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Research Article

Boundary Value Problems of Fractional Order Differential Equation with Integral Boundary Conditions and Not Instantaneous Impulses

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We investigate the existence of mild solutions for fractional order differential equations with integral boundary conditions and not instantaneous impulses. By some fixed-point theorems, we establish sufficient conditions for the existence and uniqueness of solutions. Finally, two interesting examples are given to illustrate our theory results.

1. Introduction

Impulsive differential equations are used to describe many practical dynamical systems including evolutionary processes characterized by abrupt changes of the state at certain instants. Such processes are naturally seen in biology, physics, engineering, and so forth. Due to their significance, many authors have established the solvability of impulsive differential equations. Nowadays, the theory of impulsive differential equations has received great attention. Differential equations with instantaneous impulses have been treated in several works (see, e.g., the monographs [1–3], the works on time variable impulses problem [4–7], and the references therein).

However, in almost all the papers concerning impulsive differential equations, the impulses are all instantaneous impulses, and the classical models with instantaneous impulses cannot characterize many practical problems, for example, the dynamics of evolution processes in pharmacotherapy. Let us consider the hemodynamic equilibrium of a person. The introduction of the drugs in the bloodstream and the consequent absorption for the body are gradual and continuous processes. In fact, this situation should be characterized by a new case of impulsive action, which starts at an arbitrary fixed point t_i and stays active on a finite time interval $[t_i, s_i]$. To this end, Hernández and O'Regan [8] initially offered to study a new class of abstract semilinear impulsive differential equations with not instantaneous impulses in a PC-normed Banach space. In [8], the authors discussed the following problem:

$$\begin{gathered}u'(t) = Au(t) + f(t, u(t)), \quad t \in (s_i, t_{i+1}], \ i = 0, \ldots, N, \\ u(t) = g_i(t, u(t)), \quad t \in (t_i, s_i], \ i = 1, \ldots, N, \\ u(0) = x_0,\end{gathered} \tag{1}$$

where $A : D(A) \subset X \rightarrow X$ is the generator of a C_0-semigroup of bounded linear operators $(T(t))_{t>0}$ defined on a Banach space $(X, \|\cdot\|)$, $0 = t_0 = s_0 < t_1 \leq s_1 \leq \cdots \leq t_N \leq s_N \leq t_{N+1} = a$ are prefixed numbers, $x_0 \in X$, $g_i \in C((t_i, s_i] \times X, X)$ for all $i = 1, \ldots, N$, and $f : [0, a] \times X \rightarrow X$ is a suitable function. Meanwhile, Pierri et al. [9] continued the work in [8] in a PC_α-normed Banach space.

On the one hand, the absorption of drugs has a memory effect; thus, the new class of impulsive conditions introduced by [8] may not explain this phenomenon very well. On the other hand, fractional calculus provides a powerful tool for the description of hereditary properties of various materials and memory processes [10, 11]. Fractional differential equations have recently proved to be strong tools in the modeling of medical, physics, economics, and technical sciences. For more details on fractional calculus theory, one can see the monographs of Diethelm [12], Kilbas et al. [13], Lakshmikantham et al. [14], Miller and Ross [15], Podlubny [16], and Tarasov [17]. Fractional differential equations involving the Riemann-Liouville fractional derivative or the Caputo fractional derivative have been paid more and more attention (see [10, 11, 18–22]).

The theory of boundary value problems (BVPs) with integral boundary conditions for differential equations arises in different areas of applied mathematics and physics. For example, heat conduction, chemical engineering, underground water flow, thermo-elasticity, and plasma physics can be reduced to the nonlocal problems with integral boundary conditions. For BVPs with integral boundary conditions and comments on their importance, we refer the readers to the papers by Gallardo [23], Karakostas and Tsamatos [24], Lomtatidze and Malaguti [25], and the references therein. For more information about the general theory of integral equations and their relation with BVPs, we refer to the books of Corduneanu [26] and Agarwal and O'Regan [27]. Moreover, BVPs with integral boundary conditions constitute a very interesting and important class of problems. They include two, three, multipoint, and nonlocal BVPs as special cases. The existence and multiplicity of positive solutions for such problems have received a great deal of attention. To identify a few, we refer the readers to [28–31] and references therein.

In [32], the authors consider the following problem:

$$\begin{gathered} {}^{c}D_t^q u(t) = f\left(t, u(t), \int_0^t k(t,s,u(s))\,ds\right), \\ t \in J' := J \setminus \{t_1, t_2, \ldots, t_m\}, \quad J := [0,T], \\ u(t_k^+) = u(t_k^-) + u_k, \quad u_k \in R,\ k = 0,1,\ldots,m, \\ u(0) = \int_0^1 g(s)\,u(s)\,ds, \end{gathered} \tag{2}$$

where ${}^{c}D_t^q$ is the Caputo fractional derivative of order $q \in (0,1)$ with the lower limit zero, $f : J \times X \times X \to X$ is a given function, $k : \Delta \times X \to X$, $g \in L^1([0,1], R^+)$, $g(t) \in [0,1)$, $\Delta = \{(t,s) : 0 \le s \le t \le 1\}$, and t_k satisfy $0 = t_0 < t_1 < \cdots < t_{m+1} = T$, $u(t_k^+) = \lim_{h \to 0^+} u(t_k + h)$ and $u(t_k^-) = \lim_{h \to 0^-} u(t_k + h)$ represent the right and left limits of $u(t)$ at $t = t_k$. Obviously, the impulses in (2) are instantaneous. Motivated by the work in [8, 9, 32], in this paper, we consider the following impulsive fractional differential equations with integral boundary conditions and not instantaneous impulses:

