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Research Article

Eigenvalues of Vectorial Sturm-Liouville Problems with Parameter Dependent Boundary Conditions

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We generalize the *regularized sampling method* introduced in 2005 by the author to compute the eigenvalues of scalar Sturm-Liouville problems (SLPs) to the case of vectorial SLP with parameter dependent boundary conditions. A few problems are worked out to illustrate the effectiveness of the method and show by the same token that we have indeed a general method capable of handling with ease very broad classes of SLPs, whether scalar or vectorial.

1. Introduction

In [1] we introduced the regularized sampling method, a method to compute the eigenvalues of scalar Sturm-Liouville problems (SLPs) with parameter dependent boundary conditions. We subsequently used this method to compute the eigenvalues of singular and non-self-adjoint Sturm-Liouville problems. The scope of the method was further extended to include the computation of the eigenvalues of discontinuous/impulsive, nonlocal ([2] and the references therein), and two-parameter SLPs [3]. Continuing our effort we will tackle in this paper vectorial SLP with parameter dependent nonseparated boundary conditions. Vectorial Sturm-Liouville problems have been considered in [4–13] and the references therein while corresponding inverse problems appeared in [14–17].

2. The Characteristic Function

Consider the vectorial Sturm-Liouville problem,

$$\begin{gathered} -y'' + Q(x)\,y = \mu^2 y, \quad 0 < x < 1 \\ A(\mu)\begin{pmatrix} y(0,\mu) \\ y'(0,\mu) \end{pmatrix} + B(\mu)\begin{pmatrix} y(1,\mu) \\ y'(1,\mu) \end{pmatrix} = 0, \end{gathered} \tag{1}$$

where Q is an $n \times n$ matrix function, A and B are real $2n \times 2n$ matrix functions of the parameter μ such that the matrix $[A(\mu) \mid B(\mu)]$ has full rank.

Let Y_c, Y_s be the solutions of the Sturm-Liouville matrix equation $-Y'' + Q(x)Y = \mu^2 Y$ subject to the initial conditions $Y(0,\mu) = I$, $Y'(0,\mu) = 0$ and $Y(0,\mu) = 0$, $Y'(0,\mu) = I$, respectively, I being the $n \times n$ identity matrix and 0 being the $n \times n$ zero matrix.

The general solution of the Sturm-Liouville equation

$$-Y'' + Q(x)\,Y = \mu^2 Y \tag{2}$$

is given by $Y = Y_c a + Y_s b$ with arbitrary constant vectors a and b. Replacing in the boundary conditions, we get

$$\begin{gathered} A(\mu)\begin{pmatrix} a \\ b \end{pmatrix} + B(\mu)\begin{pmatrix} Y_c(1,\mu)\,a + Y_s(1,\mu)\,b \\ Y_c'(1,\mu)\,a + Y_s'(1,\mu)\,b \end{pmatrix} = 0, \\ \left\{A(\mu) + B(\mu)\begin{pmatrix} Y_c(1,\mu) & Y_s(1,\mu) \\ Y_c'(1,\mu) & Y_s'(1,\mu) \end{pmatrix}\right\}\begin{pmatrix} a \\ b \end{pmatrix} = 0. \end{gathered} \tag{3}$$

To have a nontrivial solution $\binom{a}{b}$ a necessary and sufficient condition is that $F(\mu) = 0$ where the characteristic function is

$$F(\mu) = \det\{F\text{mat}(\mu)\}, \tag{4}$$

where

$$F\text{mat}(\mu) = A(\mu) + B(\mu)\begin{pmatrix} Y_c(1,\mu) & Y_s(1,\mu) \\ Y_c'(1,\mu) & Y_s'(1,\mu) \end{pmatrix}. \quad (5)$$

The eigenvalues of (1) are the square of the zeroes of F. It is well known that the multiplicities of these eigenvalues are at most n.

3. Main Results

Let PW_σ be the Paley-Wiener space

$$\text{PW}_\sigma = \left\{ f \text{ entire}, |f(\mu)| \le Ce^{\sigma|\mathbf{Im}\,\mu|}, \int_{-\infty}^{\infty} |f(\mu)|^2 d\mu < \infty \right\}, \quad (6)$$

and recall the celebrated Whittaker-Shannon-Kotel'nikov theorem [18].

Theorem 1. *Let $f \in PW_\sigma$; then*

$$f(\mu) = \sum_{k=-\infty}^{\infty} f\left(\frac{k\pi}{\sigma}\right) \frac{\sin\sigma(\mu - k)}{\sigma(\mu - k)}, \quad (7)$$

where the series converges uniformly on compact subset of C and in $L^2(R)$.

It is known that, in the case of scalar Sturm-Liouville problems, $y(x,\mu)$ is an entire function of μ for each fixed $x \in (0,1]$. $y(x,\mu)$ is in a Paley-Wiener space as a function of μ for each x only in the Dirichlet case. So, we had to subtract some terms from $y(x,\mu)$ to make the difference fall in an appropriate PW_σ space. We had even to subtract terms involving multiple integrals to get sharper results when it comes to computing of the eigenvalues. The regularized sampling method has been introduced recently [1] to overcome this problem; we do not have to subtract any term involving any (multiple) integration. In fact we multiplied $y(x,\mu) - \phi(x,\mu)$ and $y'(x,\mu) - \psi(x,\mu)$ by an appropriate function of μ and got the eigenvalues with much greater precision at a reduced cost. Here ϕ and ψ are known simple functions.

