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Research Article

Approximate Solutions of Fractional Riccati Equations Using the Adomian Decomposition Method

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The fractional derivative equation has extensively appeared in various applied nonlinear problems and methods for finding the model become a popular topic. Very recently, a novel way was proposed by Duan (2010) to calculate the Adomian series which is a crucial step of the Adomian decomposition method. In this paper, it was used to solve some fractional nonlinear differential equations.

1. Introduction

The fractional derivative has good memory effects compared with the ordinary calculus. In view of this point, it has been proven as a good tool in nonlinear science, for example, the anomalous diffusion [1, 2], the material's viscoelasticity [3, 4], the chaotic behaviors of biology population [5, 6], and so forth.

However, everything has two sides. The fractional derivative's memory effects also lead to the numerical solutions' accumulative errors. Many nonlinear techniques cannot perform the same role as those in ordinary differential equation. For example, the variational iteration method cannot be applied due to the fact that the integral by parts cannot hold and the Lagrange multipliers there are not easily identified; in the Adomian decomposition method, the Adomian series cannot be expanded large enough which greatly affects the solutions' accuracies and even five- or six-order approximation becomes impossible (see the analysis in [7]).

Very recently and fortunately, for the ADM, Duan [8, 9] proposed a convenient way to calculate Adomian series which is the main and crucial step of the classical ADM developed by Adomian. This way can rapidly decompose the nonlinear terms and some new high order approximation schemes for nonlinear differential equations are proposed [10]. The technique has been successfully extended to fractional differential equations and boundary value problems [11, 12].

In this paper, we investigate the following fractional nonlinear differential equation:

$$ {}_{0}^{C}D_{t}^{\alpha} x = 1 + 2x(t) - x(t)^{2}, \quad 0 < \alpha \leq 2, \tag{1}$$

where ${}_{0}^{C}D_{t}^{\alpha} x$ is the Caputo derivative with respect to $x(t)$.

The paper is organized as follows: Section 2 introduces some basics of the ADM and the fractional calculus; Section 3 considers the differential equation from the case $\alpha = 1$ in (1)

$$\frac{dx}{dt} = 1 + 2x(t) - x(t)^{2}, \quad x(0) = 0 \tag{2}$$

and gives the analytical formula.

2. Preliminaries

Definition 1 (see [13]). The Caputo derivative is defined as

$$ {}_0^C D_t^\alpha u = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{1}{(t-\tau)^{\alpha-m+1}} \frac{d^m}{d\tau^m} u(\tau)\, d\tau, \quad 0 < t, \quad 0 < \alpha, \quad m = [\alpha] + 1, \tag{3} $$

where Γ is the Gamma function.

Definition 2 (see [14]). The R-L integration of α order is defined by

$$ {}_0I_t^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} u(\tau)\, d\tau, \quad 0 < t, \quad 0 < \alpha. \tag{4} $$

Generally, consider the following nonlinear equation:

$$ L[u] + R[u] + N[u] = g(t), \tag{5} $$

where L is the highest derivative, R is the remaining linear part containing the lower order derivatives, and N is the nonlinear operator.

Apply the inverse L^{-1} of the linear operator L in (5) and we can obtain

$$ u = u(0) + u'(0)t + \cdots + u^{(m-1)}(0) \frac{t^{m-1}}{(m-1)!} + L^{-1}\left(g(t) - R[u] - N[u]\right). \tag{6} $$

Assume that

$$ u = \sum_{i=0}^{\infty} u_i \tag{7} $$

and expand the term $N[u]$ approximately as

$$ N[u] = \sum_{n=0}^{\infty} A_n, \tag{8} $$

where the A_n is calculated by

$$ A_n = \frac{1}{n} \sum_{k=0}^{n-1} (k+1) u_{k+1} \frac{dA_{n-1-k}}{du_0}. \tag{9} $$

As a result, one can obtain the analytical iteration scheme as

$$ u_{n+1} = -L^{-1}(R[u_n]) - L^{-1}(A_n[u_0, u_1, \ldots, u_n]), $$
$$ u_0 = u(0) + u'(0)t + \cdots + u^{(m-1)}(0) \frac{t^{m-1}}{(m-1)!} + L^{-1}(g(t)). \tag{10} $$

Here we conclude the application of the ADM: firstly, one needs to have equivalent integral form of the original governing equations; second, decompose the nonlinear terms into linear ones and determine the iteration schemes; third, obtain the solutions successively. For the other applications and modified versions, we do not introduce any more here. The readers who feel interested in the development of the method are referred to [7, 12, 15–19].