$$\begin{gathered} {}^{c}D_t^q u(t) = f\left(t, u(t), \int_0^t k(t,s,u(s))\,ds\right), \\ t \in (s_i, t_{i+1}], \quad i = 0,\ldots,N, \\ u(t) = g_i(t, u(t)), \quad t \in (t_i, s_i],\ i = 1,\ldots,N, \\ u(0) = \int_0^T w(s)\,u(s)\,ds, \end{gathered} \tag{3}$$

where ${}^{c}D_t^q$ is the Caputo fractional derivative of order $q \in (0,1)$ with the lower limit zero, $0 = t_0 = s_0 < t_1 \le s_1 \le \cdots \le t_N \le s_N \le t_{N+1} = T$ are prefixed numbers, $g_i \in C((t_i, s_i] \times R, R)$, for $i = 1,\ldots,N$, $J = [0,T]$, $f : J \times R \times R \to R$, $k : \Delta \times R \to R$, $w \in L^1([0,T], R^+)$, and $w \in [0, 1/T)$, $\Delta = \{(t,s) : 0 \le s \le t \le T\}$.

The rest of this paper is organized as follows. In Section 2, some lemmas which are essential to prove our main results are stated. In Section 3, we give the main results. In Section 4, two interesting examples are given to illustrate our theory results.

2. Preliminaries

At first, we present the necessary definitions for the fractional calculus theory.

Definition 1 (see [13]). The Riemann-Liouville fractional integral of order $\alpha > 0$ of a suitable function $y : (0,\infty) \to R$ is given by

$$I_{0_+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\,ds, \tag{4}$$

where the right side is pointwise defined on $(0, +\infty)$.

Definition 2 (see [13]). The Caputo fractional derivative of order $\alpha > 0$ of a suitable function $y : (0,\infty) \to R$ is given by

$${}^{c}D^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} y^{(n)}(s)\,ds, \tag{5}$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of number α and the right side is pointwise defined on $(0, +\infty)$.

Lemma 3 (see [13]). *Let $\alpha > 0$; then the fractional differential equation ${}^{c}D^{\alpha}u(t) = 0$ has solutions*

$$u(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}, \tag{6}$$

where $c_i \in R$, $i = 0,1,\ldots,n-1$, $n = [\alpha] + 1$.

Lemma 4 (see [13]). *Let $\alpha > 0$, then one has*

$$I_{0_+}^{\alpha}\, {}^{c}D^{\alpha} u(t) = u(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}, \tag{7}$$

where $c_i \in R$, $i = 0,1,\ldots,n-1$, $n = [\alpha] + 1$.

Lemma 5 (Krasnoselskii's fixed point theorem [33]). *Let M be a closed convex and nonempty subset of a Banach space X. Let A and B be two operators such that*

(1) *$Ax + By \in M$ whenever $x, y \in M$;*

(2) *A is compact and continuous;*

(3) *B is a contraction mapping.*

Then there exists $z \in M$ such that $z = Az + Bz$.

In order to study problem (3), we define $X = PC(J,R) = \{x : J \to R; x \in C((t_k, t_{k+1}], R), k = 0,1,\ldots,N$, and $x(t_k^+)$, $x(t_k^-)$ exist with $x(t_k^-) = x(t_k)$, $k = 1,\ldots,N\}$.

It is easy to check that X is a Banach space with the norm $\|x\|_{PC} = \sup_{t\in J}|x(t)|$.

Let $Bu(t) = \int_0^t k(t,s,u(s))ds$, $\sigma = \int_0^{t_1} w(t)dt$; then $0 \le \sigma \le \int_0^T w(t)dt < 1$.

If $u \in \mathrm{PC}(J,R)$ satisfies problem (3), then for $t \in (s_i, t_{i+1}]$, $i = 1,\dots,N$, integrating the first equation of (3) from s_i to t by virtue of Definition 1, one can obtain

$$u(t) = u(s_i) + \frac{1}{\Gamma(q)} \int_{s_i}^t (t-s)^{q-1} f(s,u(s),Bu(s))\,ds. \tag{8}$$

From the second equation in (3), we know $u(s_i) = g_i(s_i,u(s_i))$. Then, for $t \in (s_i,t_{i+1}]$, we have

$$u(t) = g_i(s_i,u(s_i)) + \frac{1}{\Gamma(q)} \int_{s_i}^t (t-s)^{q-1} f(s,u(s),Bu(s))\,ds. \tag{9}$$

For $t \in [0,t_1]$, integrating the first equation in (3) from 0 to t by virtue of Definition 1, one can obtain

$$u(t) = u(0) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s,u(s),Bu(s))\,ds. \tag{10}$$

By the boundary conditions, we have

$$u(t) = \int_0^T w(t)u(t)\,dt + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s,u(s),Bu(s))\,ds. \tag{11}$$

Multiplying (11) with $w(t)$ and integrating from 0 to t_1, we have

$$\begin{aligned}\int_0^{t_1} w(t)u(t)\,dt &= \int_0^{t_1} w(t)\,dt \int_0^T w(s)u(s)\,ds \\ &\quad + \frac{1}{\Gamma(q)} \int_0^{t_1} w(t) \int_0^t (t-s)^{q-1} f(s,u(s),Bu(s))\,ds\,dt,\end{aligned} \tag{12}$$