For the vectorial Sturm-Liouville problem at hand, we will use the regularized sampling method to recover the matrices $Y_c(1,\mu)$, $Y_c'(1,\mu)$, $Y_s(1,\mu)$, and $Y_s'(1,\mu)$ from which we obtain $F(\mu)$, the characteristic function whose zeroes are the square roots of the sought eigenvalues of the problem.

Consider the compatible vector and matrix norms given by

$$\|Y\| = \max_{i=1,\dots,n} |Y_i|, \qquad \|P\| = \max_{i=1,\dots,n} \sum_{j=1}^{n} |P_{ij}|, \quad (8)$$

where $Y \in R^n$, $P \in R^{n\times n}$. In the following we will make use of the standard estimate.

Lemma 2. *Consider*

$$|\cos u| \le e^{|\mathbf{Im}\,u|}, \qquad \left|\frac{\sin u}{u}\right| \le \frac{\gamma_0}{1+|u|} e^{|\mathbf{Im}\,u|}, \quad (9)$$

where γ_0 is some constant (we may take $\gamma_0 = 1.72$).

To cover both cases ($Y(0,\mu) = I$, $Y'(0,\mu) = 0$ and $Y(0,\mu) = 0$, $Y'(0,\mu) = I$) we will consider the following initial value problem:

$$Y'' + \mu^2 Y = Q(x) Y, \qquad Y(0,\mu) = E_1, \qquad Y'(0,\mu) = E_2, \quad (10)$$

where E_1 and E_2 are $n \times n$ matrices or n-vector. We have

$$Y(x,\mu) = E_1 \cos\mu x + E_2 \frac{\sin\mu x}{\mu} + \int_0^x \frac{\sin\mu(x-t)}{\mu} Q(t) Y(t,\mu)\, dt. \quad (11)$$

Our first result is the following theorem.

Theorem 3. *$Y(x,\mu)$ is an entire matrix function of μ for each fixed $x \in (0,1]$ and satisfies the growth conditions*

$$\begin{aligned}
\|Y(x,\mu)\| &\le \left(\left\{ \|E_1\| + \frac{\gamma_0}{1+|\mu|} \|E_2\| \right\} e^{\gamma_0 \int_0^1 \|Q(t)\| dt} \right) e^{x|\mathbf{Im}\,\mu|} \\
&\le \gamma_1 e^{x|\mathrm{Im}\,\mu|}, \\
&\left\| Y(x,\mu) - \left\{ E_1 \cos\mu x + E_2 \frac{\sin\mu x}{\mu} \right\} \right\| \\
&\quad \le \frac{\gamma_2}{1+|\mu|} e^{x|\mathbf{Im}\,\mu|} \le \gamma_2 e^{x|\mathrm{Im}\,\mu|}, \\
&\left\| Y'(x,\mu) - \{-\mu E_1 \sin\mu x + E_2 \cos\mu x\} \right\| \le \gamma_3 e^{x|\mathrm{Im}\,\mu|}, \\
&\Big\| Y'(x,\mu) + \mu E_1 \sin\mu x - E_2 \cos\mu x \\
&\quad - \int_0^x \cos\mu(x-t) Q(t) \left(E_1 \cos\mu t + E_2 \frac{\sin\mu t}{\mu} \right) dt \Big\| \\
&\le \frac{\gamma_4}{1+|\mu|} e^{x|\mathrm{Im}\,\mu|},
\end{aligned} \quad (12)$$

for some positive constants γ_1, γ_2, γ_3, and γ_4.

Proof. From (11) and using standard arguments, we conclude that $Y(x,\mu)$ is an entire matrix function of μ for each x in $(0,1]$. Its derivative with respect to x,

$$Y'(x,\mu) = -\mu E_1 \sin\mu x + E_2 \cos\mu x + \int_0^x \cos\mu(x-t) Q(t) Y(t,\mu)\, dt, \quad (13)$$

is also an entire matrix function of μ for each x in $(0, 1]$. Going back to (11) we get at once

$$\begin{aligned} \|Y(x,\mu)\| \le{}& \left\|E_1\cos\mu x + E_2\frac{\sin\mu x}{\mu}\right\| \\ &+ \int_0^x \left|\frac{\sin\mu(x-t)}{\mu(x-t)}\right| \\ &\cdot (x-t)\|Q(t)\|\cdot\|Y(t,\mu)\|\,dt \\ \le{}& e^{x|\mathbf{Im}\,\mu|}\left\{\|E_1\| + \frac{\gamma_0 x}{1+x|\mu|}\|E_2\|\right\} \\ &+ \int_0^x \gamma_0(x-t)e^{(x-t)|\mathbf{Im}\,\mu|}\|Q(t)\|\cdot\|Y(t,\mu)\|\,dt \\ \le{}& e^{x|\mathbf{Im}\,\mu|}\left\{\|E_1\| + \frac{\gamma_0}{1+|\mu|}\|E_2\|\right\} \\ &+ e^{x|\mathbf{Im}\,\mu|}\int_0^x \gamma_0\|Q(t)\|\cdot e^{t|\mathbf{Im}\,\mu|}\|Y(t,\mu)\|\,dt. \end{aligned} \tag{14}$$