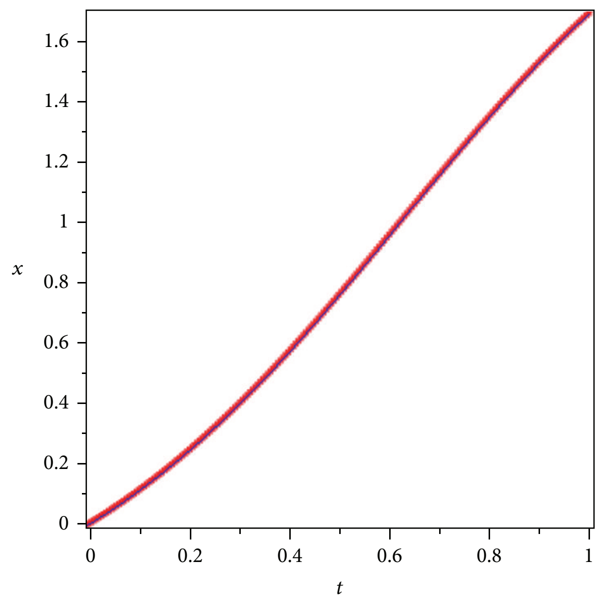

Figure 1: "$*$" the 20th order approximate solution; "–" the explicit solution.

3. Numerical Schemes of Integer Equations

According to the ADM, we first establish the integral equation as the one

$$ x(t) = x(0) + \int_0^t \left(1 + 2x(\tau) - x(\tau)^2\right) d\tau. \tag{11} $$

If we directly use the classical ADM's idea we can obtain the formula

$$ x_{n+1} = 2L^{-1}(x_n) + L^{-1}(A_n[x_0, x_1, \ldots, x_n]), \quad x(t) = \sum_{i=0}^{\infty} x_i. \tag{12} $$

The explicit solution of (2) was found to be [20]

$$ x(t) = 1 + \sqrt{2} \tanh\left(\sqrt{2}t + \frac{1}{2} \log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right). \tag{13} $$

We can assume that

$$ A_0 = -{x_0}^2, \quad x_0 = x(0) + t. \tag{14} $$

As a result, from the iteration equations (10) and (12), we can obtain the approximate solutions in comparison with (13).

We find that the approximate solution has a good agreement with the explicit solution in Figure 1. So the scheme is efficient and useful.

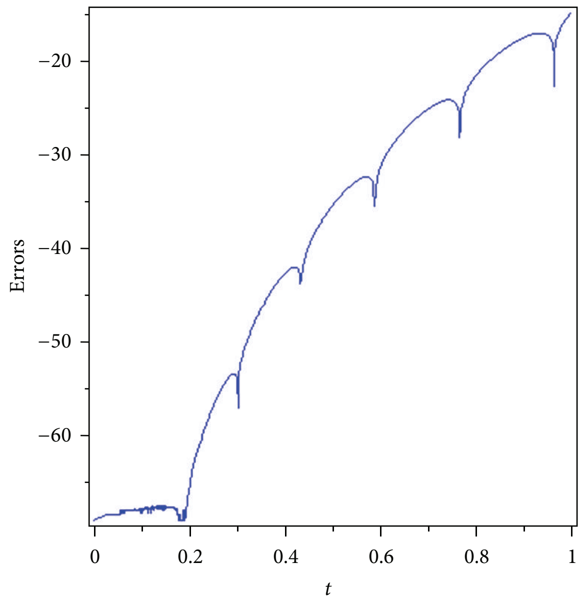

Figure 2: "–" the residual function.

