Internat. J. Math. & Math Sci.
VOL. 11 NO. 1 (1988) 43-46

RATIONAL CHOICE FUNCTION DERIVED FROM A FUZZY PREFERENCE

JIN BAI KIM

Department of Mathematics
West Virginia University
Morgantown, W. V. 26506

KERN O. KYMN

Department of Economics
West Virginia University
Morgantown, W. V. 26506

(Received September 7, 1984 and in revised form April 22, 1987)

ABSTRACT. We shall prove that every fuzzy rational choice function is fuzzy regular (see Richter [6, p. 36]), count the total number of the fuzzy rational choice functions on a set of four elements and consider a semigroup of all fuzzy rational choice functions on a set.

KEY WORDS AND PHRASES. Fuzzy relation - fuzzy binary relation - fuzzy preference - choice function - fuzzy rational choice function - fuzzy transitive - fuzzy regular - semigroup. 1985 AMS CLASSIFICATION NUMBER 03E72

1. INTRODUCTION. We have introduced a rational choice function derived from a fuzzy preference (see [2], [3], [4]). We shall establish two theorems (Theorems 1 and 2) which are motivated from the following theorems:

THEOREM 4 (Richter [6]). There exists a total rational choice which is not transitive rational.

THEOREM 6 (Richter [6]). There exists a rational choice which is not total rational.

We find that the number of all fuzzy rational choice functions on a set X = {a, b, c, d} of four elements is equal to 57751 (see [2]. We shall consider a semigroup. We note that in [4] there is a beautiful counting formula of the total number of all final choice functions on a finite set.

2. DEFINITIONS AND THEOREMS.

Let X be a finite set with more than two elements. For definitions of a choice function on X and a fuzzy binary relation (R, r) on X, we refer to [2] and [3].

DEFINITION 1 [2, p. 38]. Let (R, r) be a fuzzy relation X and let $a \in X$. Define $R(a) = \{x \in X: aRx$ and $r(a,x) \neq 0\}$ and $R_t(a) = \{x \in R(a): r(a,x) \geq \frac{1}{t}\}$ for $\frac{1}{t} \in (0,1)]$. We define a function h_R as follows: Let $a \in A \subseteq X$. Then $a \in h_R(A)$ iff $A \subseteq R_A(a)$. We add that $h_R(\emptyset) = \emptyset$, the empty set. Note that h_R is in general, not a choice function. Let h be a choice function on X. If there exists a fuzzy relation (R, r) on X such that $h_R = h$, then we shall say that h is fuzzy rational and (R, r) rationalizes h.

NOTATION 1. We denote by F(X) the set of all fuzzy binary relations on X. We define $\Sigma = 2^X$ and $C(X, \Sigma)$ denotes the set of all choice functions h on X. Let $(R, r) \in F(X)$. We use $(x, y) \in R$ and x R y when $r(x,y) \neq 0$. Let $h \in C(X, \Sigma)$ be a choice function on X. Define $F(h) = \{(R, r) \in F(X): (R,r)$ rationalizes $h\}$.

DEFINITION 2. h is said to be fuzzy transitive (total, reflexive) if there exists (R, r) in $\in F(h)$ such that (R, r) is transitive (total, reflexive). (R,r) F(X) is regular if (R,r) is reflexive, total and transitive. h is fuzzy regular if there exists $(R, r) \in F(h)$ such that (R, r) is regular.
We shall prove the following theorem.

THEOREM 1. Every fuzzy rational choice function is fuzzy transitive.

PROOF. Let h be a fuzzy rational choice function on X. Then F(h) is non-empty and let $(R, r) \in F(h)$. Then $h = h_R$. Suppose that (R, r) is not transitive. Define $\{r\} = \{r(x,y) \neq 0: x,y \in X\}$ for (R, r). We can find a positivie number $t_0 = \frac{1}{n+k}$ such that $t_0 \notin \{r\}$, where k is a positive integer. We define a fuzzy relation (S, s) as follows: If $r(x,y) \neq 0$, then we put $s(x,y)=r(x,y)$, and if $r(x,y)=0$ then we put $s(x,y)=t_0$. It is clear that (S,s) is a transitive fuzzy relation on X. We show that $h_R=h_S$. To show this, we assume that $h_R \neq h_S$. Then there exists a non-empty set A such that $B = h_R(A) \neq h_S(A)=C$. We can assume that $c \in C$ and $a \notin B$. Then $(a,x) \in S$ for all $x \in A$, $s(a,x) \geqq \frac{1}{|A|} > \frac{1}{n+k} = t_0$, and hence $s(a,x) \neq t_0$. In view of {r} and $t_0 \notin \{r\}$, it is clear that $s(a,x)=r(a,x)$ for all $x \in A$, and hence $a \in B$. This contradicts $a \notin B$. A similar proof for $b \in B$ and $b \notin C$ brings a contradiction. Therefore B=C and $h_R=h_S=h$. This proves Theorem 1.

THEOREM 2. Every fuzzy rational choice function h on X is fuzzy total.

PROOF. Let h be a fuzzy rational choice function on X. Then there exists (R, r) such that $h_R=h$. For $x, y \in X$ and $x \neq y$, it is clear that $h_R\{x,y\} \subseteq \{x,y\}$. Thus we have that either $r(x,y) \geqq \frac{1}{2}$ or $r(y,x) \geqq \frac{1}{2}$. Therefore (R,r) is total. This proves Theorem 2.