Multiplying by $e^{-x|\operatorname{Im}\mu|}$, using Gronwall's lemma, and multiplying back by $e^{x|\operatorname{Im}\mu|}$ we get

$$\begin{aligned} \|Y(x,\mu)\| &\le \left(\left\{\|E_1\| + \frac{\gamma_0}{1+|\mu|}\|E_2\|\right\}e^{\gamma_0\int_0^1\|Q(t)\|dt}\right)e^{x|\mathbf{Im}\,\mu|} \\ &\le \gamma_1 e^{x|\mathbf{Im}\,\mu|}, \end{aligned} \tag{15}$$

where $\gamma_1 = \{\|E_1\| + \gamma_0\|E_2\|\}\exp(\gamma_0\int_0^1\|Q(t)\|dt)$. Now, using the above estimate in (10), we get

$$\begin{aligned} &\left\|Y(x,\mu) - \left\{E_1\cos\mu x + E_2\frac{\sin\mu x}{\mu}\right\}\right\| \\ &\quad\le \int_0^x\left|\frac{\sin\mu(x-t)}{\mu(x-t)}\right|\cdot(x-t)\|Q(t)\|\cdot\|Y(t,\mu)\|\,dt \\ &\quad\le \int_0^x \frac{\gamma_0 e^{(x-t)|\operatorname{Im}\mu|}}{1+|\mu|(x-t)}\cdot(x-t)\|Q(t)\|\gamma_1 e^{t|\mathbf{Im}\,\mu|}dt \\ &\quad\le e^{x|\operatorname{Im}\mu|}\frac{\gamma_0\gamma_1}{1+|\mu|}\int_0^1\|Q(t)\|\,dt \\ &\quad\le \frac{\gamma_2}{1+|\mu|}e^{x|\mathbf{Im}\,\mu|} \le \gamma_2 e^{x|\mathbf{Im}\,\mu|}, \end{aligned} \tag{16}$$

where $\gamma_2 = \gamma_0\gamma_1\int_0^1\|Q(t)\|dt$. Likewise we have

$$\begin{aligned} &\|Y'(x,\mu) - \{-\mu E_1\sin\mu x + E_2\cos\mu x\}\| \\ &\quad\le \int_0^x |\cos\mu(x-t)|\cdot\|Q(t)\|\cdot\|Y(t,\mu)\|\,dt \\ &\quad\le \int_0^x e^{(x-t)|\operatorname{Im}\mu|}\|Q(t)\|\gamma_1 e^{t|\mathbf{Im}\,\mu|}dt \\ &\quad= e^{x|\operatorname{Im}\mu|}\gamma_1\int_0^1\|Q(t)\|\,dt = \gamma_3 e^{x|\operatorname{Im}\mu|}, \end{aligned} \tag{17}$$

where $\gamma_3 = \gamma_1\int_0^1\|Q(t)\|dt$. As in the scalar case, $Y(x,\mu)$ is in a Paley-Wiener space only in the Dirichlet case; however, $Y(x,\mu) - \{E_1\cos\mu x + E_2(\sin\mu x/\mu)\}$ is. As for $Y'(x,\mu)$, it is not; nor is $Y'(x,\mu) - \{-\mu E_1\sin\mu x + E_2\cos\mu x\}$ since they are not square integrable over the reals for fixed x in $(0, 1]$. Also,

$$\begin{aligned} &\Bigg\|Y'(x,\mu) + \mu E_1\sin\mu x - E_2\cos\mu x \\ &\quad- \int_0^x \cos\mu(x-t)Q(t)\left(E_1\cos\mu t + E_2\frac{\sin\mu t}{\mu}\right)dt\Bigg\| \\ &\quad\le \int_0^x |\cos\mu(x-t)|\cdot\|Q(t)\| \\ &\qquad\cdot\left\|Y(t,\mu) - \left\{E_1\cos\mu t + E_2\frac{\sin\mu t}{\mu}\right\}\right\|dt \\ &\quad\le \int_0^x e^{(x-t)|\operatorname{Im}\mu|}\|Q(t)\|\frac{\gamma_2}{1+|\mu|}e^{t|\mathbf{Im}\,\mu|}dt \\ &\quad= e^{x|\operatorname{Im}\mu|}\frac{\gamma_2}{1+|\mu|}\int_0^1\|Q(t)\|\,dt = \frac{\gamma_4}{1+|\mu|}e^{x|\operatorname{Im}\mu|}, \end{aligned} \tag{18}$$

where $\gamma_4 = \gamma_2\int_0^1\|Q(t)\|dt$. □

We get at once the following corollaries.

Corollary 4. *$Y_c(x,\mu)$, $Y_c'(x,\mu)$, $Y_s(x,\mu)$, $Y_s'(x,\mu)$, $Y_0(x,\mu)$, and $Y_0'(x,\mu)$ are entire matrix functions of μ for each fixed $x \in (0, 1]$ and satisfy the growth conditions*

$$\|Y_c(x,\mu)\| \le \left(e^{\gamma_0\int_0^1\|Q(t)\|dt}\right)e^{x|\mathbf{Im}\,\mu|},$$

$$\|Y_c(x,\mu) - I\cos\mu x\| \le \frac{\gamma_2}{1+|\mu|}e^{x|\mathbf{Im}\,\mu|} \le \gamma_2 e^{x|\operatorname{Im}\mu|},$$

$$\|Y_c'(x,\mu) + \mu I\sin\mu x\| \le \gamma_3 e^{x|\operatorname{Im}\mu|},$$

$$\left\|Y_c'(x,\mu) - \left\{-\mu I\sin\mu x + \int_0^x\cos\mu(x-t)Q(t)\cos\mu t\,dt\right\}\right\| \le \frac{\gamma_4}{1+|\mu|}e^{x|\operatorname{Im}\mu|},$$