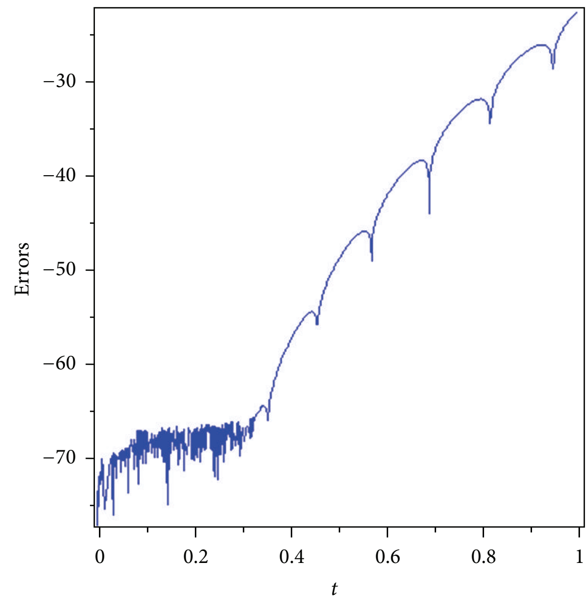

Figure 3: "–" the residual function.

4. Numerical Schemes of Fractional Equations

Case 1 ($0 < \alpha \leq 1$). Similarly, we can have the integral equation for (1):

$$x(t) = x(0) + \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left(1 + 2x(\tau) - x(\tau)^2\right) d\tau, \quad 0 < \alpha \leq 1. \tag{15}$$

If we directly use the classical ADM's idea we can obtain the formula

$$x_{n+1} = \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} A_n[x_0, x_1, \ldots, x_n]\, d\tau, \qquad x(t) = \sum_{i=0}^{\infty} x_i. \tag{16}$$

Here $x_0 = x(0)$, we give the first solutions as

$$x_0(t) = t + \frac{t^\alpha}{\Gamma(\alpha)\alpha}. \tag{17}$$

For simplicity, we define the residual function as

$$\mathrm{err} = \ln\left|{}_0^C D_t^\alpha x_n = 1 + 2x_n(t) - x_n(t)^2\right| \tag{18}$$

and illustrate the errors for $n = 30$ and $\alpha = 0.9$ in Figure 2.

Case 2 ($1 < \alpha \leq 2$). Consider the following.

$$x(t) = x(0) + x'(0)t + \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left(1 + 2x(\tau) - x(\tau)^2\right) d\tau, \quad 1 < \alpha \leq 2. \tag{19}$$

Here we need two initial conditions $x(0)$ and $x'(0)$. We set

$$x(0) = 0, \qquad x'(0) = 1. \tag{20}$$

Similarly, we can have the iteration formula as

$$x_{n+1} = \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} A_n[x_0, x_1, \ldots, x_n]\, d\tau, \qquad x_0 = x(0) + x'(0)t = t + \frac{t^\alpha}{\Gamma(\alpha)\alpha}, \qquad x(t) = \sum_{i=0}^{\infty} x_i, \tag{21}$$

whose error analysis is given in Figure 3. We can see that the approximate solution is reliable.

We plot the approximate solution in Figure 4 in the interval $[0, 1]$.

5. Conclusions

This paper applied the famous Adomian decomposition method to FDEs whose fractional order varied from 0 to 2. First, we revisit the method for integer equation. We conclude the general steps and use the method to solve FDEs analytically. Through the analysis of the residual function, it can be concluded that the ADM using Duan's way to calculate the Adomian series is very suitable and efficient for obtaining analytical solutions of fractional nonlinear differential equations.

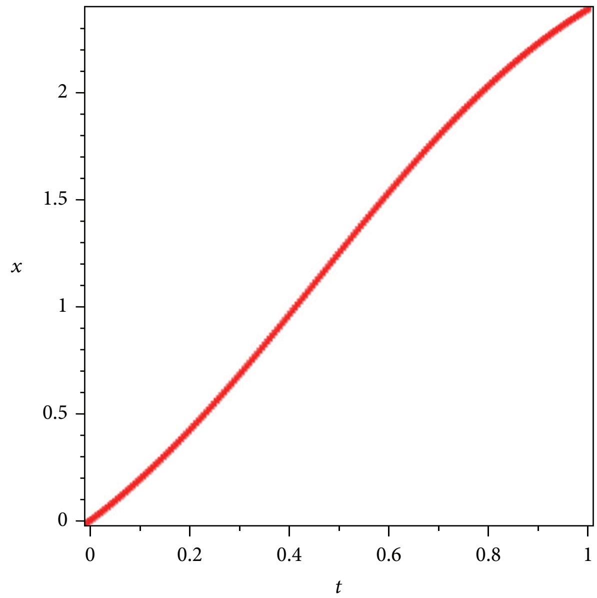

FIGURE 4: "$*$" the 30th order approximate solution of (19).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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