Multiplying (9) with $w(t)$ and integrating from s_i to t_{i+1}, $i = 1,\dots,N$, we have

$$\begin{aligned}\int_{s_i}^{t_{i+1}} w(t)u(t)\,dt &= g_i(s_i,u(s_i)) \int_{s_i}^{t_{i+1}} w(t)\,dt + \frac{1}{\Gamma(q)} \\ &\quad \cdot \int_{s_i}^{t_{i+1}} w(t) \frac{1}{\Gamma(q)} \int_{s_i}^t (t-s)^{q-1} f(s,u(s),Bu(s))\,ds\,dt.\end{aligned} \tag{13}$$

Multiplying the second equation of (3) with $w(t)$ and integrating from t_i to s_i, $i = 1,\dots,N$, we can obtain

$$\int_{t_i}^{s_i} w(t)u(t)\,dt = \int_{t_i}^{s_i} w(t) g_i(t,u(t))\,dt. \tag{14}$$

Adding (12), (13), and (14), one has

$$\begin{aligned}\int_0^T w(t)u(t)\,dt &= \int_0^{t_1} w(t)\,dt \int_0^T w(s)u(s)\,ds + \sum_{i=1}^N g_i(s_i,u(s_i)) \int_{s_i}^{t_{i+1}} w(t)\,dt \\ &\quad + \frac{1}{\Gamma(q)} \sum_{i=0}^N \int_{s_i}^{t_{i+1}} w(t) \int_{s_i}^t (t-s)^{q-1} f(s,u(s),Bu(s))\,ds\,dt \\ &\quad + \sum_{i=1}^N \int_{t_i}^{s_i} w(t) g_i(t,u(t))\,dt.\end{aligned} \tag{15}$$

Hence

$$\begin{aligned}\int_0^T w(t)u(t)\,dt &= \frac{1}{1-\sigma} \cdot \Bigg\{ \sum_{i=1}^N g_i(s_i,u(s_i)) \int_{s_i}^{t_{i+1}} w(t)\,dt \\ &\quad + \frac{1}{\Gamma(q)} \sum_{i=0}^N \int_{s_i}^{t_{i+1}} w(t) \int_{s_i}^t (t-s)^{q-1} f(s,u(s),Bu(s))\,ds\,dt \\ &\quad + \sum_{i=1}^N \int_{t_i}^{s_i} w(t) g_i(t,u(t))\,dt \Bigg\}.\end{aligned} \tag{16}$$

So, for $t \in [0,t_1]$, we have

$$\begin{aligned}u(t) &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s,u(s),Bu(s))\,ds \\ &\quad + \frac{1}{1-\sigma} \cdot \Bigg\{ \sum_{i=1}^N g_i(s_i,u(s_i)) \int_{s_i}^{t_{i+1}} w(t)\,dt + \frac{1}{\Gamma(q)} \\ &\quad \cdot \sum_{i=0}^N \int_{s_i}^{t_{i+1}} w(t) \int_{s_i}^t (t-s)^{q-1} f(s,u(s),Bu(s))\,ds\,dt \\ &\quad + \sum_{i=1}^N \int_{t_i}^{s_i} w(t) g_i(t,u(t))\,dt \Bigg\}.\end{aligned} \tag{17}$$

Then similar to Definition 2.1 in [9], we can define the mild solution for (3).

Definition 6. A function $u \in \mathrm{PC}(J,R)$ is a mild solution of problems (3) if

$$u(0) = \int_0^T w(s)\,u(s)\,ds,$$
$$u(t) = g_i(t,u(t)), \quad \forall t \in (t_i, s_i],\ i = 1,\dots,N,$$
$$\begin{aligned} u(t) &= \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s,u(s),Bu(s))\,ds + \frac{1}{1-\sigma} \\ &\cdot \left\{\sum_{i=1}^N g_i(s_i,u(s_i))\int_{s_i}^{t_{i+1}} w(t)\,dt + \frac{1}{\Gamma(q)}\right. \\ &\cdot \sum_{i=0}^N \int_{s_i}^{t_{i+1}} w(t)\int_{s_i}^t (t-s)^{q-1} f(s,u(s),Bu(s))\,ds\,dt \\ &\left. + \sum_{i=1}^N \int_{t_i}^{s_i} w(t)\,g_i(t,u(t))\,dt\right\} \end{aligned} \tag{18}$$

for all $t \in [0,t_1]$ and

$$u(t) = g_i(s_i,u(s_i)) + \frac{1}{\Gamma(q)}\int_{s_i}^t (t-s)^{q-1} f(s,u(s),Bu(s))\,ds \tag{19}$$

for all $t \in (s_i, t_{i+1}]$, $i = 1,\dots,N$.

3. Main Results

This section deals with the existence of mild solutions for (3). Before stating and proving the main results, we make the following hypotheses.

(H_1) $f : J \times R \times R \to R$ is jointly continuous. There exists a function $e \in C(J,R)$ such that

$$\begin{aligned} &\left|f(t,u_1(t),v_1(t)) - f(t,u_2(t),v_2(t))\right| \\ &\quad \le e(t)\left(\left|u_1(t)-u_2(t)\right| + \left|v_1(t)-v_2(t)\right|\right), \end{aligned} \tag{20}$$

for all $u_1, v_1, u_2, v_2 \in \mathrm{PC}(J,R)$, for all $t \in J$.