$$\|Y_s(x,\mu)\| \le \left(\frac{\gamma_0}{1+|\mu|}e^{\gamma_0\int_0^1\|Q(t)\|dt}\right)e^{x|\mathbf{Im}\,\mu|} \le \gamma_1 e^{x|\operatorname{Im}\mu|},$$

$$\left\|Y_s(x,\mu) - I\frac{\sin\mu x}{\mu}\right\| \le \frac{\gamma_2}{1+|\mu|}e^{x|\mathbf{Im}\,\mu|} \le \gamma_2 e^{x|\operatorname{Im}\mu|},$$

$$\|Y_s'(x,\mu) - I\cos\mu x\| \le \gamma_3 e^{x|\operatorname{Im}\mu|},$$

$$\left\| Y_s'(x,\mu) - \left\{ I\cos\mu x + \int_0^x \cos\mu(x-t)Q(t)\frac{\sin\mu t}{\mu}dt \right\} \right\| \le \frac{\gamma_4}{1+|\mu|}e^{x|\operatorname{Im}\mu|},$$

$$\|Y_0(x,\mu)\| \le \left(\left\{ \|D_1\| + \frac{\gamma_0}{1+|\mu|}\|D_2\| \right\} e^{\gamma_0\int_0^1\|Q(t)\|dt} \right) e^{x|\mathbf{Im}\,\mu|} \le \gamma_1 e^{x|\operatorname{Im}\mu|},$$

$$\left\| Y_0(x,\mu) - \left\{ D_1\cos\mu x + D_2\frac{\sin\mu x}{\mu} \right\} \right\| \le \frac{\gamma_2}{1+|\mu|}e^{x|\mathbf{Im}\,\mu|} \le \gamma_2 e^{x|\operatorname{Im}\mu|},$$

$$\left\| Y_0'(x,\mu) - \{-\mu D_1\sin\mu x + D_2\cos\mu x\} \right\| \le \gamma_3 e^{x|\operatorname{Im}\mu|},$$

$$\left\| Y_0'(x,\mu) + \mu D_1\sin\mu x - D_2\cos\mu x - \int_0^x \cos\mu(x-t)Q(t)\left(D_1\cos\mu t + D_2\frac{\sin\mu t}{\mu} \right)dt \right\| \le \frac{\gamma_4}{1+|\mu|}e^{x|\operatorname{Im}\mu|}, \tag{19}$$

for some generic positive constants γ_1, γ_2, γ_3, *and* γ_4.

Corollary 5. *The functions,*

$$Y_c(1,\mu) - I\cos\mu,$$
$$Y_c'(1,\mu) - \left\{ -\mu I\sin\mu + \int_0^1 \cos\mu(1-t)Q(t)\cos\mu t\,dt \right\},$$
$$Y_s(1,\mu) - I\frac{\sin\mu}{\mu},$$
$$Y_s'(1,\mu) - \left\{ I\cos\mu + \int_0^1 \cos\mu(1-t)Q(t)\frac{\sin\mu t}{\mu}dt \right\},$$
$$Y_0(1,\mu) - \left\{ D_1\cos\mu + D_2\frac{\sin\mu}{\mu} \right\},$$
$$Y_0'(1,\mu) - \left\{ -\mu D_1\sin\mu + D_2\cos\mu + \int_0^1 \cos\mu(1-t)Q(t)\left(D_1\cos\mu t + D_2\frac{\sin\mu t}{\mu} \right)dt \right\}, \tag{20}$$

belong to the Paley-Wiener space $P\dot{W}_1$ *as functions of* μ *and thus can be recovered from their samples at* $\mu_k = k\pi$, $k \in Z$ *using the WSK series.*

Theorem 6. *Let* θ *be positive real number and* $m \ge 2$ *a positive integer. Consider*

$$U_1(x,\mu) = \left(Y(x,\mu) - \left\{ E_1\cos\mu x + E_2\frac{\sin\mu x}{\mu} \right\} \right)\left(\frac{\sin(\theta\mu)}{\theta\mu} \right)^m,$$
$$U_2(x,\mu) = \left(Y'(x,\mu) - \{-\mu E_1\sin\mu x + E_2\cos\mu x\} \right)\left(\frac{\sin(\theta\mu)}{\theta\mu} \right)^m \tag{21}$$

belong to the Paley-Wiener space PW_σ *where* $\sigma = x + m\theta$ *as functions of* μ *for each fixed* $x \in (0,1]$ *for* $m \ge 1$ *and satisfy the growth condition* $\|U_1(x,\mu)\|$, $\|U_2(x,\mu)\| \le \gamma_5 e^{\sigma|\mathbf{Im}\,\mu|}/(1+\theta|\mu|)^m$ *where* γ_5 *is some positive constant* ($\gamma_5 = \gamma_0^m\max(\gamma_2,\gamma_3)$).

Proof. It is enough to note that $\sin(\theta\mu)/\theta\mu$ is an entire function of μ and satisfies the estimate in the above Lemma and the fact that $Z(x,\mu)$ is the product of two entire functions thus entire. □

Remark 7. To avoid the first singularity of $(\sin(\theta\mu)/\theta\mu)^{-m}$ we will take $\theta < (N-m)^{-1}$.