COROLLARY 1. Every fuzzy rational choice function is regular. The proof follows from Theorems 1 and 2.

3. A SEMIGROUP.

We begin with the following definition.

DEFINITION 3. Let $(R, r) \in F(X)$ be a fuzzy relation. (R, r) is completely total if $r(a,b) \neq 0$ and $r(b,a) \neq 0$ for all $a,b \in X$. A choice function h is fuzzy completely total if there exists $(R,r) \in F(X)$ such that $h_R=h$ and (R, r) is completely total. h is fuzzy completely regular if there exists (R, r) such that $h=h_R$ is fuzzy regular and fuzzy completely total.
We have considered a semigroup in [2] and [4]. We denote by CR(X) the set of all completely regular fuzzy rational choice functions on X. By Theorem 4-(i)[2], we have that $h_P h_Q \subseteq h_{P \cup Q}$, $h_P, h_Q \in CR(X)$. Thus we have the following theorem.

THEOREM 3. CR(X) forms a semigroup under the binary operation defined by $h_P h_Q = h_{P \cup Q}$, $h_P, h_Q \in CR(X)$.
We note that if $h \in CR(X)$, then there exists (P,p) such that $h=h_P$ and (P, p) is regular and completely total.

PROOF. It is clear that the binary operation is associative. It is also clear that $P \cup Q = R$ (or(R, r)) is regular and completely total. Letting $P \cup Q =$

R, $h_R(A) \subseteq A$ is a part of the definition of h_R (see Definition 1). We prove that $h_R(A)$ is non-empty when A is non-empty. We assume that $A \neq \emptyset$ and $|A| = m$. Since $h_P(A) \neq \emptyset$, there exists $a \in h_P(A)$ and hence $p(a,x) \geqq \frac{1}{m}$, for all $x \in A$. From $r(a,x)=\max\{p(a,x), q(a,x)\}$ it follows that $r(a,x) \geqq \frac{1}{m}$ for all $x \in A$. This shows that $a \in h_R(A)$. This proves Theorem 3.

The following example is to show that $h_P(h_Q)$, the composite set function, is not a fuzzy rational choice even though h_P and h_Q are both fuzzy rational choices on X.

EXAMPLE 1. Let X={a, b, c}. Let (R, r)={r(a,a)=r(b,b)=r(c,c)=1, $r(a,b)=r(a,c)=r(b,c)= \frac{1}{2}$, $r(b,a)=r(c,a)=r(c,b) = \frac{1}{4}$ } and (Q,q)={q(a,a)=q(b,b)=q(c,c)=1, $q(b,a)=q(c,a)=q(c,b)= \frac{1}{2}$, $q(b,c)= \frac{1}{3}$, $q(a,b)=q(a,c)= \frac{1}{5}$}. Then we can prove that there is not a fuzzy relation (P, p) such that $h_P=h_R(h_Q)$.

We list the following theorem.

THEOREM 4. Let (r, r) be a fuzzy relation on X. A necessary and sufficient condition for h_R to be a choice function on X is that for every non-empty subset A of X there exists at least one member a in A such that $r(a,x) \geqq \frac{1}{|A|}$ for all $x \in A$.

PROOF. We suppose that the condition holds for (R,r). Let $A \neq \emptyset$ and assume that there is a in A such that $r(a,x) \geqq \frac{1}{|A|}$ for all $x \in A$. Then $A \subseteq R_{|A|}(a)$ and $a \in h_R(A)$. $h_R(A) \subseteq A$ is a part of the definition of h_R. Thus h_R is a choice function on X. Suppose h_R is a choice on X. Then for each $A \neq \emptyset$ there is a in A such that $a \in h_r(A)$ from which we obtain that $r(a,x) \geqq \frac{1}{|A|}$. This proves Theorem 4.

4. THE NUMBER OF ALL FUZZY RATIONAL CHOICES ON {a,b,c,d}. Let X be a set of n elements. We denote the number of all fuzzy rational choice functions on X by $h_{F(X)}(n)$. In [2] we showed that $h_{F(X)}(3) = 93$. In this section we announce that $h_{F(X)}(4) = 57751$. WE shall prove this in a separate paper. A justification of $h_{F(X)}(4) = 57751$ needs several pages.

REFERENCES

[1] K. J. Arrow, Social Choice and Individual Value (Wiley, New York, 1963).

[2] Jin B. Kim, Fuzzy Rational Choice Functions, FUZZY SETS AND SYSTEMS 10(1983), 37-43.

[3] Jin B. Kim and Kern O. Kymn, Rational Choice and Gain Functions Derived From a Fuzzy Relation, ECONOMICS LETTERS 13)1983), 113-116.

[4] Jin B. Kim, Final Choice Functions, ECONOMICS LETTERS 14(1984), 143-148.

[5] Jin B. Kim, A Certain Matrix Semigroup, Mathematica Japonica 22(1978), 519-522.

[6] M. K. Richter, Rational Choice, in: J. S. Chipman, L. Hurwicz, M. K. Richter, and H. F. Sonnenschein, Eds., Preferences, Utility and Demand (Harcourt Brace Jovanovich, New York, 1971).