(H_2) $k : \Delta \times R \to R$ is continuous and there exists a function $d \in C[J,R^+]$ such that

$$\begin{aligned} &|k(t,s,u(s)) - k(t,s,v(s))| \le d(t)\left(|u(t)-v(t)|\right), \\ &\qquad \forall u,v \in \mathrm{PC}(J,R), \quad (t,s) \in \Delta. \end{aligned} \tag{21}$$

(H_3) $g_i \in C((t_i,s_i] \times R, R)$ and there exist $l_i \in C[J,R]$, $i = 1,\dots,N$ such that

$$\begin{aligned} &\left|g_i(t,x(t)) - g_i(t,y(t))\right| \le l_i(t)\left|x(t)-y(t)\right|, \\ &\qquad \forall x,y \in \mathrm{PC}(J,R), \quad \forall t \in J. \end{aligned} \tag{22}$$

Let

$$\begin{aligned} L &= \max_{1\le i\le N}\sup_{t\in J}\left|l_i(t)\right|, \qquad M = \sup_{t\in J}|e(t)|, \\ D &= \sup_{t\in J}|d(t)|. \end{aligned} \tag{23}$$

Now we are in the position to establish the main results. Our first theorem is based on contraction mapping principle.

Theorem 7. *Let (H_1)–(H_3) hold and $n < 1$; then the problem (3) has a unique mild solution, where*

$$n = \frac{T^q M(1+TD)(2-\sigma)}{\Gamma(q+1)(1-\sigma)} + \frac{2L}{1-\sigma}. \tag{24}$$

Proof. Let $A : \mathrm{PC}(J,R) \to \mathrm{PC}(J,R)$ be the map defined by

$$Au(t) = g_i(t,u(t)), \quad \text{for } t \in (t_i,s_i],\ i = 1,\dots,N,$$
$$\begin{aligned} Au(t) &= \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s,u(s),Bu(s))\,ds + \frac{1}{1-\sigma} \\ &\cdot \left\{\sum_{i=1}^N g_i(s_i,u(s_i))\int_{s_i}^{t_{i+1}} w(t)\,dt + \frac{1}{\Gamma(q)}\right. \\ &\cdot \sum_{i=0}^N \int_{s_i}^{t_{i+1}} w(t)\int_{s_i}^t (t-s)^{q-1} f(s,u(s),Bu(s))\,ds\,dt \\ &\left. + \sum_{i=1}^N \int_{t_i}^{s_i} w(t)\,g_i(t,u(t))\,dt\right\} \end{aligned} \tag{25}$$

for $t \in [0,t_1]$ and

$$\begin{aligned} Au(t) &= g_i(s_i,u(s_i)) \\ &\quad + \frac{1}{\Gamma(q)}\int_{s_i}^t (t-s)^{q-1} f(s,u(s),Bu(s))\,ds \end{aligned} \tag{26}$$

for $t \in (s_i, t_{i+1}]$, $i = 1,\dots,N$. Clearly, A is well defined.

Next we show that A is contraction on $\mathrm{PC}(J,R)$.

Fix $x, y \in \mathrm{PC}(J,R)$; we consider three cases.

Case 1. If $t \in (s_i, t_{i+1}]$, $i = 1,\dots,N$, by the assumptions (H_1)–(H_3) and the property $\Gamma(q+1) = q\Gamma(q)$, we have

$$\begin{aligned} &\left|Ax(t) - Ay(t)\right| \\ &\le \left|g_i(s_i,x(s_i)) - g_i(s_i,y(s_i))\right| \\ &\quad + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} \\ &\qquad \cdot \left|f(s,x(s),Bx(s))\,ds - f(s,y(s),By(s))\right| ds \end{aligned}$$

$$\begin{aligned}
&\leq L\|x-y\|_{PC} + \frac{M}{\Gamma(q)}\left[-\frac{(t-s)^q}{q}\right]\Bigg|_0^t \\
&\quad \cdot \left(\|x-y\|_{PC} + \|Bx-By\|_{PC}\right) \\
&= L\|x-y\|_{PC} + \frac{M}{\Gamma(q)}\frac{t^q}{q}\left(\|x-y\|_{PC} + \|Bx-By\|_{PC}\right) \\
&\leq L\|x-y\|_{PC} + \frac{T^q M}{\Gamma(q+1)}\left(\|x-y\|_{PC} + \|Bx-By\|_{PC}\right) \\
&\leq \left[L + \frac{T^q M(1+TD)}{\Gamma(q+1)}\right]\|x-y\|_{PC}.
\end{aligned} \tag{27}$$

Case 2. If $t \in [0, t_1]$, by (H_1), (H_2), one can obtain

$$\begin{aligned}
&|Ax(t) - Ay(t)| \\
&\leq \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} \\
&\qquad \cdot |f(s, x(s), Bx(s)) - f(s, y(s), By(s))|\, ds \\
&\quad + \frac{1}{1-\sigma}\left\{\sum_{i=1}^{N}|g_i(s_i, x(s_i)) - g_i(s_i, y(s_i))| \int_{s_i}^{t_{i+1}} w(t)\, dt\right. \\
&\qquad + \frac{1}{\Gamma(q)}\sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} w(t)\int_{s_i}^{t}(t-s)^{q-1} \\
&\qquad\qquad \cdot |f(s, x(s), Bx(s)) \\
&\qquad\qquad - f(s, y(s), By(s))|\, dt \\
&\qquad \left. + \sum_{i=1}^{N}\int_{t_i}^{s_i} w(t)|g_i(t, x(t)) - g_i(t, y(t))|\, dt\right\}.
\end{aligned} \tag{28}$$