The use of the WSK theorem allows us to recover $U_1(1,\mu)$ and $U_2(1,\mu)$ as

$$U_1(1,\mu) = \sum_{k\in Z}\alpha_k\frac{\sin\sigma(\mu-\mu_k)}{\sigma(\mu-\mu_k)},$$
$$U_2(1,\mu) = \sum_{k\in Z}\beta_k\frac{\sin\sigma(\mu-\mu_k)}{\sigma(\mu-\mu_k)}, \tag{22}$$

where $\alpha_k = U_1(1,\mu_k)$, $\beta_k = U_2(1,\mu_k)$, $\mu_k = k\pi/\sigma$, and $\sigma = 1 + m\theta$.

Hence, $Y(1,\mu)$ or $Y'(1,\mu)$ can be recovered as

$$Y(1,\mu) = E_1\cos\mu + E_2\frac{\sin\mu}{\mu} + \left(\frac{\sin(\theta\mu)}{\theta\mu} \right)^{-m}\sum_{k\in Z}\alpha_k\frac{\sin\sigma(\mu-\mu_k)}{\sigma(\mu-\mu_k)},$$
$$Y'(1,\mu) = -\mu E_1\sin\mu + E_2\cos\mu + \left(\frac{\sin(\theta\mu)}{\theta\mu} \right)^{-m}\sum_{k\in Z}\alpha_k\frac{\sin\sigma(\mu-\mu_k)}{\sigma(\mu-\mu_k)}. \tag{23}$$

In practice, we take $|k| \le N$ for some positive integer N, large enough, so that $F(\mu)$ can be reconstructed whose zeros are the square roots of the sought eigenvalues.

Since $\mu^{m-1}U_1(1,\mu)$ and $\mu^{m-1}U_2(1,\mu)$ are in $L^2(-\infty,\infty)$, Jagerman's result [18] is applicable and yields the following better estimate.

Lemma 8 (truncation error). *Let* $U_j^N(1,\mu) = \sum_{k=-N}^{N} U_j(1,\mu_k)(\sin\sigma(\mu-\mu_k)/\sigma(\mu-\mu_k))$ *denote the truncation of* $U_j(1,\mu)$, $j = 1,2$. *Then, for* $|\mu| < N\pi/\sigma$,

$$\left|U_j(1,\mu) - U_j^N(1,\mu)\right| \leq \frac{|\sin\gamma\mu|\,\gamma_{5,j}}{\pi(\pi/\sigma)^{m-1}\sqrt{1-4^{-m+1}}} \cdot \left[\frac{1}{\sqrt{(N\pi/\sigma)-\mu}} + \frac{1}{\sqrt{(N\pi/\sigma)+\mu}}\right]\frac{1}{(N+1)^{m-1}}, \tag{24}$$

where $\gamma_{5,j} = \|\mu^{m-1}U_j(1,\mu)\|_2$.

Lemma 9. *Consider* $|\mu| < N\pi/\sigma$,

$$|Y(1,\mu) - Y_N(1,\mu)|,$$

$$\left|Y'(1,\mu) - Y_N'(1,\mu)\right| \leq \left|\frac{\sin(\theta\mu)}{\theta\mu}\right|^{-m} \times \frac{|\sin\gamma\mu|\,\gamma_5}{\pi(\pi/\sigma)^{m-1}\sqrt{1-4^{-m+1}}} \cdot \left[\frac{1}{\sqrt{(N\pi/\sigma)-\mu}} + \frac{1}{\sqrt{(N\pi/\sigma)+\mu}}\right]\frac{1}{(N+1)^{m-1}}, \tag{25}$$

where $\gamma_5 = \max\{\|\mu^{m-1}Y(1,\mu)\|_2, \|\mu^{m-1}Y'(1,\mu)\|_2\}$.

The approximation of $Y(1,\mu)$ and $Y'(1,\mu)$ by $Y_N(1,\mu)$ and $Y_N'(1,\mu)$, respectively, induces an approximation of the characteristic function F by F_N, whose zeros are the square root of the eigenvalues of the problem.

Let $\overline{\mu}^2$ denote an eigenvalue of the problem; then independent eigenfunctions associated can be obtained using basis vectors of the null space of the matrix $F\mathrm{mat}(\overline{\mu})$ as initial conditions to the differential equation $-y'' + Q(x)y = \overline{\mu}^2 y$, $0 < x < 1$.

4. Numerical Examples

In this section we will illustrate the power of the regularized sampling method as applied to vectorial Sturm-Liouville problems with parameter dependent boundary conditions. We will take $m = 6$, $N = 40$ and a precision of 10^{-20} for the first three examples involving two dimensional SLPs. We will also work out two three-dimensional SLPs one of them involving parameter dependent boundary conditions. In these last two examples we take different values of N, namely, $N = 20, 40, 60, 80$, and 100, and take $m = 4$ and a precision of 10^{-20}. The reported multiplicities of the eigenvalues $\overline{\mu}^2$ are just the dimensions of the null space of the corresponding matrices $F\mathrm{mat}(\overline{\mu})$.

Example 1 (Chanane [1], 1D-version taken from fom Binding and Browne [19]). Consider

$$-y_1''(x) = \lambda y_1(x), \quad -y_2''(x) = \lambda y_2(x), \quad 0 \leq x \leq 1$$
$$y_1(0) + (\lambda + d)\,y_1'(0) = 0, \qquad y_2(0) + (\lambda + d)\,y_2'(0) = 0$$
$$y_1(1) - \lambda y_1'(1) = 0, \qquad y_2(1) - \lambda y_2'(1) = 0, \tag{26}$$

where $d = -4\pi^2$. The first three eigenvalues were obtained as 9.730886578213082033, 88.76331625258976337, and 157.88411043863472059 putting them at about 10^{-18} from the exact eigenvalues. All these are double eigenvalues.

