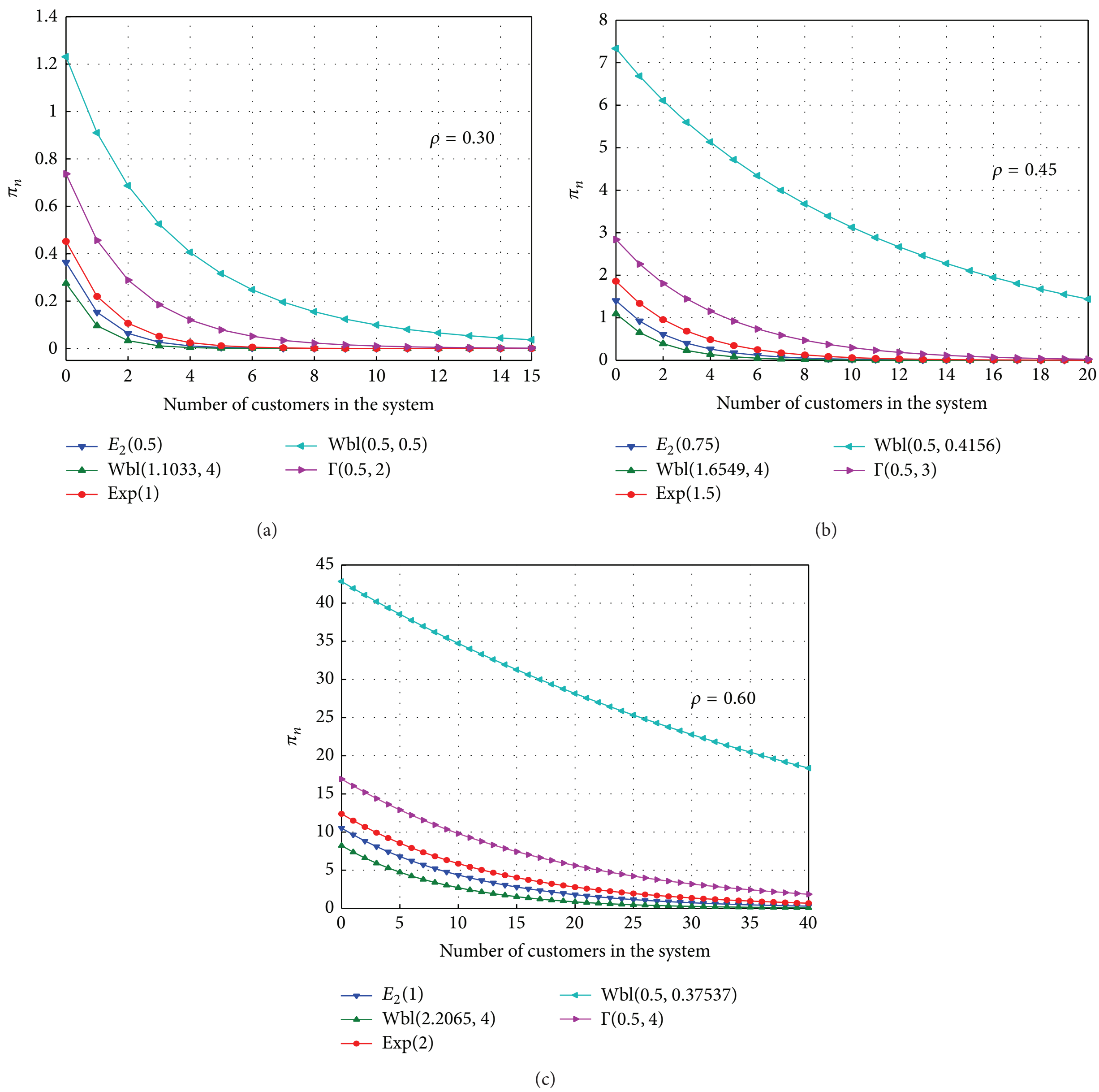

FIGURE 3: Comparison of the stationary distributions of the number of customers in the systems for the three considered cases.

7. Conclusion and Further Research

In this paper, we use a monotonicity approach to establish insensitive bounds for some performance measures of a single server retrial queue with server subject to active breakdowns by using the theory of stochastic orderings. The proposed technique is quite different from those in Djellab [18] and Wang et al. [22], in the sense that our approach provides the fact that we can come to a compromise between the role of these qualitative bounds and the complexity of resolution of some complicated systems where some parameters are not perfectly known (e.g., the service times and repair times distributions are unknown). We prove the monotonicity of the transition operator of the embedded Markov chain relative to strong stochastic ordering and convex ordering. We obtain comparability conditions for the distribution of the number of customers in the system. The main result of this paper consists in giving insensitive bounds for the stationary distribution of the considered embedded Markov chain. Such a result is confirmed by numerical illustrations.

In conclusion, the monotonicity approach holds promise for the solution of several systems with repeated attempts. Hence, it is worth noting that our approach can be further extended to more complex systems (e.g., resource allocation problems in mobile networks).

Moreover, the qualitative bounds given in this paper may have an interesting impact on "robustness analysis"; if there is insecurity on the input of a model, then our order results provide information on what kind of deviation from the nominal model to expect. In gradient estimation one has to control the growth of the cycle length as function of a change of the model.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The author thanks the referees for valuable comments and constructive suggestions. The author thanks also Dr. A. Bareche (Research Unit LaMOS, University of Bejaia) and Dr. M. Cherfaoui (Research Unit LaMOS, University of Biskra) for their help.

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