Then, by a similar argument, we can get

$$\begin{aligned}
&|Ax(t) - Ay(t)| \\
&\quad \leq \left(\frac{T^q M(1+TD)(2-\sigma)}{\Gamma(q+1)(1-\sigma)} + \frac{2L}{1-\sigma}\right)\|x-y\|_{PC}.
\end{aligned} \tag{29}$$

Case 3. If $t \in (t_i, s_i]$, $i = 1, \ldots, N$, from the assumption (H_3), we get

$$|Ax(t) - Ay(t)| \leq L\|x-y\|_{PC}. \tag{30}$$

Therefore, $\|Ax - Ay\|_{PC} \leq n\|x-y\|_{PC}$, for all $x, y \in PC(J, R)$, which implies that A is a contraction mapping. Then, there exists a unique mild solution of (3).

In order to get the second main result, we give assumption (H_4).

(H_4) The function $f : J \times R \times R \to R$ is jointly continuous and strongly measurable on J. There exist $m_f \in C(J, R^+)$ and a nondecreasing function $h_f \in C(R^+, R^+)$ such that

$$|f(t, x, y)| \leq m_f(t)\, h_f(|x+y|) \quad \forall t \in J,\ x, y \in R. \tag{31}$$

Our second result is based on Krasnoselskii's fixed point theorem. □

Theorem 8. *Assume that (H_2)–(H_4) hold; if $2L/(1-\sigma) < 1$ and there exists a constant $r > 0$ such that*

$$\begin{aligned}
&r(1-\sigma-2L) \\
&\quad \geq \frac{T^q(2-\sigma)}{\Gamma(q+1)}\|m_f\|_{PC}\, h_f[r(1+TD)+TK] + 2a,
\end{aligned} \tag{32}$$

where $a = \max_{i=1,\ldots,N}|g_i(t, 0)|$, then the problem (3) has at least a mild solution.

Proof. Let $Au(t)$ be the map introduced in the proof of Theorem 7. We consider the decomposition $Au(t) = A^1u(t) + A^2u(t)$, where

$$A^1u(t) = \sum_{i=1}^{N}A_i^1u(t), \qquad A^2u(t) = \sum_{i=1}^{N}A_i^2u(t),$$

$$A_i^1u(t) = \begin{cases}
\dfrac{1}{\Gamma(q)}\displaystyle\int_{s_i}^{t}(t-s)^{q-1} f(s, u(s), Bu(s))\, ds, \\
\qquad \text{if } t \in (s_i, t_{i+1}], \quad i \geq 1, \\
\dfrac{1}{\Gamma(q)}\displaystyle\int_{0}^{t}(t-s)^{q-1} f(s, u(s), Bu(s))\, ds \\
+\dfrac{1}{(1-\sigma)\Gamma(q)} \\
\cdot\displaystyle\sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} w(t)\int_{s_i}^{t}(t-s)^{q-1} \\
\qquad\qquad \cdot f(s, u(s), Bu(s))\, ds\, dt, \\
\qquad \text{if } t \in [0, t_1], \\
0, \quad \text{else},
\end{cases}$$

$$A_i^2u(t) = \begin{cases}
g_i(t, u(t)), & \text{if } t \in (t_i, s_i],\ i \geq 1, \\
g_i(s_i, u(s_i)), & \text{if } t \in (s_i, t_{i+1}],\ i \geq 1, \\
\dfrac{1}{1-\sigma}\left\{\displaystyle\sum_{i=1}^{N} g_i(s_i, u(s_i))\int_{s_i}^{t_{i+1}} w(t)\, dt\right. \\
\qquad \left. +\displaystyle\sum_{i=1}^{N}\int_{t_i}^{s_i} w(t)\, g_i(t, u(t))\, dt\right\} \\
& \text{if } t \in [0, t_1], \\
0, & \text{else}.
\end{cases} \tag{33}$$

Let $B_r = \{x \in PC(J, R) : \|x\|_{PC} \leq r\}$. We divide our proof into three steps.

Step 1. First we show that $A^1x+A^2y \in B_r$ whenever $x, y \in B_r$.

From (H_2), we know that $k : \Delta \times X \to X$ is continuous, and then $k(t,s,0)$ is bounded, for $(t,s) \in \Delta = \{(t,s) : 0 \le s \le t \le T\}$. Let $K := \max\{k(t,s,0) : (t,s) \in \Delta\}$.