Example 2. Consider

$$-z_1'' + xz_1 = \mu^2 z_1, \quad -z_2'' + x^2 z_2 = \mu^2 z_2, \quad 0 < x < 1$$
$$z_1(0) = 0, \quad z_2(0) = 0 \tag{27}$$
$$z_1(1) = 0, \quad z_2(1) = 0.$$

The first four eigenvalues were obtained as 10.149980317596192645, 39.426774741845613693, 88.33456043776637171, and 157.20995003768636950. Their multiplicity is two. Figure 1 illustrates the graph of the characteristic function.

In the next example we change the boundary conditions in Example 2 to a parameter dependent one.

Example 3. Consider

$$-z_1'' + xz_1 = \mu^2 z_1, \quad -z_2'' + x^2 z_2 = \mu^2 z_2, \quad 0 < x < 1$$
$$z_1(0) = 0, \qquad z_2(0) = 0$$
$$z_1(1) + \mu z_2(1) + \mu^2 z_1'(1) = 0 \tag{28}$$
$$\mu^2 z_1(1) + z_2(1) + z_1'(1) + z_2'(1) = 0.$$

The first ten eigenvalues were obtained as 1.8774711215942920040, 5.7437101320611511193, 19.343498090220217814, 27.524136648884737570, 57.53122978141141088, 68.17260384634582088, 115.66805223522017586, 128.14561531120321356, 193.70672667853536264, and 207.68609472970380538. All these are simple eigenvalues. Figure 2 illustrates the graph of the characteristic function.

Next we consider three-dimensional vectorial SLPs, with different boundary conditions.

Example 4. Consider

$$-z_1'' + x^2 z_1 = \mu^2 z_1, \qquad -z_2'' + \frac{3x}{2}z_2 - \frac{x}{2}z_3 = \mu^2 z_2,$$
$$-z_3'' - \frac{x}{2}z_2 + \frac{3x}{2}z_3 = \mu^2 z_3, \quad 0 < x < 1 \tag{29}$$
$$z_1(0) = 0, \quad z_2(0) = 0, \quad z_3(0) = 0$$
$$z_1(1) = 0, \quad z_2(1) = 0, \quad z_3(1) = 0.$$

Table 1: μ as a function of N for Example 4.

N	μ_1	μ_2	μ_3	μ_4
20	3.18255001139908084139	3.24667169525908938767	6.30074330869793149011	6.32071003783954269867
40	3.18255272015329488134	3.24667965370940216187	6.30075208373683858194	6.32073018962762953754
60	3.18255257457282246273	3.24667922487683254112	6.30075161400550151819	6.32072910805919989533
80	3.18255257724071613616	3.24667923278762738065	6.30075162260634961554	6.32072912799411873146
100	3.18255257728954451935	3.24667923293045271115	6.30075162276373835846	6.32072912835397407977

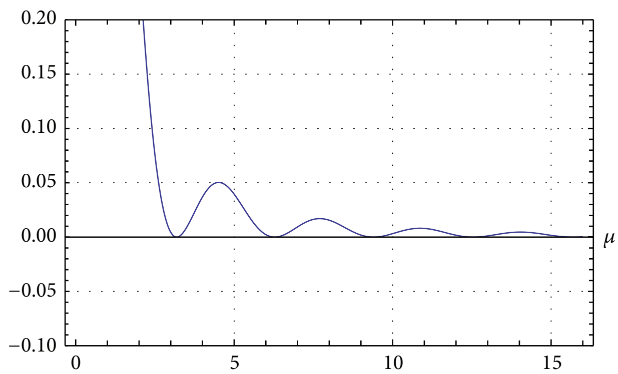

Figure 1: F for Example 2.

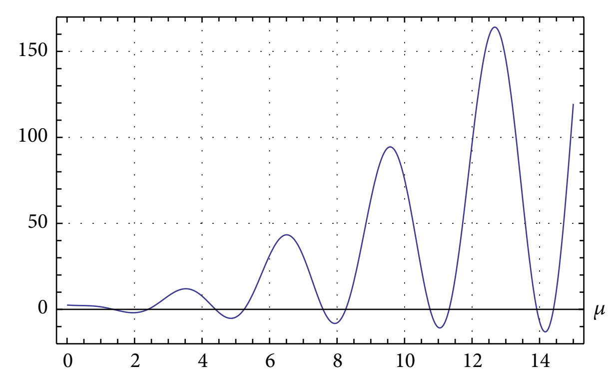

Figure 2: F for Example 3.

Here, we will take $m = 4$, $N = 20, 40, 60, 80$, and 100, and a precision of 10^{-20}. Figure 3 illustrates the characteristic function F_N over the range $[2.8, 6.9]$, while Figures 4 and 5 zoom into the regions containing the eigenvalues. Note that, in the range of interest $[2.8, 6.9]$, the graphs of F_N, $N = 20, 40, 60, 80$, and 100, are on the top of each other. A 10^{-5} precision on μ can be obtained with just $N = 40$. It appears clearly that in this example we have a simple eigenvalue $\lambda_1 = \mu_1^2$ and a double eigenvalue $\lambda_2 = \mu_2^2$, followed by a simple eigenvalue $\lambda_3 = \mu_3^2$ and a double eigenvalue $\lambda_4 = \mu_4^2$ (Figures 7, 8, 9, and 10). To obtain the double eigenvalues we look for the roots $\overline{\mu}$ of $F'(\mu)$ and then evaluate $F(\overline{\mu})$ which happened to be in each case of the order of 10^{-20}. Table 1 illustrates these μ as function of N, the number of sampling points.