Let $x \in B_r$; if $t \in (s_i, t_{i+1}]$, we have

$$\begin{aligned}
&\left|A_i^1 x(t)\right| \\
&\le \frac{1}{\Gamma(q)} \int_{s_i}^{t} (t-s)^{q-1} \left|f(s,x(s),Bx(s))\right| ds \\
&\le \frac{1}{\Gamma(q)} \left(\int_{s_i}^{t} (t-s)^{q-1} m_f(s)\, ds\right) h_f\left(\|x+Bx\|_{PC}\right) \\
&\le \frac{1}{\Gamma(q)} \left(\int_{s_i}^{t} (t-s)^{q-1} ds\right) \left\|m_f\right\|_{PC} h_f\left(\|x+Bx\|_{PC}\right) \\
&\le \frac{1}{\Gamma(q)} \left(\int_{0}^{t} (t-s)^{q-1} ds\right) \left\|m_f\right\|_{PC} h_f\left(\|x+Bx\|_{PC}\right) \\
&= \frac{1}{\Gamma(q)} \left[-\frac{(t-s)^q}{q}\right]\Bigg|_0^t \left\|m_f\right\|_{PC} h_f\left(\|x+Bx\|_{PC}\right) \\
&= \frac{1}{\Gamma(q)} \frac{t^q}{q} \left\|m_f\right\|_{PC} h_f\left(\|x+Bx\|_{PC}\right) \\
&\le \frac{T^q}{q\Gamma(q)} \left\|m_f\right\|_{PC} h_f\left(\|x\|_{PC} + \|Bx\|_{PC}\right).
\end{aligned} \tag{34}$$

By the definition of B, the integral mean value theorem, (H_2), and the property $\Gamma(q+1) = q\Gamma(q)$, we have

$$\begin{aligned}
&\left|A_i^1 x(t)\right| \\
&\le \frac{T^q}{\Gamma(q+1)} \left\|m_f\right\|_{PC} \\
&\quad \cdot h_f\left[\|x\|_{PC} + T\left(\left|k(t,\xi,x) - k(t,\xi,0)\right| + \left|k(t,\xi,0)\right|\right)\right] \\
&\le \frac{T^q}{\Gamma(q+1)} \left\|m_f\right\|_{PC} \\
&\quad \cdot h_f\left(\|x\|_{PC} + DT\|x\|_{PC} + T\left|k(t,\xi,0)\right|\right) \\
&\le \frac{T^q}{\Gamma(q+1)} \left\|m_f\right\|_{PC} h_f\left[r(1+TD) + TK\right],
\end{aligned} \tag{35}$$

where $0 \le \xi \le t \le T$.

By a similar argument, let $x \in B_r$, if $t \in [0, t_1]$; we have

$$\begin{aligned}
&\left|A_i^1 x(t)\right| \\
&\le \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s,x(s),Bx(s))\, ds \\
&\quad + \frac{1}{(1-\sigma)\Gamma(q)} \\
&\quad \cdot \sum_{i=0}^{N} \int_{s_i}^{t_{i+1}} w(t) \int_{s_i}^{t} (t-s)^{q-1} f(s,x(s),Bx(s))\, ds\, dt \\
&\le \frac{T^q}{\Gamma(q+1)} \left\|m_f\right\|_{PC} h_f\left[r(1+TD) + TK\right] \\
&\quad + \frac{1}{(1-\sigma)\Gamma(q)} \\
&\quad \cdot \int_0^T w(t) \frac{T^q}{q} \left\|m_f\right\|_{PC} h_f\left[r(1+TD) + TK\right] dt \\
&= \frac{T^q}{\Gamma(q+1)} \left(1 + \frac{\int_0^T w(t)\, dt}{1-\sigma}\right) \left\|m_f\right\|_{PC} \\
&\quad \cdot h_f\left[r(1+TD) + TK\right].
\end{aligned} \tag{36}$$

From the condition $w \in L^1([0,T], R^+)$, $w \in [0, 1/T)$, we can get

$$\left|A_i^1 x(t)\right| \le \frac{T^q(2-\sigma)}{\Gamma(q+1)(1-\sigma)} \left\|m_f\right\|_{PC} h_f\left[r(1+TD) + TK\right]. \tag{37}$$

For the other cases, from the definition of $A_i^1 u$, one can get $|A_i^1 x(t)| = 0$.

From the proof above, let $x \in B_r$; then for all $t \in [0,T]$, we have

$$\left|A_i^1 x(t)\right| \le \frac{T^q(2-\sigma)}{\Gamma(q+1)(1-\sigma)} \left\|m_f\right\|_{PC} h_f\left[r(1+TD) + TK\right]. \tag{38}$$

Let $y \in B_r$; if $t \in (t_i, s_i]$, $i \ge 1$, we can obtain

$$\left|A_i^2 y(t)\right| \le \left|g_i(t,y(t)) - g_i(t,0)\right| + \left|g_i(t,0)\right| \le Lr + a. \tag{39}$$

Let $y \in B_r$; if $t \in [0, t_1]$, we can get

$$\begin{aligned}
&\left|A_i^2 y(t)\right| \\
&\le \frac{1}{1-\sigma} \left\{\sum_{i=1}^{N} \left(\left(\left|g_i(s_i, y(s_i)) - g_i(s_i,0)\right| + \left|g_i(s_i,0)\right|\right) \int_{s_i}^{t_{i+1}} w(t)\, dt\right) \right. \\
&\quad \left. + \sum_{i=1}^{N} \int_{t_i}^{s_i} w(t) \left(\left|g_i(t,y(t)) - g_i(t,0)\right| + \left|g_i(t,0)\right|\right) dt \right\} \\
&\le \frac{2(Lr+a)}{1-\sigma}.
\end{aligned} \tag{40}$$

Proceeding as above, we obtain that $|A_i^2 y(t)| \le Lr + a$, $\forall y \in B_r$, for $t \in (s_i, t_{i+1}]$, $i \ge 1$.

Then, for all $x, y \in B_r$, we have that

$$\|Ax + By\|_{PC} \leq \frac{T^q (2-\sigma)}{\Gamma(q+1)(1-\sigma)} \|m_f\|_{PC} h_f [r(1+TD)+TK] + \frac{2(Lr+a)}{1-\sigma} \leq r. \quad (41)$$

Step 2. We show that $A^2 = \sum_{i=1}^N A_i^2$ is a contraction mapping.

From the definition of $A^2u(t)$, $A_i^2u(t)$, and (H_3), we can easily get

$$\left|A_i^2 x(t) - A_i^2 y(t)\right| \leq \frac{2L}{1-\sigma} \|x-y\|_{PC}, \quad \forall x, y \in B_r, \ \forall t \in J, \quad (42)$$

which implies that A^2 is a contraction mapping.

Step 3. Next we will prove that A^1 is compact and continuous.

We also divide the proof into 3 steps.

(I) We show that A^1 is continuous.

Let $\{x_n\}$ be a sequence such that $x_n \to x$ in $PC(J,R)$. Then for all $t \in J$, by the definition of $A^1u(t)$, $A_i^1u(t)$, we have

$$\begin{aligned}
&\left|A^1 x_n(t) - A^1 x(t)\right| \\
&\quad \leq \frac{T^q(2-\sigma)}{\Gamma(q+1)(1-\sigma)} \cdot \left|f(t, x_n(t), Bx_n(t)) - f(t, x(t), Bx(t))\right| \\
&\quad \leq \frac{T^q(2-\sigma)}{\Gamma(q+1)(1-\sigma)} \cdot \left(\left|f(t, x_n(t), Bx_n(t)) - f(t, x(t), Bx_n(t))\right| + \left|f(t, x(t), Bx_n(t)) - f(t, x(t), Bx(t))\right|\right).
\end{aligned} \quad (43)$$

From (H_2), (H_4), we can get the continuity of f and B. Then one has

$$\left\|A^1 x_n - A^1 x\right\|_{PC} \longrightarrow 0, \quad \text{as } x_n \longrightarrow x \ (n \longrightarrow \infty), \quad (44)$$

which shows that the operator A^1 is continuous.

(II) We show that A^1 maps bounded sets into bounded sets in $PC(J,R)$.

Indeed, it is enough to show that, for any $R > 0$, there exists a $R' > 0$ such that, for each $x \in B_R = \{u \in PC(J,R) : \|u\|_{PC} \leq R\}$, we have $\|A^1 x\|_{PC} \leq R'$.

For all $t \in J$, from the definition of $A^1u(t)$, $A_i^1u(t)$, and (H_2), (H_4), one can obtain

$$\left|A^1 x(t)\right| \leq \frac{T^q(2-\sigma)}{\Gamma(q+1)(1-\sigma)} \|m_f\|_{PC} h_f [R(1+TD)+TK] := R'. \quad (45)$$

Then we conclude that A^1 maps bounded sets into bounded sets in $PC(J,R)$.

(III) At last, we prove that A^1 maps bounded sets into equicontinuous sets in $PC(J,R)$.

For interval $t \in (s_i, t_{i+1}]$, $s_i \leq l_1 < l_2 \leq t_{i+1}$, $i = 1, \ldots, N$, $\forall x(t) \in B_r$, by definition of $A^1u(t)$ and (H_3), we have

$$\begin{aligned}
&\left|\left(A^1 x\right)(l_2) - \left(A^1 x\right)(l_1)\right| \\
&\quad = \left|\frac{1}{\Gamma(q)} \int_{s_i}^{l_2} (l_2 - s)^{q-1} f(s, x(s), Bx(s))\, ds - \frac{1}{\Gamma(q)} \int_{s_i}^{l_1} (l_1 - s)^{q-1} f(s, x(s), Bx(s))\, ds\right| \\
&\quad \leq \frac{1}{\Gamma(q)} \int_{l_1}^{l_2} (l_2 - s)^{q-1} \left|f(s, x(s), Bx(s))\right| ds + \frac{1}{\Gamma(q)} \int_{s_i}^{l_1} \left|f(s, x(s), Bx(s))\right| \cdot \left[(l_1 - s)^{q-1} - (l_2 - s)^{q-1}\right] ds \\
&\quad \leq \frac{\|m_f\|_{PC} h_f [r(1+TD)+TK]}{\Gamma(q)} \cdot \left\{\int_{l_1}^{l_2} (l_2 - s)^{q-1} ds + \int_{s_i}^{l_1} \left[(l_1 - s)^{q-1} - (l_2 - s)^{q-1}\right] ds\right\} \\
&\quad \leq \frac{\|m_f\|_{PC} h_f [r(1+TD)+TK]}{\Gamma(q+1)} \cdot \left[(l_2 - l_1)^q + \left|(l_1 - s_i)^q - (l_2 - s_i)^q\right|\right],
\end{aligned} \quad (46)$$

which is independent of x. As $l_1 \to l_2$, the right-hand side of the above inequality tends to zero. Therefore A^1 is equicontinuous on interval $(s_i, t_{i+1}]$, $i \geq 1$.

Proceeding as above, we can also prove that A^1 is equicontinuous for the time interval $[0, t_1]$. From the definition of A_i^1, it is easy to see that A^1 is equicontinuous for the other cases.