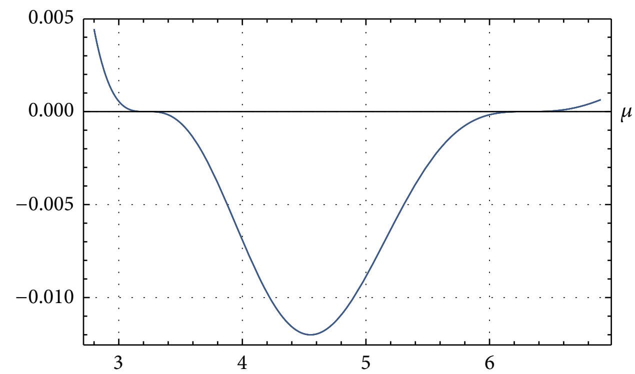

Figure 3: F for Example 4.

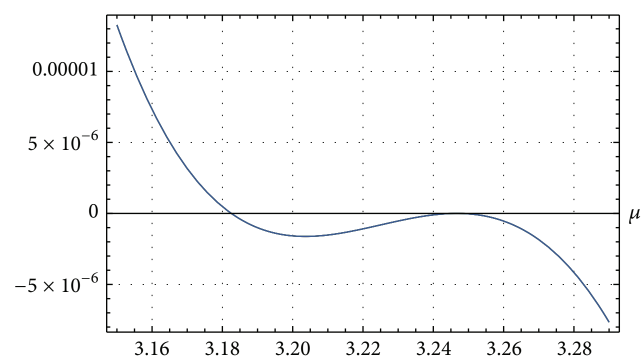

Figure 4: F around the first cluster of zeroes for Example 4.

The first few *alpha* coefficients in the cardinal series expansion of $U_1(1, \mu)$ are given as follows:

$$\alpha_0 = \begin{pmatrix} 0.0507 & 0 & 0 \\ 0 & 0.130 & -0.0447 \\ 0 & -0.0447 & 0.130 \end{pmatrix},$$
$$\alpha_1 = -\alpha_{-1} = \begin{pmatrix} 0.0165 & 0 & 0 \\ 0 & 0.0451 & -0.0157 \\ 0 & -0.0157 & 0.0451 \end{pmatrix},$$
$$\alpha_2 = -\alpha_{-2} = \begin{pmatrix} -0.00455 & 0 & 0 \\ 0 & -0.0105 & 0.00352 \\ 0 & 0.00352 & -0.0105 \end{pmatrix}, \quad (30)$$
$$\alpha_3 = -\alpha_{-3} = \begin{pmatrix} 0.00197 & 0 & 0 \\ 0 & 0.00442 & -0.00147 \\ 0 & -0.00147 & 0.00442 \end{pmatrix}.$$

The above data have been reported with only a few digits.

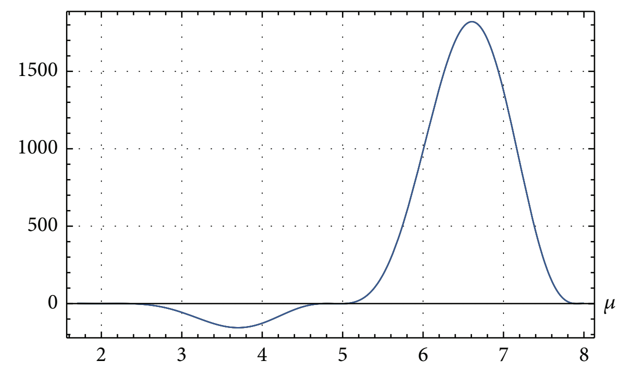

FIGURE 5: F for Example 5.

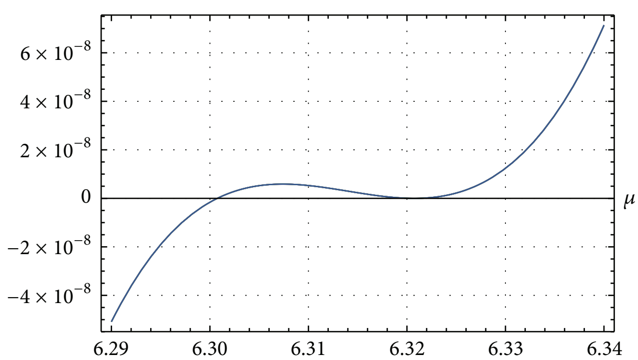

FIGURE 6: F around the second cluster of zeroes for Example 4.

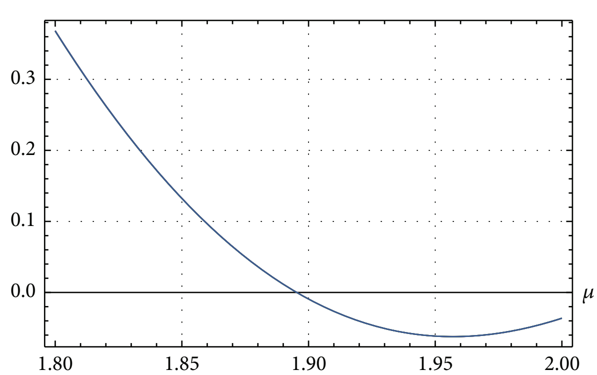

FIGURE 7: F around μ_1 in Example 5.