By Arzela-Ascoli Theorem, A^1 is continuous and compact. □

As a consequence of Lemma 5, we deduce that the operator A has at least a fixed point on B_r which means that problem (3) has at least a mild solution.

4. Examples

Consider the following impulsive system of fractional differential equations.

Example 1. Consider

$$\begin{gathered}
{}^cD_t^{1/2} u(t) = \frac{1}{16}\left(\frac{u(t)\sin t^2}{1+e^{t^2}} + \int_0^t \frac{\sin\left(t^2 + \sqrt{s}\right)}{2} u(s)\, ds\right), \\
t \in (s_i, t_{i+1}], \quad i = 1, \ldots, N,
\end{gathered}$$

$$u(t) = \frac{u(t)}{18e^t(1+|u(t)|)}, \quad t \in (t_i, s_i],\ i = 1, \dots, N,$$
$$u(0) = \int_0^1 \frac{u(t)}{3} dt, \tag{47}$$

where $0 = t_0 = s_0 < t_1 = 1/2 \le s_1 \le \cdots \le t_N \le s_N \le t_{N+1} = 1$ are pre-fixed numbers, $J = [0,1]$, $q = 1/2$, and $w(t) = 1/3 < 1$.

We prove that Example 1 satisfies all the assumptions of Theorem 7.

In Example 1, set

$$f(t,u,v) = \frac{u \sin t^2}{16\left(1+e^{t^2}\right)} + \frac{v}{16},$$
$$Bu(t) = \int_0^t \frac{\sin\left(t^2+\sqrt{s}\right)}{2} u(s)\, ds, \tag{48}$$
$$k(t,s,u) = \frac{\sin\left(t^2+\sqrt{s}\right)}{2} u.$$

It is easy to see that f is jointly continuous. We can also check that (H_1) and (H_2) are satisfied with $e(t) = 1/16$ and $d(t) = 1/2$.

For $t \in [0,1]$, $u \in R$, and $g_i(t,u) = u/18e^t(1+|u|)$, then we know

$$\left|g_i(t,u) - g_i(t,v)\right| \le \frac{1}{18}|u-v|, \tag{49}$$

with $l_i(t) = 1/36$, so (H_3) is also satisfied.

From 1, we can get $L = 1/18$, $T = 1$, $M = 1/16$, $D = 1/2$, $q = 1/2$, $\Gamma(3/2) = \sqrt{\pi}/2$, $\sigma = 1/6$, and then

$$n = \frac{T^q M(1+TD)(2-\sigma)}{\Gamma(q+1)(1-\sigma)} + \frac{2L}{1-\sigma} = \frac{33}{80\sqrt{\pi}} + \frac{2}{15} < 1. \tag{50}$$

So all the conditions of Theorem 7 are satisfied. As a consequence of Theorem 7, Example 1 has a unique mild solution.

Example 2. Consider

$$\begin{aligned} &{}^cD_t^{1/2}u(t) \\ &= \frac{1}{15\left(1+e^{\sqrt{\sin t}}\right)}\left(u(t) + \int_0^t \frac{\sin\left(t^2+\sqrt{s}\right)}{2}u(s)\,ds\right), \\ &\qquad t \in (s_i, t_{i+1}], \quad i = 1, \dots, N, \end{aligned}$$
$$u(t) = \frac{u(t)}{18(1+|u(t)|)}, \quad t \in (t_i, s_i],\ i = 1, \dots, N,$$
$$u(0) = \int_0^1 \frac{u(t)}{2} dt, \tag{51}$$

where $0 = t_0 = s_0 < t_1 = 1/3 \le s_1 \le t_2 \le \cdots \le t_N \le s_N \le t_{N+1} = 1$ are prefixed numbers, $J = [0,1]$, $q = 1/2$, and $w(t) = 1/2 < 1$.

It is easy to see that $g_i(t,0) = 0$ is bounded. Set

$$f(t,u,v) = \frac{u+v}{15\left(1+e^{\sqrt{\sin t}}\right)},$$
$$Bu(t) = \int_0^t \frac{\sin\left(t^2+\sqrt{s}\right)}{2}u(s)\,ds, \tag{52}$$
$$k(t,s,u) = \frac{\sin\left(t^2+\sqrt{s}\right)}{2}u.$$

Then we have

$$\left|f(t,u,Bu)\right| \le \frac{1}{15}|u+Bu|, \tag{53}$$

with $m_f = 1/15 \in C(J,R^+)$, $h_f(|u+Bu|) = |u+Bu| \in C(R^+,R^+)$ being nondecreasing. So (H_4) is satisfied. Similarly to the proof of Example 1, we know that (H_2) and (H_3) are satisfied.

From (51), we can obtain $T = 1$, $\sigma = 1/6$, $D = 1/2$, $L = 1/18 < 1$, $a = 0$, $K = 0$, $\Gamma(3/2) = \sqrt{\pi}/2$, $m_f = 1/15$, $q = 1/2$, and $2L/(1-\sigma) = 2/15 < 1$, and then the inequality (32) becomes $13r/18 \ge (11/30\sqrt{\pi})r$. Hence, inequality (32) holds for all $r > 0$.

Thus, all the assumptions in Theorem 8 are satisfied, and our results can be applied to Example 2. So Example 2 has at least one mild solution.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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