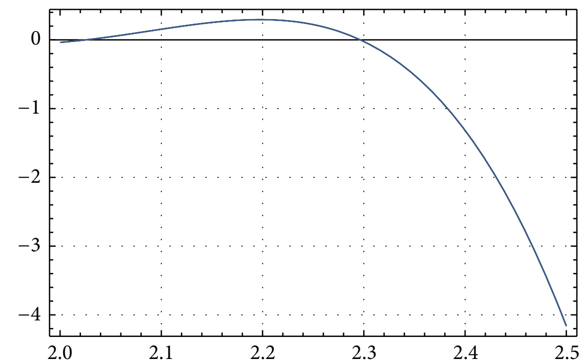

FIGURE 8: F around μ_2 and μ_3 in Example 5.

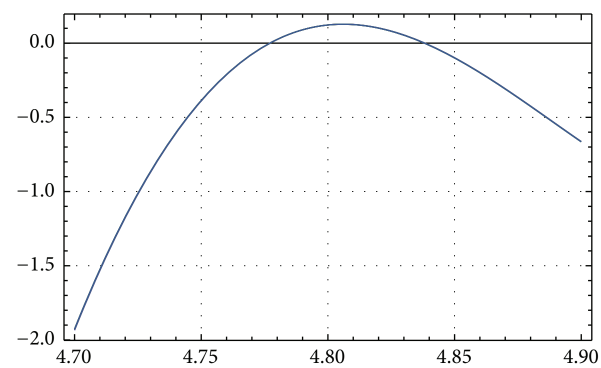

FIGURE 9: F around μ_4 and μ_5 in Example 5.

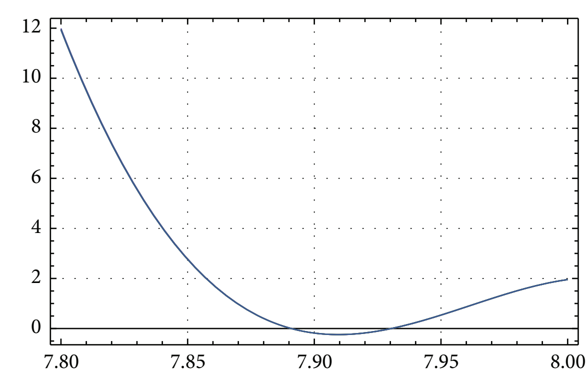

FIGURE 10: F around μ_6 and μ_7 in Example 5.

Example 5. Consider

$$\begin{gathered} -z'' + Qz = \mu^2 z, \quad 0 < x < 1 \\ z(0) = 0 \quad z(1) + Bz'(1) = 0, \end{gathered} \tag{31}$$

where

$$Q = \begin{pmatrix} x^2 & 0 & 0 \\ 0 & \dfrac{3x}{2} & -\dfrac{x}{2} \\ 0 & -\dfrac{x}{2} & \dfrac{3x}{2} \end{pmatrix}, \qquad B = \begin{pmatrix} \mu^2 & \mu & 1 \\ \mu & \mu^2 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \tag{32}$$

Here, we will take $m = 4$, $N = 20, 40, 60, 80$, and 100, and a precision of 10^{-20}. Figure 6 illustrates the characteristic function F_N over the range $[0, 8]$. In this range, the graphs of F_N, $N = 20, 40, 60, 80$, and 100, are on the top of each other. In this example the first seven (07) eigenvalues $\lambda_k = \mu_k^2$, $k = 1, \dots, 7$, are all simple. Tables 2(a) and 2(b) illustrate these μ as function of N, the number of sampling points.

5. Conclusion

In this paper we have extended the domain of application of the regularized sampling method to the case of vectorial

TABLE 2: μ as a function of N for Example 5.

(a)

N	μ_1	μ_2	μ_3	μ_4
20	1.89539417301926842876	2.02558758968702183901	2.29660628213819048323	4.77718462334715123614
40	1.89532715021073954015	2.02545891470303642172	2.29650089361152297043	4.77709199050137280895
60	1.89531503182513465256	2.02543500687372306198	2.29648223596348958117	4.77707495489416773295
80	1.89531699030377280216	2.02543884958899241306	2.29648526430437415553	4.77707769693838662656
100	1.89531696279196052243	2.02543879569567539948	2.29648522170989172904	4.77707765846484979002

(b)

N	μ_5	μ_6	μ_7
20	4.83834197825672007101	7.89099828962247999966	7.93058654329498644499
40	4.83814507749939589802	7.89090804120709347972	7.93039290552171137187
60	4.83810790006947131246	7.89089138476342607463	7.93035620848368882791
80	4.83811385300866362529	7.89089406045343050839	7.93036207289652547074
100	4.83811376961267015403	7.89089402293178420005	7.93036199078631128916

Sturm-Liouville problems with parameter dependent boundary conditions. We have presented the theoretical foundation of the method and worked out a few examples to illustrate the method and shown by the same token that we have indeed a general method capable of handling with ease very broad classes of SLPs, whether scalar or vectorial, and providing the results at a reduced cost.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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